THE H-PRINCIPLE, LECTURES 5 & 6: THE HIRSCH-SMALE THEOREM

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1. The Hirsch-Smale Theorem

We have finished proving:

Lemma 1.1. *Let $M_0 \subseteq M$ be a codimension zero submanifold, where both M and M_0 are compact, and let N be a smooth manifold without boundary. Then the natural map*

$$\mathrm{Map}^{\mathrm{sm}}(M, N) \to \mathrm{Map}^{\mathrm{sm}}(M_0, N)$$

is a Serre fibration.

Corollary 1.2. $\mathrm{Imm}^{\mathrm{f}}(D^k \times D^{n-k}, N) \to \mathrm{Imm}^{\mathrm{f}}(S^{k-1} \times [0,1] \times D^{n-k}, N)$ *is a Serre fibration.*

Proof. This follows directly from the lemma as soon as we notice that

$$\mathrm{Imm}^{\mathrm{f}}(D^k \times D^{n-k}, N) \cong \mathrm{Map}^{\mathrm{sm}}(D^k \times D^{n-k}, V_n(T_N)$$

and

$$\mathrm{Imm}^{\mathrm{f}}(S^{k-1} \times [0,1] \times D^{n-k}, N) \cong \mathrm{Map}^{\mathrm{sm}}(S^{k-1} \times [0,1] \times D^{n-k}, V_n(T_N)).$$

□

The following lemma is the technical heart of the Hirsch-Smale theorem.

Lemma 1.3 (Hirsch-Smale Fibration Lemma)**.** *Restricting along a collar of the boundary of an n-disk, $S^{k-1} \times [0,1] \hookrightarrow D^k$, induces map*

$$\mathrm{Imm}(D^k \times D^{n-k}, N^n) \to \mathrm{Imm}(S^{k-1} \times [0,1] \times D^{n-k}, N^n)$$

which is a Serre fibration provided $n > k$.

Note 1.4. Why is the $n > k$ condition necessary? Suppose $n = k$. Then we're considering the map

$$\mathrm{Imm}(D^n, N) \to \mathrm{Imm}(S^{n-1} \times [0.1], N)$$

where $\dim(N) = n$. Let's look at this situation where $n = 1$. So we're trying to find lifts of the form

$$\begin{array}{ccc} \mathrm{pt} & \longrightarrow & \mathrm{Imm}(D^1, \mathbb{R}) \\ \downarrow & \overset{\widetilde{f}}{\nearrow} & \downarrow \\ \mathrm{pt} \times [0,1] & \xrightarrow{f} & \mathrm{Imm}(D^1_0 \sqcup D^1_1, \mathbb{R}) \end{array} .$$

Choose the natural immersion ι of $D^1_0 \sqcup D^1_1$ into $\mathbb{R}$ that sends D^1_0 to $[0, 1/3]$ and D^1_1 to $[2/3, 1]$ say, and let f be the homotopy that swaps the two discs over. Then try to lift f to a homotopy $\widetilde{f}$ from the natural embedding $[0,1] \to \mathbb{R}$ with itself, extending f. Then there will have to be some value t where $\widetilde{f}(t)$ is *not* an immersion. Note that as soon as we are immersing into a higher-dimensional Euclidean space, such as $\mathbb{R}^2$, then this problem goes away and we can find such a $\widetilde{f}$.

In this lecture, we will prove the Hirsch-Smale theorem assuming the lemma above. Next time, we will

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Theorem 1.5 (Hirsch-Smale, first form)**.** *If M and N are n-manifolds, where M is open and compact and N is without boundary, then*

$$\mathrm{Imm}(M,N) \xrightarrow{d} \mathrm{Imm}^{\mathrm{f}}(M,N)$$

is a weak homotopy equivalence.

