MATH 465, LECTURE 21: BOTT PERIODICITY

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Recall the theorem from the previous lecture:

Theorem 0.1. *If M is a Riemannian manifold and $x, y \in M$ are points on M satisfying certain conditions, then the map*

$$\Omega^{min}_{x,y} M \to \Omega_{x,y} M$$

is $(k-1)$-connected, where k is the smallest index of a non-minimal geodesic between x and y.

In this lecture we will use the theory of non-monimal geodesics on a Riemannian manifold to prove Bott's result about periodicity of homotopy groups of infinite unitary and orthogonal groups.

1. Minimal geodesics

Recall from the last time that we have the exponential map

$$\begin{array}{ccc} T_I U(n) & \xrightarrow{exp} & U(n) \\ A & \longmapsto & \Sigma \frac{A^k}{k!}, \end{array}$$

from the Lie algebra of $U(n)$ (which consists of skew Hermitian matrices) to $U(n)$, which sends A to $\Sigma \frac{A^k}{k!}$. This map is invariant with respect to the adjoint action.

Question. For which matrices $A \in T_I U(n)$ is $expA = -I$? (If we can find this space, it would be the space of minimal geodesics.)

First notice that any matrix A can be diagonalized by conjugation by some element $g \in U(n)$ and this conjugation does not change the value of exponential map, since $exp(gAG^{-1}) = g \cdot exp(A) \cdot g^{-1} = g(-I)g^{-1} = -I$. So we can assume that A is diagonal, hence it is a matrix of the form $A = diag(ia_1, ia_2, \dots, ia_n)$, where a_i are real numbers (because we know that A is skew Hermitian). Then $exp(A) = diag(e^{ia_1}, \dots e^{ia_n})$ hence $e^{ia_j} = 1$ and $a_j = \pi k_j$, where k_j is an odd integer. The length of geodesics will be given by $\pi\sqrt{\Sigma k_j^2}$ and it is minimized when $k_j = \pm 1$. The eigenspace of this matrix is a direct sum of negative eigenspace (given by $k_j = -1$) and positive eigenspace(orthogonal complement of the negative). The positive eigenspace can form any subspace of $\mathbb{C}^n$. Hence

$$\Omega^{min}_{I,-I} U(n) \cong \coprod_{0 \le k \le n} Gr_k(\mathbb{C}^n).$$

Recall also that $Lie(SU(n)) \subset Lie(U(n))$ and let $n = 2m$. Then

$$\Omega^{min}_{I,-I} SU(2m) \cong Gr_m(\mathbb{C}^{2m})$$

by the same argument as before, taking into account that matrices in $Lie(SU(n))$ have zero trace (so negative and positive eigenspaces have equal dimensions).

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2. Bott periodicity

We will use the following lemma which will be given without proof.

Lemma 2.1. *The smallest index of a non-minimal geodesic in* $SU(2m)$ *is* $2m+2$.

Corollary 2.2. *The space of minimal geodesics, which we just identified with the Grassmannian is (2m+1)-connected.*

$$Gr_m(\mathbb{C}^{2m}) \cong \Omega^{min}_{I,-I} SU(2m) \to \Omega SU(2m)$$

$\mathbb{Z} \times \Omega SU \cong \Omega U$ and it follows from the corollary that

$$\mathbb{Z} \times Gr_m(\mathbb{C}^{2m}) \to \Omega U(2m)$$

is $(2m+1)$-connected. On the other hand

$$\varinjlim Gr_m(\mathbb{C}^{2m}) = BU.$$

So we proved that $\mathbb{Z} \times BU$ is homotopy equivalent to ΩU and we have

$$\Omega^2 U \cong \Omega(\mathbb{Z} \times BU) \cong \Omega BU \cong U$$

We know that $\pi_0 U = 0$ and $\pi_1 U = \mathbb{Z}$, so $\pi_i U = \begin{cases} 0, & i \text{ even} \\ \mathbb{Z}, & i \text{ odd} \end{cases}$.

We can also apply these methods to the orthogonal group to get:

$$\begin{aligned} \Omega O &= O/U \\ \Omega^2 O &= U/Sp \\ \Omega^3 O &= \mathbb{Z} \times BSp \\ \Omega^4 O &= Sp \\ \Omega^5 O &= Sp/U \\ \Omega^6 O &= U/O \\ \Omega^7 O &= \mathbb{Z} \times BO \\ \Omega^8 O &= O \end{aligned}$$

$\pi_0 Sp = \pi_0(Sp/U) = \pi_0(U/O) = \pi_0(U/Sp) = 0$, because they are connected.
$\pi_0 O = \pi_0(O/U) = \mathbb{Z}/2$, because they have two components.
$\pi_0(\mathbb{Z} \times BSp) = \pi_0(\mathbb{Z} \times BO) = \mathbb{Z}$ and the result follows:

$$\pi_k O = \pi_0 \Omega^k O = \begin{cases} 0, & k = 2, 4, 5 \text{ or } 6 \\ \mathbb{Z}, & k = 3 \text{ or } 7 \\ \mathbb{Z}/2, & k = 1 \text{ or } 8 \end{cases}$$