MATH 465, LECTURE 11: THE WHITNEY TRICK, SECOND PART

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Recall from the preceding lecture that we gave a proof of the Whitney trick predicated on the existence of a fairly specific embedding of $U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1}$ into our ambient manifold V, Lemma 6.7 of [1]. In order to construct this embedding, we wanted to prove a lemma that allowed us to bring to bear some Riemannian geometry to the construction. This lemma was the following:

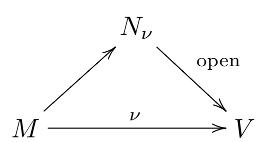
Lemma 0.1 (Lemma 6.8 of [1])**.** *There exists a Riemannian metric on V such that*

- *M and M' are totally geodesic submanifolds;*
- *There exist coordinate neighborhoods N_x and N_y of x and y where the metric is Euclidean, and in which the line segments $N_x \cap C_0$, $N_x \cap C_0'$, $N_y \cap C_0$, $N_y \cap C_0'$ are all straight.*

It appeared that the surprising part of this result was an arbitrary submanifold can be made to be totally geodesic, given an appropriate choice of metric. I'll just prove that result here, which can be fine-tuned to yield this more involved result. Again, see [1].

Lemma 0.2. *For any closed submanifold $M \subset V$, one can choose a Riemannian metric on V with respect to which M is a totally geodesic submanifold.*

Proof. First, choose any Riemannian metric $\langle -,- \rangle$ on V. Additionally, choose a tubular neighborhood N_ν of M:



We have an action of $O(1) = \mathbb{Z}/2$ on N_ν given by the antipodal map, σ. Let us define a new metric $\langle -,- \rangle_\sigma$ on the tubular neighborhood N_ν, defined by averaging

$$\langle v, w \rangle_\sigma := \frac{1}{2}\langle v, w \rangle + \frac{1}{2}\langle \sigma v, \sigma w \rangle.$$

By choosing a partition of unity subordinate to the cover of V by N_ν and $V - N_\nu$, this metric $\langle -,- \rangle_\sigma$ can be extended over all of V by scaling the contribution of the term $\langle \sigma -, \sigma - \rangle$ to zero at the periphery of N_ν, away from M. Consequently, to establish the lemma it now suffices to show that M is a totally geodesic submanifold of $(N_\nu, \langle -,- \rangle_\sigma)$.

With respect to this new metric, σ is clearly an isometry. Now let ω be a geodesic of N_ν, $\omega : [0,1] \to N_\nu$, such that there is a single $t_0 \in [0,1]$ such that both the value $\omega(t_0)$ is contained in M, and the derivative vector $d\omega(t_0)$ is contained in the subspace and $T_{\omega(t_0)}M$ of $T_{\omega(t_0)}N_\nu$. We now show that the geodesic ω must lie entirely in M, and thus that M is totally geodesic.

Since σ is an isometry, it sends geodesics to geodesics. Thus, the antipodal reflection of the geodesic ω, $\sigma(\omega)$, is also a geodesic. Since the point $\omega(t_0)$ lies in M, which is the fixed points $N_\nu^{O(1)}$ of the action of $O(1)$, thus the geodesics ω and $\sigma(\omega)$ intersect at the point $\omega(t_0) = \sigma\omega(t_0)$. Furthermore, the tangent vector at this point is also the same: $d(\sigma\omega(t_0) = d\omega(t_0)) \in TM$. If two geodesics share the same tangent vector at the same point, they must be the same geodesic, so ω and $\sigma(\omega)$ are the same. Therefore ω is fixed by the map σ, implying that ω lies entirely in M. This shows M is totally geodesic. $\square$

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Our aim in the rest of this lecture is the prove the following very plausible looking result. First, let V^{r+s}, M^r, M'^s, and $x, y \in M \cap M'$ satisfy the hypotheses stated for the Whitney trick. I.e., M and N are transversally intersecting submanifolds of V; the tangent bundle TM and normal bundle of M' are both oriented; $r + s \geq 5$ and $s \geq 3$; if $r = 1$ or $r = 2$, then $\pi_1 V - M' \to \pi_1 V$ is assumed injective; points $x, y \in M \cap M'$ have opposite intersection signs; $C \subset M$ and $C' \subset M'$ are appropriate paths connecting x and y and the loop $C \cup C'$ is contractible V. Then:

Lemma 0.3 (Lemma 6.7 of [1])**.** *Given C_0 and C_0' smooth paths bounding a disk D in the plane, and a map ϕ_1 so that*

$$\begin{array}{ccc} C_0 & \longrightarrow & C \\ \downarrow & & \downarrow \\ C_0 \cup C_0' & \xrightarrow{\phi_1} & V \\ \uparrow & & \uparrow \\ C_0' & \longrightarrow & C' \end{array}$$

then there is an extension ϕ of ϕ_1

$$\phi : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \hookrightarrow V$$

such that

$$\phi^{-1}(M) = C_0 \times \mathbb{R}^{r-1} \times \{0\}$$
$$\phi^{-1}(M') = C_0' \times \{0\} \times \mathbb{R}^{s-1}$$

where U is an open neighborhood of the disk D.

