MATH 465, LECTURE 14: THE H-COBORDISM THEOREM AND THE GENERALIZED POINCARÉ CONJECTURE

J. FRANCIS, NOTES BY I. BOBKOVA

1. The h-cobordism theorem

Theorem 1.1 (The h-cobordism theorem)**.** *Let W be an h-cobordism of dimension at least 6. If W is simply connected, then W is diffeomorphic relative $\partial_0 W$ to a product $\partial_0 W \times [0,1]$.*

We will require the following lemma.

Lemma 1.2. *Any smooth cobordism has a handlebody structure.*

Proof. Deferred until our treatment of Morse theory next week. □

We now prove the h-cobordism theorem:

Proof. Applying the normal form lemma, we can alter the handle presentation of W so that all of the handles have index either q or $q+1$ for a choice of a single q, $2 \leq q \leq n-3$. Let us now consider the cellular differential d_{q+1}

$$H_{q+1}(W_{q+1}, W_q) = C^{\text{cell}}_{q+1}(W, \partial_0 W) \xrightarrow{d_{q+1}} C^{\text{cell}}_q(W, \partial_0 W) = H_q(W_q, W_{q-1}) \ .$$

Choosing the basis for each $H_m(W_m, W_{m-1})$ formed by the m-handles $[\varphi^m_j]$, then the differential d_{q+1} is then given by a matrix $A = (a_{ij})$, where the coefficient a_{ij} is equal to the algebraic intersection number of the transverse sphere of the ith q-handle φ^q_i and attaching sphere of the jth $(q+1)$-handle, $\varphi^{q+1}_j(S^q \times \{0\})$.

Trivial case: Consdier the fortuitous case in which A is a diagonal matrix. Then we can apply the cancellation lemma sequentially to each pair of q- and $(q+1)$-handles which have intersection number equal ± 1. This thereby gets rid of all of the handles, and we obtain the desired diffeomorphism $W \cong \partial_0 W \times [0,1]$.

General case: In general, the matrix A may be any invertible matrix with integer coefficients. Of course, any matrix over the integers can be diagonalized by elementary matrix operations, so if we can isotope the attaching maps for our handles so as to make their intersection theory performa elementary row reduction, we will be able to maneuver to the trivial case. We can do exactly this, as summarized by the following table:

Row reduction operation	Handle operation
$A \sim \begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix}$	Cancellation lemma
Multiply the kth row by x and add to the lth row	Modification lemma: modify $\varphi^{q+1}_l\vert_{S^q \times \{0\}}$ by the boundary $x d_{q+1}[\varphi^{q+1}_k]$. Modify the embedding to $\varphi^{q+1}_l\vert_{S^q \times \{0\}}$ so that $[\varphi^{q+1}_l\vert_{S^q}] = [\varphi^{q+1}_l\vert_{S^q}] + x d_{q+1}[\varphi^{q+1}_k]$
Multiply the kth row by $x \in \mathbb{Z}^\times$	Precompose with degree -1 map on S^q $\varphi^{q+1}_k : S^q \times D^{n-q-1} \to \partial_k W$
Switch rows and columns	Relabel handles

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Thus, we can reduce the handle presentation of any such cobordism W to a cylinder by handle operations corresponding to row reduction operations. This proves the h-cobordism theorem. $\square$

2. Poincaré conjecture

The h-cobordism theorem has many powerful consequences. Before addressing the Poincaré conjecture, on manifold structures on the homotopy type of the n-sphere, we first consider the case of the n-disk.

Theorem 2.1. *Let W be a smooth compact manifold of dimension $n \geqslant 6$ with simply connected boundary. If W is contractible, then W is diffeomorphic to the standard n-disk.*

Proof. Let us choose a smooth embedding g of an n-disk D into the interior of W and remove the interior D° of this n-disk from W. $W - g(D^\circ)$ then defines a cobordism between ∂W and S^{n-1}. Since W can be glued back together from this cobordism and D along the common boundary S^{n-1}, we have the following pushout diagram:

$$\begin{tikzcd} & W & \\ W - g(D^\circ) \arrow[ur, hook] & & D \arrow[ul, hook', "g"'] \\ & S^{n-1} \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

All the maps are cofibrations, so the diagram is a homotopy pushout. Since D is contractible, this implies that $S^{n-1} \to W - g(D^\circ) \to W$ is a homotopy cofiber sequence. By the long exact sequence of cofibration

$$\ldots \to H_*(S^{n-1}) \to H_*(W - g(D^\circ)) \to H_*(W) \to H_{*-1}(S^{n-1}) \to \ldots$$

$W - g(D^\circ)$ is simply connected, therefore $S^{n-1} \to W - g(D^\circ)$ is a homotopy equivalence. Applying the h-cobordism theorem to $W - g(D^\circ)$ we obtain that $W - g(D^\circ)$ is diffeomorphic to $S^{n-1} \times [0, 1]$ relative the boundary component $\partial_0 W = g(\partial D)$. By gluing the n-disk back on along $\partial_0 W$, we thus obtain a diffeomorphism $W \cong D$. $\square$

The following is an easy, but key, fact.

