MATH 516: THE H-PRINCIPLE IN TOPOLOGY

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Gromov formulated the homotopy principle, or h-principle, as a condition for when topology controls the analysis of a differential equation. It provides a criterion for when the space of solutions is equivalent to a space of *formal* solutions, which can be analyzed by homotopy theory without the usual hard labor of analysis. The h-principle is a core local-to-global property in studying manifolds.

The h-principle arose from a question in immersion theory: When can a map $g : M \to N$ be deformed to be an immersion? The Smale-Hirsch theorem gave a surprisingly *soft* condition, under certain restrictions: If there's a fiberwise injection of bundles $\widetilde{g} : T_M \to T_N$ covering g, then g can be deformed to an immersion. Smale thereby classified immersions of spheres into Euclidean spaces; in particular, he proved the then implausible result that S^2 can be turned inside-out in $\mathbb{R}^3$.

This core example of immersion theory has a vast generalization, beyond the world of differential equations. A topological version of the h-principle can be formulated and applied to produce homotopy theoretic models of objects from geometric topology, such as configurations spaces, spaces of foliations, cobordism categories, and spaces of manifold structures. The h-principle is also a powerful tool in symplectic and Riemannian geometry, and can be used to prove such varied results as Nash's isometric embedding theorem (that every Riemannian manifold embeds isometrically in Euclidean space) and Gromov's theorem on the existence of symplectic forms on open manifolds.

This class will first develop the general theory of Gromov's h-principle alongside the example of immersion theory. We will then establish instances of the h-principle in other subjects, each of which will have profound implications. Each result will then serve as a jumping off point for a more extended study of that subject. These topics will be drawn from the provisional list below, based on participants' interests:

- Differential topology. Smale-Hirsch immersion theory. Phillips-Gromov submersion theory. Applications: eversion of the 2-sphere; $O(n) \xrightarrow{\simeq} \mathrm{Immers}(\mathbb{R}^n, \mathbb{R}^n)$. The h-principle for microflexible sheaves, generalizing the main theorems of immersion-submersion theory.

- Foliations. Haefliger's theorem on existence and classifying spaces for foliations of open manifolds. Thurston's generalization to closed manifolds. Characteristic classes of foliations. Foliations of 3-manifolds. Segal, Mather, and Thurston's theorems on classifying spaces of foliations and diffeomorphism groups.

- Configuration spaces. Configuration space models of mapping spaces à la McDuff, Segal. The James construction: $J(X) \simeq \Omega\Sigma X$. The Barratt-Priddy-Quillen theorem.

- Cobordism categories and spaces of manifolds. The h-principle proof of the Galatius-Madsen-Tillmann-Weiss theorem, that $|\mathrm{Cob}_n| \simeq \Omega^{\infty-1} MTO(n)$, and generalizations.

- Symplectic and Riemannian geometry. Nash-Kuiper isometric embedding theorem. Existence of metrics of positive and negative sectional curvature on open manifolds. Existence of symplectic forms on open almost complex manifolds. Symplectic and Lagrangian foliations.

- Beyond the h-principle: Goodwillie-Weiss's manifold calculus of functors. The space of embeddings $\mathrm{Emb}(M, N)$. Applications to knot theory and finite-type invariants.

Prerequisites: Basic knowledge of algebraic topology, manifolds, vector and fiber bundles, cohomology, and characteristic classes.

References

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