THE H-PRINCIPLE, LECTURE 7: FIBRATIONS IN IMMERSION THEORY, PART 1

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We now address the final part of the proof of the Hirsch-Smale theorem, of the weak homotopy equivalence between immersions and formal immersions. The following lemma is the crux of the result. Let N be any smooth n-dimensional manifold without boundary.

Notation 0.1. We will denote the sheaf of immersions into N by

$$\mathcal{I} := \mathrm{Imm}(-, N).$$

We will further abbreviate $V := D^k \times D^{n-k}$ and $U := S^{k-1} \times [0,1] \times D^{n-k}$.

Lemma 0.2 (Smale-Hirsch Fibration Lemma)**.** *The map on immersions into N induced by restricting along a standard embedding of a thickened annulus into a thickened disk, $U = S^{k-1} \times [0,1] \times D^{n-k} \hookrightarrow D^k \times D^{n-k} = V$,*

$$\mathcal{I}(V) \longrightarrow \mathcal{I}(U)$$

is a Serre fibration, provided that n is strictly larger than k.

In the last lecture, we proved the Hirsch-Smale theorem assuming this lemma. The rest of this lecture will be devoted to proving this lemma. The following proof is based on Thom's simplification of Smale's proof of the original lemma (which was for the case of $N = \mathbb{R}^n$). See [2], [3].

Remark 0.3. The corresponding lemma for embeddings, rather than immersions, is much easier and more general (true without the $n > k$ provision). It is an immediate consequence of the parametrized isotopy extension theorem.

Outline of the proof for immersions:
Step 1. First, given the problem of lifting a map $D^i \times [0,1] \to \mathcal{I}(U)$, we will show the existence of a partial lift, i.e., a lift of $D^i \times [0,\epsilon] \to \mathcal{I}(V)$, for some sufficiently small, but positive, number ϵ.
Step 2. Second, we will introduce the notion of "being in good position" (GP) and modify our partial lift from Step 1 so that it became GP.
For the lift the property GP will imply the existence of extension of the lift from $[0,\epsilon] \times D^i$ to the whole $[0,1] \times D^i$, thus the lemma will follow.

Lemma 0.4 ("Microfibration" property)**.** *Let $M_0 \subset M$ be a codimension zero submanifold, where M_0 and M are compact. For any lifting problem*

$$\begin{tikzcd}
D^i \times \{0\} \arrow[r] \arrow[d] & \mathcal{I}(M) \arrow[d] \\
D^i \times [0,1] \arrow[r, "f"] & \mathcal{I}(M_0)
\end{tikzcd}$$

there exists ϵ sufficiently small such that a lift exists after restriction to $D^i \times [0,\epsilon]$:

$$\begin{tikzcd}
& D^i \times \{0\} \arrow[r] \arrow[d] & \mathcal{I}(M) \arrow[d] \\
D^i \times [0,\epsilon] \arrow[r] \arrow[urr, dashed] & D^i \times [0,1] \arrow[r] & \mathcal{I}(M_0)
\end{tikzcd}$$

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Before proving this lemma, it is helpful to again give the definition of the topology on the space $\mathrm{Map}^{\mathrm{sm}}(M,N)$ and, more generally, the space of smooth sections $\Gamma^{\mathrm{sm}}(M,E)$ of a smooth fiber bundle $E \to M$. First, recall that the bundle $E^{(r)}$, of r-jets of E, is a fiber bundle over M, and whose fiber over a point $x \in M$ is an equivalence class of a section f defined in a neighborhood of x, and where f and g are equivalent if the first k derivatives of $f-g$ vanishes at the point x. $E^{(r)}$ is a smooth manifold. The r-jet prolongation map, $j^{(r)}$, sends a section of E to the associated section of $E^{(r)}$ given by imposing the preceding equivalence relation.

Definition 0.5. The compact-open C^∞ topology on $\Gamma^{\mathrm{sm}}(M,E)$ is defined by the basis given of open sets $S(K,U)$, where $K \subset M$ is compact, $U \subset E^{(r)}$ is an open subspace for some r, and

$$S(K,U) := \{f \in \Gamma^{\mathrm{sm}}(M,E) |\ j^{(r)}f(K) \subset U\}$$

is the subset of all sections f of E whose associated r-jet, $j^{(r)}f : M \to E^{(r)}$, send every element of K to an element of U. This specializes to the compact-open C^∞ topology on $\mathrm{Map}^{\mathrm{sm}}(M,N)$ by setting E to be the trivial bundle $M \times N$.

