MATH 465, LECTURE 1: OVERVIEW

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The driving question for this class will be to classify manifolds. There are two obvious features to address in making this question more precise:

(1) What kinds of manifolds?
(2) Classify up to what notion of equivalence?

1. What kinds of manifolds?

Topological manifolds form the most basic choice. A topological manifold is just a particularly nice type of topological space: being locally Euclidean is a property of a space (as is being Hausdorff and paracompact). There's no need to choose additional structure, such as a topological atlas.

Of course, one may want to do analysis on manifolds, so we may also want to consider smooth structures (i.e., a smooth atlas) on topological manifolds, particularly since all the canonical examples of manifolds come with such a structure.

However, there are reasons to make a slightly different choice, that of piecewise linear (PL) manifolds. For one, our interest in analysis might be slight, and it might be focused on the large class of analytic invariants that admit more topological interpretations (e.g., the de Rham complex). For another, more historical, reason, the origins of topology were very combinatorial, so the notion of a PL manifold might well be that closest to Poincaré's original ideas. Finally, and most compellingly, we might want to restrict attention to those invariants that are readily computable from a triangulation – TFT invariants should look like this, according to the Baez-Dolan hypothesis. The structure of topological manifolds is insufficient (in part since it is not obvious that they are even triangulable), while the PL structure is fundamental. PL manifolds have fallen out of style recently, but they may make a comeback.

So the three basic cases we'll consider are topological, smooth, and PL manifolds.

2. Classify up to what?

The most obvious classification would be to ask for a list of equivalence classes of Cat manifolds up to Cat homeomorphism, where Cat refers to smooth, PL, or topological. This has the practical drawback that it is hard, since the equivalence relation is so restrictive, and the theoretical drawback that it is not possible to produce a sensible list, for both practical and list-theoretic reasons. We can deal with the first issue by dealing with a coarser classification problem first: classify manifolds up to cobordism. Thom's cobordism theory gives a simple classification, amenable to the list format: The cobordism group of n-manifolds is a finitely generated abelian group for any n, and it is possible to make explicit choices of generators. This will be our first subject.

But, for example, all orientable 2-manifolds bound a 3-manifold, and all orientable 3-manifolds are parallelizable, and hence bound a 4-manifold. So the cobordism classification of orientable 1-, 2-, and 3-manifolds is trivial, since they are all null-cobordant, even though the theory of 3-manifolds is, of course, very interesting.

So how does one refine this coarse classification? Well, cobordism can alter the homotopy type of a manifold (for instance, an orientable 2-manifold does not have the homotopy type of the empty set, even though they are cobordant). So what if we ask for cobordisms also preserve homotopy classes? This is the notion of an h-cobordism.

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Definition 2.1. M is h-cobordant to N if there exists a manifold W with boundary $i \amalg j : M \amalg N \xrightarrow{\cong} \partial W$ for which the maps $i : M \hookrightarrow W$, $j : N \hookrightarrow W$ are both homotopy equivalences.

This radically throws out the grossness of previous classification, so much so that it's not clear that the notion of h-cobordism is different from the notion of Cat homeomorphism. That is, the first example of an h-cobordism is a cylinder, $W = M \times [0, 1]$; but are there any nontrivial h-cobordisms?

However, the problem of classifying manifolds up to h-cobordism is more immediately tractable. For example, Kervaire and Milnor mostly did this for spheres S^k for all k (except k=3 or 4, and modulo a factor in certain dimensions depending on the Kervaire invariant problem, solved by Hill-Hopkins-Ravenel). By the h-cobordism theorem, their work simultaneously classified exotic smooth structures on spheres.

Theorem 2.2 (The h-cobordism theorem)**.** *Let M be simply-connected Cat manifold of dimension greater than* 4*. Let W be an h-cobordism from M to N. Then W is Cat-homeomorphic to the product $M \times [0, 1]$*

This has a corollary.

Corollary 2.3 (Higher dimensional Poincaré conjecture)**.** *If M is a smooth or PL manifold of dimension greater than* 4*, and M is homotopy equivalent to some S^n, then M is PL homeomorphic (and, in particular, homeomorphic) to S^n.*

The result was first proved by Smale in the smooth case for dimensions greater than 6, then shortly thereafter by Stallings for PL manifolds, then by Zeeman for dimensions 5 and 6. It is *false* that M need be diffeomorphic to S^n, as Milnor's construction of an exotic 7-sphere first showed.

So, there is no unique section of the forgetful functor $\mathrm{Mflds}^{\mathrm{Diff}} \to \mathrm{Mflds}^{\mathrm{Top}}$. In fact, there is no section at all, because there exist unsmoothable topological manifolds.

What about manifolds with nontrivial fundamental groups? Here, the h-cobordism theorem fails: There exist h-cobordisms that are not products. But this is fixable by refining our notion of "homotopy" to correct this equivalence. The required notion is that of a simple homotopy.

The idea is as follows: For M a smooth or PL manifolds, M has a CW structure. Thus we can ask that homotopies be of a certain type—if it is built up by a nice kind of inclusion (like a circle into a cylinder, which looks like a fattening or an expansion) or a nice kind of surjection (like a cylinder down to a circle, which looks like contracting or collapsing), we'll call this a simple homotopy equivalence.

Definition 2.4. An s-cobordism is an h-cobordism for which i and j are simple homotopy equivalences.

It is *not* true that any homotopy equivalence can be expressed as a composite of these nice kinds of maps. So not every homotopy equivalence of CW complexes is simple. But there is a classification of all *types* of homotopy equivalences of CW complexes, given by something called the Whitehead torsion. This is a purely algebraic invariant determined by π_1.

