THE H-PRINCIPLE, LECTURE 4: FLEXIBLE SHEAVES

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1. Flexible sheaves

We might say that a (pre)sheaf $\mathcal{F}$ on M is *very flexible* if all the restriction maps $\mathcal{F}(V) \to \mathcal{F}(U)$ are Serre fibrations, for every inclusion for every inclusion of opens $U \to V$. Unfortunately, even very well-behaved sheaves, as the sheaf of functions $\mathrm{Map}(-,\mathbb{R})$, typically fail to satsify this very strong condition: For instance, for an open subspace of a disk $U \subset D^n$, the restriction $\mathrm{Map}(D^n\mathbb{R}) \to \mathrm{Map}(U,\mathbb{R})$ is never a Serre fibration. We can ask for a slightly weaker condition: Recall from the last lecture that a (pre)sheaf of spaces $\mathcal{F}$ is flexible if all restriction maps $\mathcal{F}(L) \to \mathcal{F}(K)$ are Serre fibrations for any inclusion of compact subspaces $K \subset L$. Some of the importance of this notion is due to the following key observation (which also holds for very flexible sheaves):

Proposition 1.1. *Let $\mathcal{F} \to \mathcal{F}'$ be a map of flexible sheaves on a compact n-manifold M, possibly with boundary. If $\mathcal{F}(U) \to \mathcal{F}'(U)$ is a weak homotopy equivalence for every $U \subset M$ which is contractible, $U \simeq \mathrm{pt}$, then $\mathcal{F}(M) \to \mathcal{F}'(M)$ is a weak homotopy equivalence.*

Proof. Since M is compact and smooth, we can choose a finite cover of a M by n-disks D_i, $1 \leq i \leq k$, such that the intersection of D_{j+1} with $M_j := \bigcup_{i\leq j} D_i$ is a thickened sphere, $M_j \cap D_{j+1} \cong S^l \times D^{n-l}$. (Such a decomposition of M is essentially provided by a handlebody decomposition, and can be obtained from any Morse function on M.)

First we prove that $\mathcal{F}(S^l \times D^{n-l}) \to \mathcal{F}'(S^l \times D^{n-l})$ is a weak homotopy equivalence for any submanifold $S^l \times D^{n-l} \subset M$. We prove this sequentially on increasing l. For the initial case of $l = 0$, the combination of the sheaf property and the fact that weak homotopy equivalences are preserved by taking products implies the chain of equivalences

$$\mathcal{F}(S^0 \times D^n) \cong \mathcal{F}(\{-1\} \times D^n) \times \mathcal{F}(\{1\} \times D^n) \simeq \mathcal{F}'(\{-1\} \times D^n) \times \mathcal{F}'(\{1\} \times D^n) \cong \mathcal{F}'(S^0 \times D^n).$$

Now, assuming the equivalence of $\mathcal{F}$ and $\mathcal{F}'$ on all submanifolds of M diffeomorphic to $S^l \times D^{n-l}$ we show the equivalence on submanifolds diffeomorphic to $S^{l+1} \times D^{n-l-1}$. Choose a submanifold $S^{l+1} \times D^{n-l-1} \subset M$, and consider a decomposition of the $(l+1)$-sphere as a union of Euclidean spaces over a neighborhood of the equator, $S^{l+1} \cong D^{l+1} \cup_{S^k \times D} D^{l+1}$. Applying $\mathcal{F} \to \mathcal{F}'$ to these pushout squares of submanifolds, we obtain a map of pullback squares of spaces:

$$\begin{tikzcd}
\mathcal{F}(S^{l+1} \times D^{n-l-1}) \arrow[r] \arrow[d] & \mathcal{F}(D^{l+1} \times D^{n-l-1}) \arrow[d] \arrow[r, phantom, "\Longrightarrow"] & \mathcal{F}'(S^{l+1} \times D^{n-l-1}) \arrow[r] \arrow[d] & \mathcal{F}'(D^{l+1} \times D^{n-l-1}) \arrow[d] \\
\mathcal{F}(D^{l+1} \times D^{n-l-1}) \arrow[r] & \mathcal{F}(S^l \times D^{n-l}) & \mathcal{F}'(D^{l+1} \times D^{n-l-1}) \arrow[r] & \mathcal{F}'(S^l \times D^{n-l})
\end{tikzcd}$$