To prove this, we will make use of two lemmas:

Lemma 0.4. *Let $M_1 \hookrightarrow V_1$ be a submanifold of codimension at least three. Then the induced map $\pi_1(V_1 - M_1) \to \pi_1 V_1$ is injective.*

This will allow the embedding of a certain disk D in V. Once D has been embedded, we'll also use:

Lemma 0.5. *There exists an orthonormal basis*

$$\xi_1, \ldots, \xi_{r-1}, \eta_1, \ldots, \eta_{s-1}$$

of sections of the normal bundle N of the disk $D \subset V$, where $N \cong TV|_D - TD$ is the orthogonal complement of the tangent bundle of D in TV, and such that:

- *$\xi_1, \ldots, \xi_{r-1}$ restricted to C form a basis for $TM|_C - TC$;*
- *$\eta_1, \ldots, \eta_{s-1}$ restricted to C' form a basis for $TM'|_C - TC'$.*

Remark 0.6 (Remark on Lemma 0.5). Once we have these bases, we will construct our embedding by exponentiation along these tangent vectors.

Proof of Lemma 6.7. We first construct an embedding of the disk $D \hookrightarrow V$ so that D°, the interior of the disk, misses the image of M and M', and such that the boundary ∂D identifies with $C \cup C'$. (An embedded disk that intersects M and M' at other places will assuredly not satisfy the conditions of Lemma 6.7.) This is where a couple of the dimension requirements come in. (Look at the proof of Lemma 0.4.)

As a corollary of Lemma 0.4, we first show that the map

$$\pi_1(V - (M \cup M')) \to \pi_1(V)$$

is an injection. so that we can find an nullhomotopy D of the loop $\partial D \cong C_0 \cup C_0'$, and so that D misses M and M'.

To prove this, we will apply Lemma 0.4 twice, to each of the successive embeddings

$$V - (M \cup M') \hookrightarrow V - M' \hookrightarrow V.$$

Let us first consider the second map: If $r = \dim M$ is greater than or equal 3, then the codimension of M' is likewise greater than or equal to 3, so we can apply Lemma 0.4 for $V_1 = V$ and $M_1 = M'$ obtain that the map $\pi_1(V - M') \to \pi_1 V$ is injective. If, to the contrary, the dimension of M is equal to 1 or 2, then this map was assumed to be injective in the assumptions listed above.

Now consider the first map: Since the dimension $s = \dim M'$ was assumed to be at least 3, therefore the codimension of $M - (M \cap M')$ in $V - (M \cup M')$ is at least 3, and by applying the lemma to case of $V_1 = V - (M \cup M')$ and $M_1 = M - (M \cap M')$, we again obtain that that the induced map on π_1 is injective.

Thus, we we have that composite map

$$\pi_1(V - (M \cap M')) \hookrightarrow \pi_1 V$$

is an injection. Since the dimension of V is at least 5, we can chose the nullhomotopy $D \hookrightarrow V$ so as to be an embedding. (Select this embedding so that its tangent vectors are orthogonal at the boundary to M and M'.)

We now construct the embedding of $U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1}$ into V, which will make use of Lemma 0.5. The construction is merely by exponentiation along the vector fields ξ_i and η_j. That is, define the embedding

$$\phi : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \hookrightarrow V$$

in two steps. (Here, U is an open neighborhood of the disk D) First, to triplet of $u \in U, a = (a_1, \ldots, a_{r-1}) \in \mathbb{R}^{r-1}, b = (b_1, \ldots, b_{s-1}) \in \mathbb{R}^{s-1})$, make the assignment of

$$(u, a, b) \rightsquigarrow \exp_u\left(\sum a_i \xi_i + \sum b_j \eta_j\right).$$

This defines a diffeomorphism in a neighborhood of $0 \in \mathbb{R}^{r-1} \times \mathbb{R}^{s-1}$. Now choose a diffeomorphism

$$\alpha : N_\epsilon \cong \mathbb{R}^{r-1} \times \mathbb{R}^{s-1}$$

and then define the long desired ϕ as the composite $\exp \circ \alpha$.

