EXOTIC 7-SPHERES, PART III

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1. Summary of last time

(1) We first constructed an invariant of oriented 7-manifolds. We showed (using Thom's cobordism theory) that the oriented cobordism ring of 7-manifolds is zero. That is, $\Omega_y^{SO} = 0$. That is, every oriented 7-manifold bounds some 8-manifold. Then, using the Hirzebruch signature theorem, we showed "given a 7-manifold M, if you choose some 8-manifold B with $\partial B = M$, then the number

$$\lambda(M) := 2\langle p_1^2(B), [B, \partial M]\rangle - \mathrm{Sig}(B)$$

modulo 7 is independent of B. We called this invariant $\lambda(M)$. (Here, $[B, \partial M]$ is the relative fundamental class of B, and the pairing is that of cohomology to homology. p_1 is the first pontrjagin class. Sig is the signature of B.

(2) We constructed 7-manifolds by fibering S^3 over S^4. We used that fact that $\pi_4 BSO_f = \mathbb{Z} \times \mathbb{Z}$. (This is a special n—$\pi_4 BSO(n) = \pi_4 BSO(5) = \mathbb{Z}$ for all $n \geq 5$; we're just out of stable range with $n = 4$.) Choosing a class (i, j) such that $i - j = 1$,

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d] & S^3 \arrow[d] \\
M_{i,j} \arrow[r] \arrow[d] & BSO(3) \arrow[d] \\
S^4 \arrow[r, "f_{ij}"'] & BSO(4)
\end{tikzcd}$$

from the long exact sequence on π_*, we see that M_{ij} is 6-connected. By the h-cobordism theorem, M^7 is hence homeomorphic to S^7.

(3) We calculated $\lambda(M_{ij})$, knowing

$$M_{ij} = \partial \mathrm{Disk}(\xi_{ij})$$

where ξ_{ij} is the vector bundle classified by f_{ij}, and $B = \mathrm{Disk}(\xi_{ij})$. The signature of B is equal to 1 since $H^4 B$ is 1-dimensional. (It's a disk bundle over S^4). Then we calculated that $p_1(B) = \pm 2(i+j)\iota$.

2. Finishing the proof

Now let's calculate some more. Setting $k = i + j$, we have that

$$\lambda(M_{ij}) = 2pi_1^2(B) - \mathrm{Sig}(B) = 8k^2 - 1$$

which equals $k^2 - 1$ modulo 7.

Then, for $k^2 \neq 1$ modulo 7, M_k is *not* diffeomorphic to S^7, because the value of this invariant for S^7 is zero.

So there exist exotic spheres, QED.

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3. More Exotic Spheres. Kervaire-Milnor's Computation of Θ_n

How is the entire theory of exotic spheres supposed to work? It's not clear; we used some trick that used how a low-dimensional homotopy group behaved for $SO(4)$. Moreover, there may be other exotic 7-spheres. So what we should ask for now is how to do this systematically. This is work of Kervaire and Milnor.

Definition 3.1. Let Θ_n denote the collection of diffeomorphism classes of exotic n-spheres.

By the h-cobordism theorem, this collection is in bijection with another(a priori different) set of equivalence classes.

$$\Theta_n = \{\text{exotic } n\text{-spheres}\}/\text{h-cobordisms}$$

4. Proof of Theorem

There's a fundamental theorem that we'll partially consider:

Theorem 4.1. *Every exotic n-sphere M is stably parallelizable. That is, $TM \oplus \underline{\mathbb{R}}^1$ is trivial.*

Remark 4.2. This is clearly true for the standard n-sphere. Also, While stably parallelizable usually means we can add a trivial bundle $\underline{\mathbb{R}}^k$ of *some* dimension n, the homotopy groups of $SO(n)$ tells us that $k = 1$ is enough. This is because TM is classified by some map

$$M \to BO(n)$$

and tacking on a trivial bundle of dimension k is the same as a map

$$M \to BO(n+k)$$

which is just saying that the composition

$$M \to BO(n) \to BO(n+k)$$

becomes null-homotopic. But the long exact sequence on π_* associated to the fibration

$$\begin{tikzcd} O(n+1) \arrow[r] & O(n+2) \arrow[d] \\ & S^{n+1} \end{tikzcd}$$

tells us that $\pi_i O(n+1) \cong \pi_i O(n+2)$ for $i \leq n$. So $V \oplus \underline{\mathbb{R}}^k$ is trivial if and only if $V \oplus \underline{\mathbb{R}}^1$ is trivial. Note this fact is true of any n-dimensional manifold M.

Remark 4.3. If M is homeomorphic to S^n, TM is classified by an element of $\pi_n BSO$. Bott periodicity helps us here. It tells us that for O,

$$\pi_i(\mathbb{Z} \times BO) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ \mathbb{Z}/2 & i = 2 \\ 0 & i = 3 \\ \mathbb{Z} & i = 4 \\ 0 & i = 5 \\ 0 & i = 6 \\ 0 & i = 7 \end{cases}$$

where all the i equations are modulo 8.[1]

[1]Note that $\mathbb{Z} \times BO \cong \Omega O$. This even says that $\Omega^8 O \cong O$. It's very rare that one can compute all homotopy groups of a nice geometric space, so Bott Periodicity is a very amazing theorem. We black-box it for the moment.