Proof. Our argument exactly follows the proof from Lecture 4 that local equivalences of flexible sheaves imply global equivalences, applied to the case where the map $\mathcal{F} \to \mathcal{F}'$ is $\mathrm{Imm}(-,N) \to \mathrm{Imm}^{\mathrm{f}}(-,N)$. The idea here is to build M as a handlebody, then prove the result inductively on filtration defined by adding handles. Namely, we have a filtration of M

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{n-1} = M$$

where M_{q+1} is built from M_q by attaching $(q+1)$-handles, i.e., by a pushout

$$\begin{array}{ccccc} \coprod_\alpha S^q_\alpha \times D^{n-q-1}_\alpha & \hookrightarrow & \partial M_q & \hookrightarrow & M_q \\ \downarrow & & & & \downarrow \\ \coprod_\alpha D^{q+1}_\alpha \times D^{n-q-1}_\alpha & & \longrightarrow & & M_{q+1} \end{array} \quad .$$

Note that we stop at $(n-1)$-handles. This is necessary because of the condition in our lemma that required $n > k$. So we prove the result inductively on this filtration. We have a map of pullback squares:

$$\begin{array}{ccccccc} \mathrm{Imm}(M_{j+1},N) & \longrightarrow & \mathrm{Imm}(D^{q+1}\times D^{n-q-1},N) & & \mathrm{Imm}^{\mathrm{f}}(M_{j+1},N) & \longrightarrow & \mathrm{Imm}^{\mathrm{f}}(D^{q+1}\times D^{n-q-1},N) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Imm}(M_j,N) & \longrightarrow & \mathrm{Imm}(S^q\times[0,1]\times D^{n-q-1},N) & & \mathrm{Imm}^{\mathrm{f}}(M_j,N) & \longrightarrow & \mathrm{Imm}^{\mathrm{f}}(S^q\times[0,1]\times D^{n-q-1},N) \end{array}$$

Our lemma tells us that the left-hand square is actually a homotopy pullback square, as its right-hand vertial map is a Serre fibration for $q+1 < n$. Now we apply an induction argument to deduce that the map on the bottom right corners of square is always a homotopy equivalence. Then the induction step on j follows immediately. ☐

This proves that the map $\mathrm{Imm}(M,N) \to \mathrm{Imm}^{\mathrm{f}}(M,N)$ is a weak homotopy equivalence if M has a handle decomposition with no handles of index n. What does this actually mean concretely?

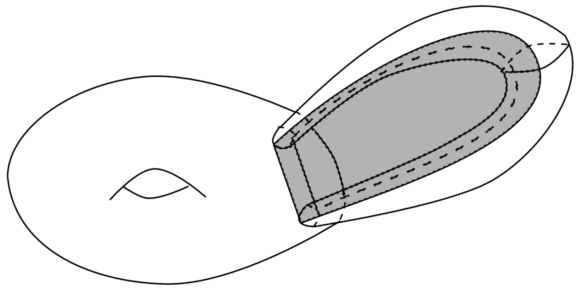
Lemma 1.6. *A manifold M of dimension n has a handle decomposition without n-handles if and only if M is open.*

Proof. First note M is open if and only if $H_n(M) = 0$.

$\Longrightarrow$:
Recall, given a handlebody decomposition one has an associated CW complex by collapsing all the thickenings of k-handles. This CW complex has an associated cellular chain complex. The fact that there are n-handles implies our chain complex has no generators for H_n, so $H_n(M) = 0$ and M is open.

$\Longleftarrow$:
Proceed by cancellation of handles. We won't go into details here because some technical machinery is required – see, for instance, the notes on handle cancellation from last year's surgery class. Suppose $H_n(M) = 0$. Choose any handle decomposition of M. We'll show that we can get rid of all the n-handles. We know $C^{\mathrm{cell}}_n(M) \xrightarrow{d} C^{\mathrm{cell}}_{n-1}(M)$ is injective. Choose $[D^n_\alpha] \in C^{\mathrm{cell}}_n$. It pairs nontrivially

with an element $[D_\beta^{n-1}]$, $\beta \in \Upsilon$, where $d[D_\alpha^n] = \sum_\Upsilon [D_\upsilon^{n-1}]$, and one can cancel them, i.e., omit them both from the handle presentation of M without changing the diffeomorphism type of M. The following is a picture of the case of canceling a 2-handle a 1-handle:



$\square$

Thus we've proved the Hirsch-Smale theorem in the case $\dim M = \dim N$, with M open and compact. What about if $\dim M < \dim N$?