By the original assumption together with the inductive assumption, this map is a weak homotopy equivalence for all three of the subspaces of $S^{l+1} \times D^{n-l-1}$. Since the diagram maps are Serre fibrations, we obtain that the map $\mathcal{F}(S^{l+1} \times D^{n-l-1}) \to \mathcal{F}'(S^{l+1} \times D^{n-l-1})$ is a weak homotopy equivalence.

We now prove the proposition by induction on the submanifolds M_j. Consider the inductive step, where we assume the proposition for M_j and then infer it for M_{j+1}. Expressing M_{j+1} as a union of M_j and D_{j+1}, we then have a map of pullback squares

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$$\begin{array}{ccc} \mathcal{F}(M_{j+1}) \longrightarrow \mathcal{F}(D_{j+1}) & \implies & \mathcal{F}'(M_{j+1}) \longrightarrow \mathcal{F}'(D_{j+1}) \\ \downarrow \qquad\qquad \downarrow & & \downarrow \qquad\qquad \downarrow \\ \mathcal{F}(M_j) \longrightarrow \mathcal{F}(M_j \cap D_{j+1}) & & \mathcal{F}'(M_j) \longrightarrow \mathcal{F}'(M_j \cap D_{j+1}) \end{array}$$

Since $M_j \cap D_{j+1}$ is diffeomorphic to a thickened sphere, the map $\mathcal{F}(M_j \cap D_{j+1}) \to \mathcal{F}'(M_j \cap D_{j+1})$ is a weak homotopy equivalence, using the first part of the proof. Thus, the maps on the three subspaces of M_{j+1} are all weak homotopy equivalences, and since the diagram maps are fibrations, the induced map $\mathcal{F}(M_{j+1}) \to \mathcal{F}'(M_{j+1})$ is a weak homotopy equivalence. Hence, the map $\mathcal{F}(M) \to \mathcal{F}'(M)$ is a weak homotopy equivalence.

□

Question 1.2. This proposition should absolutely be valid for topological manifolds, but smoothness is unfortunately used in the proof because of the use of Morse theory. This shortcoming is fundamental: For instance, nonsmoothable topological 4-manifolds don't have handlebody structures. Can you find a proof that works for topological manifolds?

The following is an important observation: Note that in the proof of the proposition, we didn't truly use that the maps $\mathcal{F}(L) \to \mathcal{F}(K)$ were Serre fibrations for *all* inclusions $K \subset L$: The argument goes through exactly as well only using the particular case when L is diffeomorphic to a disk D^n, K is a thickened sphere, and the inclusion is a standard embedding $S^l \times D^{n-l} \subset D^n$.

2. Fibrations of mapping spaces

The preceding proposition clearly shows that it's critical to understand when restriction induces a fibration on various mapping spaces. Before moving on to immersions and formal immersions, it's natural to first fit the more basic study of mapping spaces such as $\mathrm{Map}(M, N)$ and $\mathrm{Map}^{\mathrm{sm}}(M, N)$, into the previous rubric of flexible sheaves.

Recall that the space of continuous maps $\mathrm{Map}(M, N)$ has the compact-open open topology, a subbasis of which is formed by those subsets $S(K, U) = \{f : M \to N | f(K) \subset U\}$, K a compact subspace of M, U an open subspace of N. If M is compact, then this topology is the same as the topology of uniform convergence, so that a sequence of maps f_n converges to f if the supremum of $d(f(x), f_n(x))$ tends to zero, for any choice of a metric d on N. (If M is not compact, you could instead consider the topology of uniform convergence: This is horrific from the point of view of homotopy types.) The compact-open topology on $\mathrm{Map}(M, N)$ is okay if M and N are compactly generated spaces (see [**?**]), and it's even better if M is compact. We have the following basic result from algebraic topology

