MATH 465, LECTURE 20: MORSE THEORY ON LOOP SPACES

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We have shown that, given a Morse function f on a manifold M, we obtain a CW-complex homotopy equivalent to M which has a k-cell for each critical point of index k. Here M was a finite dimensional manifold and the CW-complex obtained from f a finite CW-complex. We can ask wether or not this generalizes to infinite CW-complexes so as to account for spaces like $Gr_n(\mathbb{C}^\infty)$ or U. The happy fact is that if we restrict to loop spaces, Morse theory caries over without too much work. In this lecture we will generalize the main theorem of Morse theory to loop spaces. We will then use this result to analyze $\Omega U(n)$ and $\Omega SU(n)$.

1. The Main theorem of Morse theory for loop spaces

Let M be a smooth Riemannian manifold, and x, y be points on M. Define $\Omega_{x,y}M$ to be the space of piecewise smooth paths $f : [0,1] \to M$ such that $f(0) = x$ and $f(1) = y$. This loop space is homotopy equivalent to ΩM. We define an energy function on $\Omega_{x,y}M$ as follows.

Definition 1.1. The *energy* E is the function $E : \Omega_{x,y}M \to \mathbb{R}$ given by

$$\gamma \mapsto \int_0^1 |\frac{\mathrm{d}\gamma}{\mathrm{d}t}|^2 \mathrm{d}t.$$

If you imagine the path γ to be a stretched elastic band, the energy E corresponds to the potential energy of the band. Alternatively, if you're familiar with the calculus of variations, E can be given another interpretation as the *action*.

Our hope is to use E as a Morse function, so we need to understand what we mean by a critical point of E, i.e., we want to define $\mathrm{d}E$. We already have a definition of tangent spaces for a manifolds. A natural way to generalize it to loop spaces is to consider tangent vectors of M along a path γ. In other words, to take a section of the tangent bundle of M along our path.

Definition 1.2. Let γ be in $\Omega_{x,y}M$. The tangent space $T_\gamma \Omega_{x,y}M$ is the vector space of piecewise smooth vector fields along γ.

Now we would like to define

$$\mathrm{d}E : T_\gamma \Omega \to T_{E(r)}\mathbb{R}.$$

But really, what we are after is the critical points, so it is sufficient to know when this map should be zero. A map of vector spaces is zero if it vanishes on every one dimensional subspaces. We can reduce the question to a finite dimensional problem.

Definition 1.3. Let γ be in $\Omega_{x,y}M$. A *variation* for γ is a map

$$\overline{\alpha} : (-\epsilon, \epsilon) \to \Omega_{x,y}M$$

with $\overline{\alpha}(0) = \gamma$.

A variation is a one parameter family, so passing to tangent spaces should give one dimensional subspaces. We will want to define the critical points to be those for which dE vanishes on all variations, i.e., the composite of the moral maps

$$T_0(-\epsilon, \epsilon) \xrightarrow{d\overline{\alpha}} T_\gamma \Omega_{x,y}M \xrightarrow{dE} T_{E(\gamma)}\mathbb{R}$$

must be zero. Since we can define the map on tangent spaces for the composite $E \circ \overline{\alpha}$, we can bypass dE.

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Definition 1.4. The point γ in $\Omega_{x,y}M$ is a critical point of E if, for any variation $\overline{\alpha}$ of γ, the map

$$d(E \circ \overline{\alpha}) : T_0(-\epsilon, \epsilon) \to T_{E(\gamma)}\mathbb{R}$$

is zero.

Now we turn to the index. Recall that the index of f at a critical point x is the bilinear form given by the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

Another description of the bilinear form $f_{*,*}(V_1, V_2)$ for V_1 and V_2 vector fields can be given by choosing a two parameter variation $\alpha : \mathbb{R}^2 \to M$ such that $\partial\alpha/\partial u_i = V_i$. Then

$$f_{*,*}(V_1, V_2) = \frac{\partial^2 (f \circ \alpha)}{\partial u_1 \partial u_2}.$$

One can check that the two definitions coincide, but this one is taylored for generalization.

Definition 1.5. For γ a critical point of E, the bilinear form

$$E_{*,*} : T_\gamma \Omega_{x,y}M \times T_\gamma \Omega_{x,y}M \to \mathbb{R}$$

is defined by

$$E_{*,*}(V_1, V_2) = \frac{\partial^2 (E \circ \overline{\alpha})}{\partial u_1 \partial u_2}(0, 0).$$

Here $\overline{\alpha} : \mathbb{R}^2 \to \Omega_{x,y}M$ is a two parameter family such that $\overline{\alpha}(0, 0) = \gamma$ and $\partial\overline{\alpha}/\partial u_i = V_i$.

Going back to our analogy with the elastic band, if γ is a geodesic, pushing the band in any direction increases the energy. Hence the energy is at its minimum precisely in this case. This is an intuitive explanation for the following two lemmas.

Lemma 1.6. *A geodesic is a critical point γ of E.*

Lemma 1.7. *A minimal geodesic is critical point of index zero, where a minimal geodesic is one minimizing the distance between x and y.*

Now we have all the ingredients to generalize the main theorem of Morse theory.

Theorem 1.8 (Main Theorem of Morse theory on loop spaces)**.** *Let M be a Riemannian manifold and x, y be points on M, satisfying certain conditions. Under certain conditions the space $\Omega_{x,y}M$ is homotopy equivalent to a countable CW complex with a k-cell for each index k critical point of E.*

From this we get a some homotopical information.

Theorem 1.9. *If the subspace $\Omega^{\min}_{x,y}M$ of minimal geodesics in $\Omega_{x,y}M$ is a topological manifold and the smallest index of a non-minimal cirtical point of E is equal to k, then*

$$\pi_i(\Omega_{x,y}M, \Omega^{\min}_{x,y}M) = 0 \textit{ if } i < k,$$

i.e., the inclusion $\Omega^{\min}_{x,y}M \to \Omega_{x,y}M$ is $k-1$ connected.

We already have an example. Consider S^{n+1} with antipodal points x and y. A minimal geodesic is a meridian from x to y. It intersects the equator, S^n, in a unique point. Hence there is a homeomorphism between $\Omega^{\min}_{x,y}M$ and S^n. Our theorem tells us that $S^n \to \Omega_{x,y}S^{n+1} \simeq \Omega\Sigma S^n$ has a certain connectivity. In fact, this map is the unit of the adjunction between loop and suspension. We already know from Freudenthal's suspension theorem that it is $2n-1$ connected. Another geodesic is one that wraps around the sphere a certain number of times. You can check that index and connectivity match in this example. We could prove Freudenthal's Suspension Theorem this way. But we would not have the full generality since it holds for any space given a certain connectivity.

2. Understanding ΩU and ΩSU

In this section we describe $\Omega_{I,-I}U(n)$ and $\Omega_{I,-I}SU(2n)$. From Lie Theory we have a usual metric on $U(n)$ given by $Tr(AB^t)$, having chosen the left invariant framing of $T_I U(n)$. We can describe $T_I U(n)$ as the set of $n \times n$ Hermitian matrices, i.e., those matrices (a_{ij}) satisfying $a_{ij} = -\overline{a_{ji}}$.

We have an exponential map $exp : T_I U(n) \to U(n)$, given by

$$\exp(A) = \sum_{k \geq 0} \frac{A^k}{k!}.$$

It is a standard fact of Lie theory that this map is surjective.

Next time, we will use this to understand the minimal geodesics of $U(n)$ and $SU(2n)$.

References