So we can refine our classification to obtain:

Theorem 2.5 (The s-cobordism theorem)**.** *(Barden, Mazur, Stallings) For dim $M > 4$, an s-cobordism W of M is equivalent to a product $W \cong M \times [0, 1]$. Furthermore, the collection of equivalence classes h-cobordisms with source M are uniquely classified by Whitehead torsion: $H - \mathrm{Cob}(M) \cong \mathrm{Wh}(\pi_1 M)$.*

Remark 2.6. For topological 4-manifolds, this is due to Freedman. Donaldson showed the h-cobordism theorem fails for smooth 4-manifolds.

That is, h-cobordisms of M are in one-to-one correspondence with elements of the Whitehead group of $\pi_1 M$ (which is a quotient of the algebraic K-group $K_1(\mathbb{Z}[\pi_1 M])$).

Now we have some ways of classifying manifolds in terms of homotopy theory. For instance, the results of Smale, Kervaire, and Milnor classified exotic spheres by first specifying the *homotopy* *type* of S^n, then classifying smooth manifolds within this homotopy type (which all happened to be homeomorphic). This idea could be applied more generally: Instead of asking for a list of *all* smooth manifolds, we could ask for a mechanism for listing smooth manifolds within a homotopy type, once that homotopy type has been specified. The ideas that went into these theorems can then be applied quite generally, as was done by C.T.C. Wall (Browder and Novikov in the simply-connected case).

Theorem 2.7 (The fundamental theorem of surgery)**.** *There exists an exact sequence (of sets) which we call the surgery exact sequence for any , as follows:*

$$\widetilde{\mathcal{S}}^{\mathrm{Cat}}(X) \to \mathcal{N}^{\mathrm{Cat}}(X) \to L[\pi_1 X]$$

Here $\widetilde{\mathcal{S}}^{\mathrm{Cat}}(X)$ is the set of equivalence classes of Cat manifolds M such that M is homotopy equivalent to X; $\mathcal{N}^{\mathrm{Cat}}(X)$ is a set reflecting data about the possible normal bundles of X; the algebraic L-theory group $L[\pi_1 X]$ determines an obstruction. (This sequence can be continued to the left, which we will also discuss later.)

3. Second half

The preceding will form the core of the course. In the latter half of the course, we tackle some subset of the following: smoothing theory, low-dimensional manifolds, and topological field theories.

First, instead of *set* ways of giving manifold structures, what is the *space* of ways?

Version One: Let X be a space. Then define $\mathcal{S}^{\mathrm{Cat}}(X)$ to be the Cat structure space of X (the space of all Cat manifolds homotopy equivalent to X) which we will define to be $B\mathcal{C}_X$, the classifying space of a topological category $\mathcal{C}_X$. The category $\mathcal{C}_X$ is defined to have as objects the Cat manifolds M homotopy equivalent to X, and morphism spaces consist of Cat homeomorphisms, $\mathrm{Map}_{\mathcal{C}_X}(M,N) = \mathrm{Homeo}^{\mathrm{Cat}}(M,N)$.

So $\pi_0\mathcal{S}^{\mathrm{Cat}}(X)$ is the set of equivalence classes of manifolds homotopy equivalent to X. Each connected component is the classifying space of $\mathrm{Diff}(M)$ (or $\mathrm{Top}(M)$ or $\mathrm{PL}(M)$).

Version 2 (relative) By a theorem of Whitehead, every smooth manifold has an essentially unique underlying PL manifold structure. We therefore have a hierarchy of structures

$$\begin{array}{ccc}
\mathrm{Mflds}^{\mathrm{Diff}} & \xrightarrow{\text{Whitehead}} & \mathrm{Mflds}^{\mathrm{PL}} \\
& \searrow \quad \swarrow & \\
& \mathrm{Mflds}^{\mathrm{Top}} & \\
& \downarrow & \\
& \mathrm{Spaces}^{\mathrm{htpy}} &
\end{array}$$

And, given an object M of one of these categories, a Cat_0 manifold, we could ask for the *space* of lifts of that object to the category of Cat_1 manifolds: This space of lifts is equivalent to the fiber of the map $\mathcal{S}^{\mathrm{Cat}_1}(M) \to \mathcal{S}^{\mathrm{Cat}_0}(M)$ over the point $M \in \mathcal{S}^{\mathrm{Cat}_0}(M)$. Smoothing theory addresses how to analyze the homotopy type of this space of lifts.

The main theorem of smoothing theory, due to Kirby-Siebenmann (building on Cairns-Hirsch and Hirsch-Mazur for the PL to smooth case), is that one can build a fiber bundle over a M whose sections classify Cat_1 manifold refinements of the Cat_0 manifold structure on M (except in the case where the dimension is 4 and $\mathrm{Cat}_0 = \mathrm{Top}$).

This theory shows that there are obstructions to PL smoothing a topological manifold M living in $H^4(M, \mathbb{Z}/2)$, which can be used to disprove the Hauptvermutung, that every topological manifold has a unique PL manifold structure. When this obstruction vanished for a topological manifold M, then distinct PL structures are classified by elements of $H^3(M, \mathbb{Z}/2)$.

Finally, in low dimensions, smoothing theory contributes to the proof of the following results:

Dimension 3: Topological 3-manifolds have unique PL and smooth structures in strong sense, in that any automorphism lifts in a unique way to a smooth map;

Dimension 4: PL 4 manifolds have a unique smoothing.