Lemma 2.1. *Take X_0, X, Y be compactly generated spaces, and let $X_0 \hookrightarrow X$ be a cofibration (or Serre cofibration, respectively). Then the restriction map $\mathrm{Map}(X, Y) \to \mathrm{Map}(X_0, Y)$ is a fibration (Serre fibration, respectively).*

Proof. For any K, we want to solve the lifting problem

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Map}(X, Y) \arrow[d] \\ K \times [0,1] \arrow[r] \arrow[ur, dashed] & \mathrm{Map}(X_0, Y) \end{tikzcd}$$

This lifting problem above is equivalent to the lifting problem

$$\begin{tikzcd} K \times X_0 \arrow[r] \arrow[d] & \mathrm{Map}([0,1], Y) \arrow[d] \\ K \times X \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Now the product $K \times X_0 \to K \times X$ is a cofibration. (This can been seen using the explicit condition for a cofibration, that $A \to A'$ is a cofibration if and only if $A' \times [0,1]$ retracts onto the subspace $A \times [0,1] \cup_{A \times \{0\}} A' \times \{0\}$.) And the restriction map $\mathrm{Map}([0,1], Y) \to Y$ is a fibration and a homotopy equivalence, so therefore the lift in this diagram exists.

□

Now, how about the case of smooth functions: when does restriction to a submanifold induce a fibration of mapping spaces? We will prove the following:

Lemma 2.2. *Let M_0 be a codimension zero submanifold of M, a compact manifold with boundary, and let N be a manifold without boundary. Then the restriction map $\mathrm{Map}^{\mathrm{sm}}(M, N) \to \mathrm{Map}^{\mathrm{sm}}(M_0, N)$ is a Serre fibration.*

Remark 2.3. The precise condition is that there exists a smooth function $f : M \to \mathbb{R}$ such that $M_0 = f^{-1}\mathbb{R}_{\leq 0}$ and both M and ∂M are transverse to $0 \in \mathbb{R}$.

We will first prove this lemma in the special case of the inclusion $M_0 = L \times [-1, 0] \to L \times [-1, 1] = M$, where L is any smooth manifold, possibly with boundary, and $N = \mathbb{R}$. We will use Emile Borel's Lemma:

Lemma 2.4. *The map $\mathrm{Map}^{\mathrm{sm}}(L \times \mathbb{R}, \mathbb{R}) \to \prod_{i \geq 0} \mathrm{Map}^{\mathrm{sm}}(L, \mathbb{R})$, which assigns to F the collection of partial derivatives $\left\{ \dfrac{\partial^i F}{\partial t^i}|_{t=0} \right\}$, has a continuous inverse.*

Sketch proof: Choose a bump function $\mu(t)$. To a collection $\{f_i\}$, define the function $\sum_{i \geq 0} \frac{t^i}{i!} \mu_i(t) f_i(x)$, where fidget with the bump function $\mu_i(t)$ so as to ensure convergence of all derivatives. □

Now we turn to the special case above, that $\mathrm{Map}^{\mathrm{sm}}(L \times [-1, 1], \mathbb{R}) \to \mathrm{Map}^{\mathrm{sm}}(L \times [-1, 0], \mathbb{R})$ is a Serre fibration.

Proof of special case. We are given a lifting problem:

$$\begin{tikzcd} K \arrow[r, "\tilde{f}_0"] \arrow[d] & \mathrm{Map}^{\mathrm{sm}}(L \times [-1,1], \mathbb{R}) \arrow[d] \\ K \times [0,1] \arrow[r, "f"'] \arrow[ur, dashed] & \mathrm{Map}^{\mathrm{sm}}(L \times [-1,0], \mathbb{R}) \end{tikzcd}$$