Remark 0.6. Alternatively, one can put the usual compact-open on the spaces of r-jets $\Gamma(M,E^{(r)})$ and define the compact-open C^∞ topology on $\Gamma^{\mathrm{sm}}(M,E)$ as the weakest topology (i.e., fewest open sets) such that each map $j^{(r)} : \Gamma^{\mathrm{sm}}(M,E) \to \Gamma(M,E^{(r)})$ is continuous, for every r.

From this definition, we can see that the space of immersions $\mathcal{I}(M) \in \mathrm{Map}^{\mathrm{sm}}(M,N)$ is open if M is compact, since it is of the form $S(M,U)$, where U is the open subspace of 1-jets consisting of maps that injective on the fibers. We now, finally, prove the lemma.

Proof. In Lecture 4 we proved that $\mathrm{Map}^{\mathrm{sm}}(M,N) \to \mathrm{Map}^{\mathrm{sm}}(M_0,N)$ is a Serre fibration. Thus, we have a lift g:

$$\begin{tikzcd}
D^i \times \{0\} \arrow[r] \arrow[d] & \mathcal{I}(M) \arrow[r] \arrow[d] & \mathrm{Map}^{\mathrm{sm}}(M,N) \arrow[d] \\
D^i \times [0,1] \arrow[r] \arrow[urr, dashed, "g"] & \mathcal{I}(M_0) \arrow[r] & \mathrm{Map}^{\mathrm{sm}}(M_0,N)
\end{tikzcd}$$

By commutativity of the diagram, $g|_{D^i\times\{0\}}$ lies in the image of $\mathcal{I}(M)$. Then, since $\mathcal{I}(M) \subset \mathrm{Map}^{\mathrm{sm}}(M,N)$ is open and $D^i \times [0,1]$ is compact, there exists a closed neighborhood $D^i \times [0,\epsilon]$ of $D^i \times \{0\}$ such that $g|_{D^i\times[0,\epsilon]}$ is contained in $\mathcal{I}(M)$.

$$\begin{tikzcd}
 & D^i \times \{0\} \arrow[r] \arrow[d] \arrow[dl] & \mathcal{I}(M) \arrow[r] \arrow[d] & \mathrm{Map}^{\mathrm{sm}}(M,N) \arrow[d] \\
D^i \times [0,\epsilon] \arrow[r] \arrow[urr, dashed, "g|_{D^i\times[0,\epsilon]}"] & D^i \times [0,1] \arrow[r] \arrow[urr, dashed, "g"] & \mathcal{I}(M_0) \arrow[r] & \mathrm{Map}^{\mathrm{sm}}(M_0,N)
\end{tikzcd}$$

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Step 2. In the next lecture we will introduce the notion of "being in good position." Let us show how it helps to prove the Smale-Hirsch Fibration Lemma.

We will use the same idea, that was used in proving the fact that $\mathrm{Map}^{\mathrm{sm}}(V,N) \to \mathrm{Map}^{\mathrm{sm}}(U,N)$ is a Serre fibration. We had a sequence of submanifolds $M_0 \subset M_1 \subset M$, where $M_1 = M_0 \cup \partial_M M_0 \times [0,1]$ is a collar neighbourhood of M_0 in M. Then we used the fact that this situation is simple to find lift from M_1 and, using the extra room, we extended the lift so as to be constant in the t-direction ($t \in [0,1]$ parameterizes the interval) outside of M_1.

Our goal is to implement the same tactic here, i.e., we would like to modify the lift $g|_{[0,\epsilon]\times D}$ to be constant in the t-direction, outside of a collar neighborhood of U. The only thing that remains is to find a condition on the lift that will mean that the lift can, indeed, be modified in such manner. This condition will be called "being in good position" and will be discussed in the next lecture.

References

[1] Hirsch, Morris. Immersions of manifolds. Transactions A.M.S. 93 (1959), 242-276.

[2] Phillips, Anthony. Submersions of open manifolds. Topology 6 1967 171206.

[3] Poénaru, Valentin. Regular homotopy and isotopy. Mimeographed notes. Harvard University, Cambridge, Mass., 1964.

[4] Smale, Stephen. The classification of immersions of spheres in Euclidean spaces. Ann. Math. 69 (1959), 327-344.