Proposition 2.2. *Every homeomorphism $S^{n-1} \xrightarrow{f} S^{n-1}$ from a sphere to itself extends over the disk (in the following diagram the dotted arrow exists and is a homeomorphism):*

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & S^{n-1} \arrow[d, hook] \\ D^n \arrow[r, dotted, "\exists \tilde{f}"] & D^n \end{tikzcd}$$

Proof. Radially extend the map f through the interior of the disk by defining $\tilde{f}(x) = |x| f(\frac{x}{|x|})$. $\square$

This fact is actually a feature of something much stronger, which is not strictly necessary for the proof of the Poincaré conjecture, but it is easy, important, and pretty, so we include it:

Proposition 2.3 (Alexander trick)**.** *The map* $\mathrm{Res}_\partial : \mathrm{Homeo}(D^n) \to \mathrm{Homeo}(S^{n-1})$ *is a homotopy equivalence.*

Proof. The map Res_∂ is a fibration, where each of the fibers are homeomorphic to $\mathrm{Homeo}(D^n, S^{n-1})$, the space of homeomorphisms of the n-disk which restrict to the identity on the boundary S^{n-1}. The proposition is thus equivalent to statement that $\mathrm{Homeo}(D^n, S^{n-1})$ is contractible, which we shall now show be exhibiting an explicit contraction. That is, we will exhibit $\mathrm{Homeo}(D^n, S^{n-1})$ as a retraction of a contractible space, the cone $\mathrm{Cone}(\mathrm{Homeo}(D^n, S^{n-1}))$, by constructing a map $\mathrm{Cone}(\mathrm{Homeo}(D^n, S^{n-1})) \to \mathrm{Homeo}(D^n, S^{n-1})$ such that composite

$$\mathrm{Homeo}(D^n, S^{n-1}) \to \mathrm{Cone}(\mathrm{Homeo}(D^n, S^{n-1})) \to \mathrm{Homeo}(D^n, S^{n-1})$$

is the identity. This map deforms a homeomorphism f, as t varies from 0 to 1, from f to the identity map. I.e., f_t, as t varies, is a topological isotopy of f and id.

Define f_t by shrinking the homeomorphism f to take place in a disk of radius $1-t$, and outside to be the identity. A formula is given by $f_t(x) = x$, for $|x| \geq t$ and $f_t(x) = t \cdot f(x/t)$ for $|x| \leq t$. $\square$

We finally come to the coup de grâce.

Theorem 2.4 (Generalized Poincaré conjecture)**.** *Let M be a smooth manifold of dimension n at least 5, and suppose M is homotopy equivalent to S^n. Then M is homeomorphic to S^n.*

Remark 2.5. The proof below only applies to the case where n is at least 6. We will prove the case of $n = 5$ later, in our discussion of Kervaire-Milnor's work on exotic spheres. In that dimension, we will prove the stronger result that M must additionally be diffeomorphic to S^5 if they are homotopy equivalent.

Proof. Choose a smooth embedding g of an n-disk in the interior of M, $g : D \hookrightarrow M$, and remove the image of the interior of the disk from the manifold. The resulting manifold with boundary, $M - g(D^\circ)$ is contractible by the long exact sequence in homology. Applying the previous theorem to $W := M - g(D^\circ)$, we obtain that $M - g(D^\circ)$ is diffeomorphic to D^n. Hence we see that is M constructed from two n-dimensional disks D^n_1 and D^n_0 glued along their boundaries by some unspecified diffeomorphism f of the $(n-1)$-sphere:

$$\begin{tikzcd}
S^{n-1} \arrow[r, hook] \arrow[dd, "\cong"', "f"] & D^n_0 \arrow[dr, hook] & \\
& & M \\
S^{n-1} \arrow[r, hook] & D^n_1 \arrow[ur, hook] &
\end{tikzcd}$$

Then, by the Alexander trick, f extends to a homeomorphism $\tilde{f}$ of the disk D^n. We now define a homeomorphism $S^n \to M$ by gluing together the two maps of the n-disks:

$$\begin{tikzcd}
D^n_0 \arrow[r, "\mathrm{id}"] \arrow[d, hook] & D^n_0 \arrow[d, hook] \\
S^n \arrow[r, dotted] & M \\
D^n_1 \arrow[r, "\tilde{f}"] \arrow[u, hook] & D^n_1 \arrow[u, hook]
\end{tikzcd}$$

$\square$

Remark 2.6. The proof establishes a homeomorphism, rather than a diffeomorphism, between M and S^n because the Alexander trick is only available in the topological (or PL) setting. Radial extension has manifestly singular behavior at the origin. Let $\mathscr{S}^{\mathrm{sm}}(S^n)$ denote the space of smooth manifold structures on the topological n-sphere. The argument above shows that any smooth structure on S^n is given by gluing a pair of n-disks along a diffeomorphism of their boundaries. Thus, we obtain a surjective map $\mathrm{Diff}(S^{n-1}) \to \mathscr{S}^{\mathrm{sm}}(S^n)$. A smooth manifold structure will be equivalent to the usual S^n if the diffeomorphism extends over the disk. We thus obtain a sequence of spaces

$$\mathrm{Diff}(D^n) \to \mathrm{Diff}(S^{n-1}) \to \mathscr{S}^{\mathrm{sm}}(S^n)$$

that is exact at the level of π_0. This remark is only valid outside dimension 4: The possibility of exotic 4-spheres exists for completely different reasons, not by twists along diffeomorphisms of S^3.

References

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