where we try to construct a map $K \times [0,1] \to \mathrm{Map}^{\mathrm{sm}}(L \times [-1, 1], \mathbb{R})$ making the above diagram commute. First, using Borel's Lemma, let us construct a section of the right vertical map, $s : \mathrm{Map}^{\mathrm{sm}}(L \times [-1, 0], \mathbb{R}) \to \mathrm{Map}^{\mathrm{sm}}(L \times [-1, 1], \mathbb{R})$. By differentiating a smooth $F : L \times [-1, 0] \to \mathbb{R}$ at $t = 0$, we obtain a collection of functions $\{\dfrac{\partial^i F}{\partial t^i}|_{t=0}\}$, which using Borel's Lemma we can use to define a new function $\tilde{F} : L \times \mathbb{R} \to \mathbb{R}$, all of whose derivatives agree with F at $t = 0$. Thus, we can define the value of the section $s(F) : L \times [-1, 1] \to \mathbb{R}$ to be F along $L \times [-1, 0]$ and $\tilde{F}$ along $L \times [0, 1]$.

Choosing a lift $s \circ f$, we obtain that the bottom part of above the diagram commutes:

$$\begin{tikzcd} & \mathrm{Map}^{\mathrm{sm}}(L \times [-1,1], \mathbb{R}) \arrow[d] \\ K \times [0,1] \arrow[ur, "s \circ f"] \arrow[r, "f"'] & \mathrm{Map}^{\mathrm{sm}}(L \times [-1,0], \mathbb{R}) \end{tikzcd}$$

However, $s \circ f$ probably doesn't agree with $\tilde{f}_0$ over $K \times \{0\}$. Using that $\mathrm{Map}^{\mathrm{sm}}(L \times [-1,1], \mathbb{R})$ is a topological vector space, we can fix the section by adding the difference between these two maps. That is, define the lift as

$$\tilde{f} = s \circ f + [(\tilde{f}_0 - s \circ f|_{t=0}) \circ \mathrm{proj}],$$

proj is the projection map $K \times [0,1] \to K$. It's easy to see this is still a lift of f and agrees with $\tilde{f}_0$ on $K \times \{0\}$, which completes the proof of the lemma in this special case.

□

Now we deduce the general case, of restricting smooth maps into N along $M_0 \subset M$, from preceding special case.

Reduction of general case to the special case. Firstly, we can immediately see that the case of $N = \mathbb{R}$ implies the case of general Euclidean spaces, $N = \mathbb{R}^n$, since we have

$$\mathrm{Map}^{\mathrm{sm}}(M, \mathbb{R}^n) \cong \prod_n \mathrm{Map}^{\mathrm{sm}}(M, \mathbb{R})$$

and a product of Serre fibrations remains a Serre fibration (since a product of lifts remains a lift).

Secondly, we reduce the case of general N to that of Euclidean space. Any manifold N admits an embedding $\eta : N \hookrightarrow \mathbb{R}^k$. Let E_η denote the normal bundle of the inclusion, the cokernel of the map $d\eta : T_N \hookrightarrow \eta^* T_{\mathbb{R}^k}$. By the tubular neighborhood theorem, the inclusion η factors as the inclusion of the zero section $z : N \to E_\eta$ followed by an open embedding $\tilde{\eta} : E_\eta \to \mathbb{R}^k$, $\eta = \tilde{\eta} \circ z$. Now, by assumption we can solve the problem of lifting the map $f : K \times [0,1] \to \mathrm{Map}^{\mathrm{sm}}(M_0, N)$ below after we have embedded N in $\mathbb{R}^k$:

$$\begin{tikzcd} K \arrow[r, "\tilde{f}_0"] \arrow[d] & \mathrm{Map}^{\mathrm{sm}}(M, N) \arrow[d] & \mathrm{Map}^{\mathrm{sm}}(M, E_\eta) \arrow[l, "\pi"'] \arrow[r, "\tilde{\eta}"] & \mathrm{Map}^{\mathrm{sm}}(M, \mathbb{R}^k) \arrow[d] \\ K \times [0,1] \arrow[r, "f"'] \arrow[urrr, dashed, "\widetilde{\eta f}"] & \mathrm{Map}^{\mathrm{sm}}(M_0, N) \arrow[rr, "\eta"'] & & \mathrm{Map}^{\mathrm{sm}}(M_0, \mathbb{R}^k) \end{tikzcd}$$