Corollary 4.4. *(1) For $n = 3, 5, 6, 7$, $\pi_n BO$ is zero. Hence the class $[TM \oplus \underline{\mathbb{R}}^1] \in \pi_n BO$ is also zero.*

Proof. (2) Now assume that $n = 0 \bmod 4$. Then if the map $S^n \to BO$ classifying TM is non-zero, then the pontrjagin class of $p_{n/4}(TM)$ is non-zero. But the Hirzebruch signature theorem would then imply that the signature of M is non-zero, since $p_i(M) = 0$ for all $i < n/4$. But the signature of any manifold is a topological invariant, so if M is homeomoprhic to a sphere, its signature is zero. Hence $[TM \oplus \mathbb{R}] = 0$.

(3) This is hard. We'll say a few words, then come back to it later. If $n = 1, 2 \bmod 8$, we have these $\mathbb{Z}/2$ showing up. This depends on an understanding of the j homomorphism. $\square$

Definition 4.5 (J-homomorphism)**.** We define a map

$$J_k : \pi_k 0 \to \pi_k \mathbb{S}^0)$$

as follows. There is a map

$$O(n) \to Maps_{pairs}((D^n, S^{n-1}), (D^n, S^{n-1}))$$

given by the action of $O(n)$ on (D^n, S^{n-1}), and collapsing

$$\begin{tikzcd} O(n) \arrow[r] \arrow[dr, "J"'] & Map_{pairs}((D^n, S^{n-1}), (D^n, S^{n-1})) \arrow[d] \\ & Map_*(S^n, S^n) \end{tikzcd}$$

gives an unstable version of J.[2] It gives us a map

$$\pi_k O(n) \to \pi_k \Omega^n S^n \cong \pi_{n+k} S^n.$$

Letting $n \to \infty$, we have the stable version, a map

$$J_k : \pi_k O \to \pi_k \mathbb{S}^0.$$

Now the J-homomorphism on homotopy groups is injective.

Theorem 4.6 (Adams, $J(X)$ IV)**.** *The map*

$$J_k : \pi_k O \to \pi_k \mathbb{S}^0$$

is injective for $k = 1, 2$ modulo 8.

Corollary 4.7. *$TM \oplus \underline{\mathbb{R}}$ is trivial if and only if the associated bundle of spheres is trivial. This can be shown to be trivial.*[3]

So we have a surjective map from stabley framed exotic n-spheres to exotic n-spheres, and the corresponding maps in the cobordism theory

$$\begin{tikzcd} \Theta_n \arrow[r] & \Omega_n^{\mathrm{fr}}/\{\text{framings}\} \\ \Theta_n^{\mathrm{fr}} \arrow[u] \arrow[r] & \Omega_n^{\mathrm{fr}} \arrow[u] \end{tikzcd}$$

And Pontrjagin-Thom says that

[2] Owen points out this is the same thing as embedding $O(n) \hookrightarrow O(n+1)$.

[3] Given a vector bundle, we can form a disk bundle. Then we quotient out by the boundaries of these disks, fiberwise, to form a sphere bundle. This is what the J homomorphism is doing. So the vector bundle is trivial if and only if the associated bundle of spheres is trivial. We'll give an argument later about why this sphere bundle is trivial.

$$\begin{tikzcd} \Omega_*^B \arrow[r] \arrow[d, "B=*"'] & \pi_* MB \arrow[d, "MB=\mathbb{S}^0"'] \\ \Omega_*^{\mathrm{fr}} \arrow[r] & \pi_*\mathbb{S}^0 \end{tikzcd}$$

But we get that

$$\Omega_*^{\mathrm{fr}} = \pi_*\mathbb{S}^0/\mathrm{image}(\pi_* O) = \mathrm{Coker}(J_*).$$

So

$$\begin{tikzcd} \Theta_n \arrow[r] & \mathrm{Coker}(J_n) \\ \Theta_n^{\mathrm{fr}} \arrow[r] \arrow[u] & \Omega_n^{\mathrm{fr}} \arrow[u] \end{tikzcd}$$

By defining bP_{n+1} to be the kernel of the map $\Theta_n \to \Omega_n^{\mathrm{fr}}/\{\text{framing}\}$ (i.e., those exotic n-spheres that bound a parallelizable $(n+1)$-manifold) we need to understand

$$bP_{n+1} \to \Theta_n \to \Theta_n/bP_{n+1} \to \mathrm{Coker}(J_n)$$

and there is a theorem that almost always, this last map is an isomorphism. So studying Θ_n/bP_{n+1} reduces to understanding homotopy theory and the image of J_*.[4] On the other hand, understanding bP_{n+1} involves studying framed surgery. This is the somewhat more tractable problem.

So the next order of business is the J-homomorphism. In the end we'll get some results, like Θ_n is zero for $n = 5, 6$ and Θ_7 has 28 elements.

[4]This is hard, this involves knowing the homotopy groups of $\mathbb{S}^0$ and O.