MATH 465, LECTURE 11: THE WHITNEY TRICK, FIRST PART

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In the last lecture, we established one of the key technical lemmas in the proof the the s-cobordism theorem, a homological condition on a smooth embedding of a q-sphere that determines whether it it can isotoped so as to cancel an individual q-handle, leaving the other handles unchanged. Our proof relied on the assertion of a particular geometric maneuver, the Whitney trick, which desribed when a pair of intersection points between the q-sphere and the transverse sphere of opposite intersection sign could be simultaneously removed by isotopy of the map.

This is a technical result, but one which very much seems to underly the behavior of higher-dimensional topology: The failure of the trick in dimension 4 seems to likewise underlie the special behavior of low-dimensional topology. So I'm happy this class gives me the opportunity to really understand the proof, our treatment of which will follow that of Milnor in [1]. This treatment is a little laborious compared with Milnor's usual clarity and fluency, but that's probably necessary. For ease of cross-reference, we will use the same notation as [1] as much as possible.

Theorem 0.1 (The Whitney Trick)**.** *Let V^{r+s} be a smooth manifold of dimension $r+s$ with transversally intersecting closed submanifolds M and M' of complementary dimensions r and s, and in which both M and the normal bundle of M' are oriented, so that the intersection sign $\epsilon_x = \pm 1$ of a point $x \in M \cap M'$ can be defined. Assume:*

- *The dimension $r+s$ is at least 5;*
- *If $r = 1$ or 2, then assume further that the map $\pi_1(V - M') \to \pi_1(V)$ is injective.*

Let $x, y \in M \cap M'$ be two points with opposite intersection numbers, $\epsilon_x = -\epsilon_y$. Finally, assume there exits paths $C \subset M$, $C \subset M'$, both connecting x and y, and such that

- *both C and C' miss the other points in the intersection (i.e., C and C' miss $M \cap M' - \{x, y\}$);*
- *The loop $C \cup C'$ is contractible in V.*

Under these conditions, then there exists an isotopy $f_t : [0,1] \times V \to V$, where $f_0 = \mathrm{id}_V$, and such that:

- *$f_1(M) \cap M' = M \cap M' - \{x, y\}$;*
- *The isotopy is constant in a neighborhood of $M \cap M' - \{x, y\}$ and outside an open ball containing $C \cup C'$.*

In other words, one can select an isotopy that moves M so as to simultaneously eliminate the two points of intersection with M' at x and y, but where everything else about the intersection $M \cap M'$ does not change at all.

For example, in the case that M and M' have zero algebraic intersection number, with finite intersection, this just says that we can move M off of M'.

See the figure below, of a sinusoidal surface intersecting a line in $V = \mathbb{R}^3$.)

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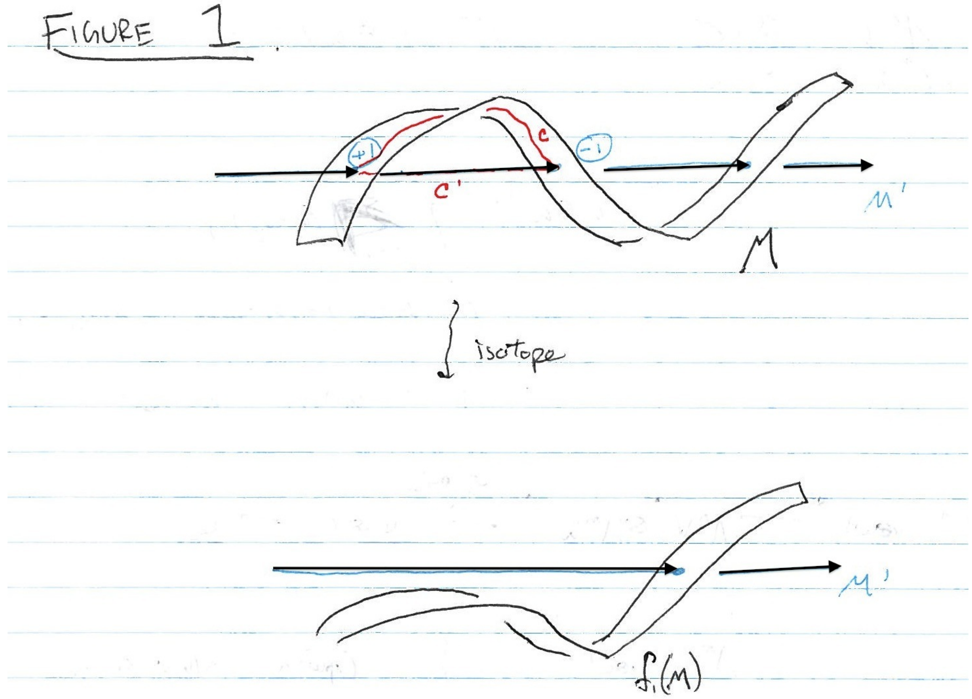