That is, a lift $\widetilde{\eta f}$ exists, adjoint to a map $M \times K \times [0,1] \to \mathbb{R}^k$ that is smooth for each choice of $(k,t) \in K \times [0,1]$. However, this map may not factor through $E_\eta \subset \mathbb{R}^k$. We fix this as follows, by scaling the lift along so as to bring it within E_η:

Choose a smooth function $\epsilon : M \to (0,1]$, where $\epsilon(x) = 1$ for all $x \in M_0$, and such that for each $x \in M$, the image of the map $\widetilde{\eta f} : \{x\} \times K \times [0, \epsilon(x)]$ lies in E_η. Finally, for each $x \in M$ choose a diffeomorphism $\phi_x : [0,1] \to [0, \epsilon(x)]$ varying continuously in M, and such that $\phi_x : [0,1] \to [0, \epsilon(x) = 1]$ is the identity map for $x \in M_0$. Now the composite of restriction with scaling defines, for each $x \in M$, a map $\widetilde{\eta f} \circ \phi_x : \{x\} \times K \times [0,1] \to E_\eta \subset \mathbb{R}^k$, whose image lies in E_η. These assemble to define a map $\widetilde{\eta f} \circ \phi : M \times K \times [0,1] \to E_\eta$, and, because all choices were made to vary smoothly, this is adjoint to a map $\widetilde{\eta f} \circ \phi : K \times [0,1] \to \mathrm{Map}^{\mathrm{sm}}(M, E_\eta)$. Now, define the lift $\tilde{f}$ to be composite of the modified lift $\widetilde{\eta f} \circ \phi$ with the projection map $\pi : E_\eta \to N$, which clearly defines a lift of f.

Lastly, we reduce the general case of a codimension zero submanifold $M_0 \subset M$ to the case of the inclusion $L \times [-1,0] \subset L \times [-1,1]$. This has two steps. First, we can delete any subspace of the interior of M_0 without having an effect on the lifting problem. Thus, after deleting open complement of some ϵ neighborhood U of the boundary of M_0, we have the situation $M_0 - U \cong L \times [-1,0]$. Second, we now assume the special case where the manifold $M_1 \cong L \times [-1,1]$ is a closed neighborhood of $L \times [-1,0] \cong M_0$ in M, $M_0 \subset M_1 \subset M$. Now, given a lifting problem as above, we have the following intermediate lift, as the result in the special case:

$$\begin{tikzcd}
K \arrow[r, "\tilde{f}_0"] \arrow[dd] & \mathrm{Map}^{\mathrm{sm}}(M,N) \arrow[d] \\
& \mathrm{Map}^{\mathrm{sm}}(M_1,N) \arrow[d] \\
K \times [0,1] \arrow[r, "f"'] \arrow[ur, dashed, "g"] & \mathrm{Map}^{\mathrm{sm}}(M_0,N)
\end{tikzcd}$$

Now, we construct the desired lift $\tilde{f}$. The idea is that outside of M_1, $\tilde{f}$ will be constant in the t direction, identically equal to $\tilde{f}_0$. We will construct this by scaling g by choosing a smooth function $\psi : M \to [0,1]$ which is identically 1 on M_0 and identically 0 on $M - M_1$. Now define the smooth map $\tilde{f}(k,t) : M \to N$, for $k \in K$ and $t \in [0,1]$, as

$$\tilde{f}(k,t)(x) := \begin{cases} \tilde{f}_0(k)(x) & x \in M - M_1 \\ g(k, \psi(x) \cdot t)(x) & x \in M_1 \end{cases}$$

which is clearly a lift of f and agrees with $\tilde{f}_0$. $\square$

References

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