THE H-PRINCIPLE, LECTURE 23: GOODWILLIE-WEISS CALCULUS

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Let us start with an example:

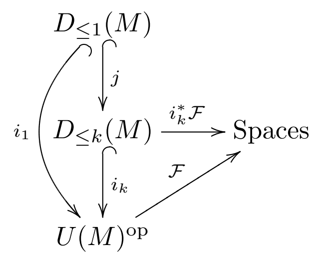
Claim 0.1. The functor $\mathrm{Mfld}_n^{\mathrm{op}} \xrightarrow{\mathrm{Map}((-)^k, X)} \mathrm{Spaces}$ is polynomial of degree k.

Proof. Recall that this means that for any $k+1$-cube of spaces $A_i \subset M \mid 1 \leq i \leq k+1$, (where A_i are pairwise disjoint, open subspaces which are the interiors of codim 1 submanifolds) the map $\mathcal{F}(M) \to \mathrm{holim}_{S \subset k+1}(\mathcal{F}(M - \cup_S A_i))$ is a weak homotopy equivalence. So we just have to check that $M^k \leftarrow \mathrm{colim}(M - \cup A_i)^k$ is a weak homotopy equivalence (exercise). □

This functor is a good example of how this theory behaves. We will come back to it later.

Proposition 0.2. *If there exists a functor* $\mathrm{Mfld}_n^{\mathrm{op}} \to \mathrm{Spaces}$ *such that* $\mathcal{F}(\coprod D_i^n) \to \prod \mathcal{F}(D_i^n)$ *is a weak homotopy equivalence, then* $T_1\mathcal{F} \simeq T_k\mathcal{F}$ *for all* $k \geq 1$.

Proof. Consider the commutative diagram:



$T_k\mathcal{F} = i_{k*}i_k^*\mathcal{F}$, where i_{1*} is the right Kan extension. Notice that $i_k^*\mathcal{F} = j_*i_1^*\mathcal{F}$ because they both agree on the disks and preserve products. Then we get:

$$T_k\mathcal{F} = i_{k*}i_k^*\mathcal{F} = i_{k*}j_*i_1^*\mathcal{F} = i_{1*}i_1^*\mathcal{F} = T_1\mathcal{F}$$

□

So, for many examples that we considered, Goodwillie-Weiss calculus doesn't really help us. For example:

1. $T_k \mathrm{Imm}_N = T_1 \mathrm{Imm}_N = \mathrm{Imm}^f$
2. $T_k \mathrm{Subm}_N = T_1 \mathrm{Subm}_N = \mathrm{Subm}_N^f$. Note, that we still do not understand $\mathrm{Subm}(M, N)$ when M is closed.
3. $\mathcal{F} = \mathrm{Map}((-)^k, X)$. In this case $T_{k-1}\mathcal{F} = \mathrm{Map}(\triangle_{\leq k-1}(-)^k, X)$, so $T_{k-1}\mathrm{Map}(M^k, N) = \mathrm{Map}(\triangle_{\leq k-1}(M)^k, X)$.

Definition 0.3. The functor $\mathcal{F}$ is called k-homogeneous if it is a polynomial of degree less than or equal to k and $T_{k-1}\mathcal{F} \simeq *$.

By analogy with the Postnikov tower methods we should now study the fibers of the maps $T_k\mathcal{F} \to T_{k-1}\mathcal{F}$.

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Definition 0.4. Given a fixed point in $\mathcal{F}(M)$ define $L_k\mathcal{F}$ as the functor

$$\begin{array}{rcl} U(M)^{\mathrm{op}} & \longrightarrow & \text{Spaces} \\ U & \longmapsto & L_k\mathcal{F}(U) \end{array}$$

which assigns to every cover U the value of the homotopy pullback

$$\begin{array}{ccc} T_k\mathcal{F}(U) & \longleftarrow & L_k\mathcal{F}(U) \\ \downarrow & & \downarrow \\ T_{k-1}\mathcal{F}(U) & \longleftarrow & * \end{array}$$

Lemma 0.5. *$L_k\mathcal{F}$ is homogeneous polynomial of degree k.*

Proof. Let us apply T_{k-1} to the square that defines $L_k\mathcal{F}$. Then in the homotopy pullback diagram

$$\begin{array}{ccc} T_{k-1}\mathcal{F} = T_{k-1}T_k\mathcal{F} & \longleftarrow & T_{k-1}L_k\mathcal{F} \\ \downarrow \simeq & & \downarrow \simeq \\ T_{k-1}\mathcal{F} = T_{k-1}T_{k-1}\mathcal{F} & \longleftarrow & * \end{array}$$

the left arrow is homotopy equivalence, so the right arrow has to be homotopy equivalence as well and $T_{k-1}L_k\mathcal{F} \simeq *$ □

Example 0.6. Let us try to understand the homogeneous layers of the functor $\mathcal{F} = \mathrm{Map}((-)^k, X)$.

$$\begin{array}{ccccc} \Delta_{k-1}(M^k) & \hookrightarrow & M^k & \hookleftarrow & \mathrm{Conf}_k(M) \\ \downarrow & & \downarrow & & \\ * & \hookrightarrow & M/\Delta_{\leq k-1} & = & \mathrm{Conf}_k(M) \end{array}$$

Applying the functor $\mathrm{Map}(-, X)$ to the above diagram we get:

$$\begin{array}{ccccc} T_{k-1}\mathcal{F}(M) & = & \mathrm{Map}(\Delta_{\leq k-1}M^k, X) & \longleftarrow & \mathrm{Map}(M^k, X) \\ & & \uparrow & & \uparrow \\ & & * & \longleftarrow & L_k\mathcal{F}(M) \end{array}$$

And, by the universal property of pullback $\mathrm{Map}_c(\mathrm{Conf}_k(M), X) \simeq L_k\mathcal{F}(M)$.

This observation leads us to the following theorem which will be proved in the next lecture:

Theorem 0.7. *Any homogeneous of degree k functor is equivalent to $\Gamma_c(C_k(M), Z)$, where $C_k(M) = \mathrm{Conf}_k(M)_{\Sigma_k}$ and $Z \to C_k(M)$ is a Serre fibration with a section.*

References

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