Corollary 0.2. *Let V, M, M' be as in the theorem, but assume additionally that all are oriented compact manifolds, V is simply-connected, and that the inclusions of M and M' are orientation preserving maps. Let $[M]$ and $[M']$ represent the classes in $H_*(V, \mathbb{Z})$ given by the pushforward of the fundamental classes. Define the* algebraic intersection number *of M and M' by*

$$\#(M, M') := \langle \mathrm{Pd}[M] \smile \mathrm{Pd}[M'], [V] \rangle.$$

Then there exists an isotopy f_t of the embedding of M such that the actual *number of intersection points in $f_1(M) \cap M'$ is equal to the algebraic intersection number. i.e., we have an equality*

$$\#(f_1(M) \cap M') = \#(M, M').$$

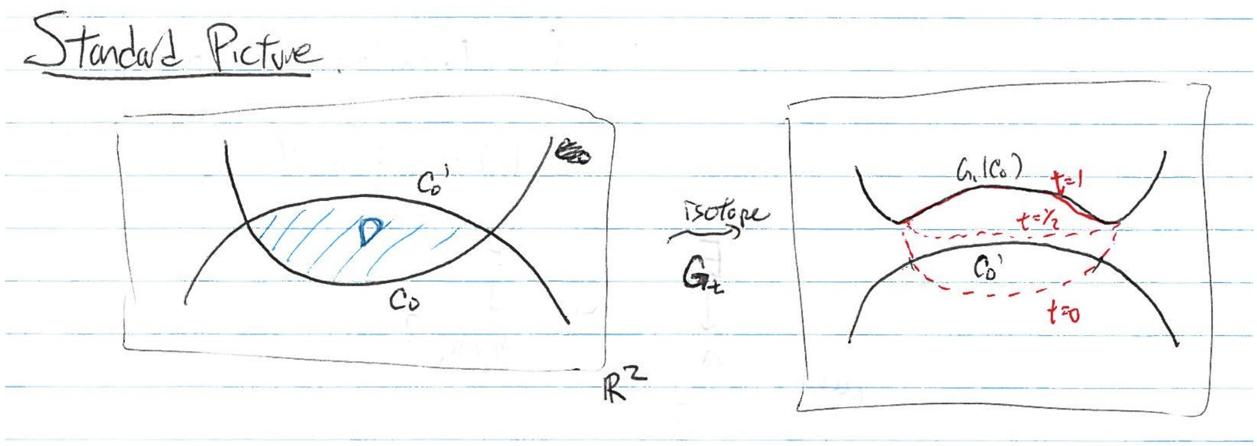
This is, of course, also relies on the following essential result of intersection theory, that the cup product counts the sum of the signed intersection points:

Lemma 0.3. *For V, M and M' as above, then following numbers are equal*

$$\sum_{x \in M \cap M'} \epsilon_x = \#(M, M').$$

We now turn to the proof the Whitney trick, which will occupy the rest of this and next lectures

Proof. We will prove the theorem by first formulating a "standard picture," prove the theorem in this particular case, and then finally prove a Lemma 6.7 which shows that the standard picture is in fact universal, and can always be embedded. Consider the standard picture of the diagram below. Here we have two smooth curves (one upward, one downward) intersecting in $\mathbb{R}^2$. D is a disk, and C_0 and C_0' are the arcs of the curves bounding the disc. Let U be an open neighborhood of D.