Theorem 1.7 (Hirsch-Smale, final form)**.** *If M and N are smooth manifolds with M compact and N without boundary, and either 1) M open, or 2) $\dim M < \dim N$, then the map*

$$\mathrm{Imm}(M,N) \to \mathrm{Imm}^{\mathrm{f}}(M,N)$$

is a weak homotopy equivalence.

Proof. Suppose $m = \dim M < \dim N = n$. Any immersion $M \to N$ factors through the disk bundle of some vector bundle of dimension $n - m$

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & \nearrow & \\ D(V) & & \end{array}$$

where $V = \operatorname{coker}\left(T_M \overset{df}{\to} f^* T_N\right)$, the normal bundle. We know the result for such thickenings of M, so we'll work backwards to the result for M.

Definition 1.8. For $V \to M$ a vector bundle of dimension $n - m$, define

$$\mathrm{Imm}_V(M,N) = \{f \in \mathrm{Imm}(M,N) \text{ with a bundle isomorphism with the normal bundle } V \cong \operatorname{coker}(df)\}.$$

Observe that we have a natural maps

$$\mathrm{Imm}(D(V),N) \overset{d}{\to} \mathrm{Imm}_V(M,N)$$

a special case of which is familiar to us. Indeed, if M is a point then this becomes

$$\mathrm{Imm}(\mathrm{pt},N) \to \mathrm{Imm}_{\underline{\mathbb{R}^n}}(\mathrm{pt},N) = V_n(T_M).$$

Lemma 1.9. *This map $\mathrm{Imm}(D(V),N) \overset{d}{\to} \mathrm{Imm}_V(M,N)$ is a weak homotopy equivalence.*

Sketch. We will generalize our proof that

$$\mathrm{Imm}(D^{n-m},N) \cong V_{n-m}(N)$$

i.e., we construct a section going back. Choose a Riemannian metric on N and construct a section using the exponential map. Our data gives a preferred includion $V \hookrightarrow T_N$, which we can compose with $\exp\colon T_N \to N$. $\square$

Using this lemma, we can conclude the proof. First, note that the forgetful map $\mathrm{Imm}_V(M,N) \to \mathrm{Imm}(M,N)$ is a fiber bundle, and thus a Serre fibration. Now, observe that the following is a pullback square

$$\begin{array}{ccc} \mathrm{Imm}_V(M,N) & \xrightarrow{d} & \mathrm{Imm^f}_V(M,N) \\ \downarrow & & \downarrow \scriptstyle{\text{Serre fibration}} \\ \mathrm{Imm}(M,N) & \xrightarrow{d} & \mathrm{Imm^f}(M,N) \end{array}$$

where the top right object is defined in the obvious way:

$$\mathrm{Imm^f}_V(M,N) \cong \left\{ T_M \overset{F}{\to} T_N \in \mathrm{Imm^f}(M,N) \text{ with an isomorphism } \operatorname{coker}(F) \cong (V) \right\}.$$

This pullback square maps to the square

$$\begin{array}{ccc} \mathrm{Imm}(D(V),N) & \xrightarrow{d} & \mathrm{Imm^f}(D(V),N) \\ \downarrow & & \downarrow \\ \mathrm{Imm}(M,N) & \xrightarrow{d} & \mathrm{Imm^f}(M,N) \end{array}$$

where the maps on the top row are weak homotopy equivalences by the previous lemma. Thus the square is a homotopy pullback square, and

$$\mathrm{Imm}(D(V),N) \to \mathrm{Imm^f}(D(V),N)$$

is a weak homotopy equivalence. Using the long exact sequence on homotopy groups associated to the two vertical fibrations, we find that the map

$$\mathrm{Imm}(M,N) \to \mathrm{Imm^f}(M,N)$$

induces an isomorphism on homotopy groups π_* so long as the basepoint chosen lies in component which is in the image of $\pi_0 \mathrm{Imm}_V(M,N)$. Finally, by using all the possible $(n-m)$-dimensional vector bundles V, we obtain the isomorphism $\pi_* \mathrm{Imm}(M,N) \to \pi_* \mathrm{Imm^f}(M,N)$ for all choices of basepoints, and so we conclude that

$$\mathrm{Imm}(M,N) \to \mathrm{Imm^f}(M,N)$$

is a weak homotopy equivalence. □

References

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