To conclude the proof, the final point is to show that $\phi^{-1}M$ and $\phi^{-1}(M')$ do indeed lie in the subspaces we claimed, $C_0 \times \mathbb{R}^{r-1} \times \{0\}$ and $C_0' \times \{0\} \times \mathbb{R}^{s-1}$, respectively. It suffices to demonstrate just one of them, the case being identical for the other. Recall that $M \subset V$ is totally geodesic (an unused property, until this point). This implies that, for each $x \in M$, the following diagram commutes:

$$\begin{array}{ccc} T_x M & \xrightarrow{\exp} & M \\ \downarrow & & \downarrow \\ T_x V & \xrightarrow{\exp} & V \end{array}$$

That is, since the exponential map is defined by tracing tangent vectors along their associated geodesics, and since geodesics in M are the same as those of V in M, the two exponential maps coincide for tangent vectors in the subspace $T_x M \subset T_x V$.

By the condition that $\xi_i \in T_x M$, we see that the image of ϕ

$$\phi(C \times \mathbb{R}^{r-1} \times \{0\}) \subset M.$$

This, finally, proves the lemma. $\square$

We now turn to the subsidiary results, Lemmas 0.4 and 0.5, used in the proof above.

Proof of Lemma 0.4. Let the composite $\ell : S^1 \hookrightarrow V_1 - M_1 \hookrightarrow V_1$ be nullhomotopic in V_1. We show that l is nullhomotopic in $V_1 - M_1$. Choose a nullhomotopy in V_1 given by a map of pairs

$$(D^2, S^1) \to (V_1, V_1 - M_1).$$

By transversality, we can find an arbitrarily close map of D^2 into V_1 such that D^2 intersects M transversally. By the assumption that the codimension of M_1 was at least 3, a transverse intersection of D with M_1 must be empty, the sum of dimensions of M_1 and D^2 being less than that of V_1. $\square$

Remark 0.7. The same proof implies that if M_1 has codimension at least k, then $\pi_i(V_1 - M_1) \to \pi_i V_i$ is an injection for $i \leq k-2$.

And now, lastly:

Proof of Lemma 0.5. Choose the sections $\xi_1|_C, \ldots, \xi_{r-1}|_C$. (For instance, one can first select a basis of tangent vectors at $T_x M - T_x C$, and then parallel transport these vectors along the curve.) For each point $c \in C$, this defines an element of the Stiefel manifold of $(r-1)$-framed subspaces in $\mathbb{R}^{r+s-2}$. This gives a map

$$C \to V_{r-1}(\mathbb{R}^{r+2-1}) \cong O(r+s-2)/O(s-1)$$

The only conditions on the restriction of $\xi_1 \ldots, \xi_{r-1}$ to C' was that they lie in the complement of $TM'|_{C'}$. So we may arbitrarily choose a basis for this complement, and assign the restrictions of $\xi_1, \ldots, \xi_{r-1}$ to be this basis. Thus, we obtain a map

$$S^1 \cong C \cup C' \to V_{r-1}(\mathbb{R}^{r+s-2}).$$

To define the ξ_i over the entire disk D, we need to extend this map to the disk. This map will always extend if the fundamental group of $V_{r-1}(\mathbb{R}^{r+s-2})$ is zero. For this to be the case, in the fiber sequence

$$O(s-1) \to O(r+s-2) \to V_{r-1}(\mathbb{R}^{r+s-2})$$

it suffices for the first map on π_1 to be surjective (by looking at the long exact sequence on homotopy groups). Since it was assumed that $s \geq 3$, and $\pi_1 O(2) \to \pi_1 O(k)$ is surjective for $k \geq 2$, it follows that the group $\pi_1 V_{r-1}(\mathbb{R}^{r+s-2})$ is zero. Thus, we can extend $\xi_1|_{C \cup C'}, \ldots \xi_{r-1}|_{C \cup C'}$ over all of D.

Finally, we can just choose $\eta_1, \ldots, \eta_{s-1}$ as a basis for the orthogonal complement to the subspace spanned by the ξ_i, and this basis will necessarily satisfy the stipulated conditions. □

This completes our treatment of the Whitney trick.

Remark 0.8. Note that for $r+s=4$ and $s=2$, the argument at end of our proof of Lemma 0.5 breaks down, and $V_1(\mathbb{R}^2)$ has a nontrivial fundamental group. This technical failure underlies the radically different behavior of smooth 4-manifold topology versus higher dimensions.

References

[1] Milnor, John. Lectures on the h-cobordism theorem. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J. 1965 v+116 pp. Available from `http://www.maths.ed.ac.uk/~aar/surgery/hcobord.pdf/`.