Now assume we have an embedding of this picture $\phi_1 : U \hookrightarrow V$.

$$\begin{array}{ccc} C_0 & \xrightarrow{\phi_1} & C \\ \downarrow & & \downarrow \\ U & \xrightarrow{\phi_1} & V \\ \uparrow & & \uparrow \\ C_0' & \xrightarrow{\phi_1} & C' \end{array}$$

and such that the interior D° maps to $V - (M \cup M')$.

Assume that the embedding ϕ_1 extends to an open embedding

$$\phi_2 : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \hookrightarrow V$$

so that we can write

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & V \\ & \searrow \quad \nearrow_{\phi_2 \text{ open}} & \\ & U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} & \end{array}.$$

Further assume we can choose ϕ_2 such that

$$\phi_2^{-1}(M) = C_0 \times \mathbb{R}^{r-1} \times \{0\}$$

and that

$$\phi_2^{-1}(M') = C_0' \times \{0\} \times \mathbb{R}^{s-1}.$$

Making these assumptions, the rest of the proof is easy. We will construct the requisite isotopy on $U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1}$ and then import it to V.

In the standard picture, we choose an isotopy G_t of U so that

$$G_1(C_0) \cap C_0' = \phi.$$

The upward parabola, labeled C_0, is just moving upward and upward, until it goes completely above C_0', which is the downward parabola.

Now we extend G_t to $U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1}$. We normalize it so that as it gets far away in the $\mathbb{R}^{r-1}$ or $\mathbb{R}^{s-1}$ direction, it goes to the identity map. For instance, just choose a bump function $\rho : \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \to [0, 1]$ and apply to make the extension.

Then the isotopy extends to all of V via the identity.
We then have $G_1(M) \cap M' = M \cap M' - \{x, y\}$. QED. $\square$

So the proof is done up until the choice of a certain embedding. This is the 'very intuitive lemma' that we need to prove, that would make the rest of the proof easy. Unfortunately, this lemma *doesn't exist*.

Lemma 0.4 (Lemma 6.7 of [1]). *Fix V, M, and M' as before. (All assumptions of the theorem are necessary.) Then there exists an embedding*

$$\phi_2 : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \hookrightarrow V$$

extending the embedding of $\phi_1 : U \to V$ taking C_0' and C_0 to C' and C, respectively, and such that

$$\phi_2^{-1} M = C_0 \times \mathbb{R}^{r-1} \times \{0\}$$

$$\phi_2^{-1} M' = C_0' \times \{0\} \times \mathbb{R}^{s-1}.$$

The embedding of Lemma 6.7 will be constructed using Riemannian geometry. The following Lemma, 6.8, says that our problem can be placed in a nice Riemannian setting. First recall the following definition.

Definition 0.5. $M \subset V$ is totally geodesic if for any geodesic $f : [0, 1] \hookrightarrow V$ in which a single point $f(t)$ lies in M and for which the tangent vector $df(t)$ lies in the subspace $T_{f(t)}M \subset T_{f(t)}V$, then the geodesic itself lies entirely in M.

Lemma 0.6 (Lemma 6.8 of [1]). *There exists a Riemannian metric on V such that*

- *M and M' are totally geodesic submanifolds;*
- *There exist neighborhoods N_x and N_y of x and y where the metric is Euclidean, and in which the line segments $N_x \cap C_0$, $N_x \cap C_0'$, $N_y \cap C_0$, $N_y \cap C_0'$ are all straight.*

We will prove these lemmas in the next class.

References

[1] Milnor, John. Lectures on the h-cobordism theorem. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J. 1965 v+116 pp. Available from http://www.maths.ed.ac.uk/~aar/surgery/hcobord.pdf/.