MATH 465, LECTURE 16: MORSE FUNCTIONS AND HANDLES

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We'll follow Milnor's book, chapter one and refer to it for proofs and superior exposition. Smooth means C^∞, though everything would be true for C^2. All spaces are smooth manifolds with boundary and all maps are smooth.

1. Critical points and critical values

Definition 1.1. Let M and N be smooth manifolds, $f : M \to N$ a smooth map, and $x \in M$. If $Tf : T_xM \to T_{f(x)}N$ is surjective, we call x a regular point and $f(x)$ a regular value. Otherwise we call x a critical point and $f(x)$ a critical value.

Definition 1.2. If $f : M \to \mathbb{R}$ is a smooth function and x is a critical point of f, we call x a nondegenerate critical point if the Hessian of f is nondegenerate at that point.

Lemma 1.3 (Morse lemma)**.** *Let M^n be a smooth manifold, f a smooth function, and p a nondegenerate critical point. Then there exists a coordinate neighborhood centered at p such that we can write $f(x) = f(p) - x_0^2 - \ldots - x_q^2 + x_{q+1}^2 + \ldots + x_n^2$*

Corollary 1.4. *Nondegenerate critical points are isolated.*

Definition 1.5. We define the index of a nondegenerate critical point to be q in the above formula. We can also define the index to be the dimension of the maximal negative definite subspace of T_xM with respect to the quadratic form defined by the Hessian.

Lemma 1.6. *The two definitions of Morse index are the same.*

2. Topology of level and sublevel sets

Theorem 2.1 (Ehresmann)**.** *Any proper surjective submersion is a smooth fiber bundle.*

Remark 2.2. The idea is to lift coordinate vector fields from the base and use the flows to identify nearby fibers. Propriety is essential for the existence of these flows. In the case where the base has dimension one, we don't even have to worry about commutativity.

Corollary 2.3. *Let $f : M \to \mathbb{R}$ be smooth, proper and without critical points, then $M \equiv f^{-1}(\{0\}) \times \mathbb{R}$*

Corollary 2.4. *If f is a proper smooth function with no critical values in $[a, a+\epsilon]$ then $f^{-1}(-\infty, a]$ and $f^{-1}(-\infty, a+\epsilon]$ are diffeomorphic as manifolds with boundary.*

It remains only to describe how the topology changes as we cross a critical value. In general this could be very drastic, but we have the following lemma.

Lemma 2.5. *Fix $f : M \to \mathbb{R}$ and let $x \in M$ be a nondegenerate critical point of index q. Say $f(x) = 0$. Assume that there are no other critical points y with $f(y) \in [-\epsilon, \epsilon]$. Then $f^{-1}(-\infty, \epsilon]$ is obtained from $f^{-1}(-\infty, -\epsilon]$ by attaching a q-handle.*

sketch. The idea is to produce a function F such that

- $F \equiv f$ outside of a small neighborhood of f
- F and f have the same critical points.

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- $F^{-1}(-\infty, \epsilon] = f^{-1}(-\infty, \epsilon]$
- $F^{-1}(-\infty, -\epsilon]$ is obtained from $f^{-1}(-\infty, -\epsilon]$ by attaching a q-handle.
- F has no critical values in $[-\epsilon, \epsilon]$

This is a tall order, but we can see in this case that $f^{-1}(-\infty, \epsilon] = F^{-1}(-\infty, \epsilon]$ is diffeomorphic to $F^{-1}(-\infty, -\epsilon]$ which is just $f^{-1}(-\infty, -\epsilon]$ plus a q-handle. To see the construction, have a look in Milnor.

□

References

[1] Milnor, John. Morse theory. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N.J. 1963 vi+153 pp.