# MATH 465, LECTURE 22: EXOTIC SPHERES THAT BOUND PARALLELIZABLE MANIFOLDS

J. FRANCIS, NOTES BY H. TANAKA

Our goal is to understand $\Theta_n$, the group of exotic $n$-spheres. The group operation is connect sum, where clearly the connect sum of two topological spheres is again a topological sphere.

We'll write this by $M\sharp M'$. There is a cobordism $M\sharp M'$ to $M\coprod M'$, given by the surgery we perform on $M$ and $M'$ to obtain the connect sum. This is because

$$M\sharp M' = \partial_1(M\coprod M' \times [0,1] + \phi^1).$$

We've shown that any exotic sphere is stably parallelizable, in that we can always direct sum a trivial line bundle to the tangent bundle to obtain a trivial vector bundle.

If we consider the set of framed exotic $n$-spheres, $\Theta_n^{\mathrm{fr}}$, we can consider the Pontrjagin construction

$$\Theta_n^{\mathrm{fr}} \to \Omega_n^{\mathrm{fr}}$$

but this is going to factor through the set of framed exotic spheres modlo those which are boundaries of framed $n+1$-manifolds, $bP_{n+1}^{\mathrm{fr}}$. In fact this map is a gorup homomorphism becasue disjoint union is cobordant to connect sum!

$$\begin{array}{ccccc}
\Theta_n^{\mathrm{fr}} & & \longrightarrow & & \Omega_n^{\mathrm{fr}} \\
 & \searrow & & \nearrow & \\
\downarrow & & \Theta_n^{\mathrm{fr}}/bP_{n+1}^{\mathrm{fr}} & & \downarrow \\
\Theta_n/bP_{n+1} & & \longrightarrow & & \pi_n\mathcal{S}^0/\pi_n(O)
\end{array}$$

The map from $\Theta_n/bP_{n+1}$ to $\pi_n\mathcal{S}^0/\pi_n(O)$ is an injection, and this latter set is the ckernel of the $J_n$ homomorphism. We also have a map from $\Omega_n^{\mathrm{fr}}$ to $\pi_n\mathcal{S}^0/\pi_n O$ by Pontrjagin-Thom.

(Note also that the map $\Theta_n^{\mathrm{fr}}/bP_{n+1}^{\mathrm{fr}} \to \Omega_n^{\mathrm{fr}}$ is injective.)

$$bP_{n+1} \longrightarrow \Theta_n \longrightarrow \Theta_n/bP_{n+1} \subset Coker(J_n)$$

We know from stable homotopy theory the following homotopy groups

$$\pi_n\mathcal{S}^0 = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n=1 \\ \mathbb{Z}/2 & n=2 \\ \mathbb{Z}/24 & n=3 \\ 0 & n=4 \\ 0 & n=5 \\ \mathbb{Z}/2 & n=6 \\ \mathbb{Z}/240 & n=7 \end{cases}$$

For example the Hopf map is the generator of $\pi_3 S^2$ and it surjects onto $\pi_1\mathcal{S}^0 = \mathbb{Z}/2$.

---

*Date*: Lecture May 19, 2010. Not yet edited.

Similarly we have what is most relevant to us: The Hopf fibration $S^7 \to S^{15} \to S^8$, which is given by the octonions. There is an action by the (unit) octonions on (units in) octonion 2-space, and this quotient is $S^8$. And this gives a surjection $\pi_{15}S^8 \to \pi_7\mathcal{S}^0$, where $\pi_{15}S^8$ is $\mathbb{Z}$.

And the surjection from $\pi_7 0$ to $\mathbb{Z}/240$ is given by the $J_7$ homomorphism.

Conclusion: This implies that the $J$ homomorphism is surjective in degree 7. i.e., $J_7$ is surjective.

We are interested in the cokernel of the $J$-homomorphism.

Well, we see that the coker $J_n$ is given by 0 at n=5,7 and $\mathbb{Z}/2$ at $n = 6$. This implies that $\Theta_n/bP_{n+1}$ is equal to 0 in $n = 5, 7$ and 0 or $\mathbb{Z}/2$ in $n = 6$.

So by homotopy theory, we know that this is zero in some cases.

# 1. How do we analyze $bP_{n+1}$?

## 1.1. The case of $bP_{2k+1}$.

**Proposition 1.1.** *Let $M^n \in bP_{n+1}$. Then there exists a parallelizable $W^{n+1}$ such that $\pi_i W = 0$ for $i < n/2$, such that $\partial W = M^n$.*

In other words, we can always find a highly-connected manifold whose boundary is $M^n$.

*Remark* 1.2. Note that $M$ is just stably parallelizable, but $W$ is straight-up parallelizable. As it turns out,

**Lemma 1.3.** *If $W$ is a manifold with non-empty boundary, then $W$ is parallelizable if and only if it is stably so.*

*Proof.* Top homology vanishes because $W$ has non-empty boundary. We can choose a $CW$ structure on $W$ such that $W$ is homotopy equivalent to its $n$-skeleton.

$$\begin{array}{ccccc} W & \xrightarrow[T_W]{} & BO(n+1) & \xrightarrow[i]{} & BO \\ & \searrow^{\exists \tilde{T_W}} & \uparrow & & \\ & & BO(n) & & \end{array}$$

since $W$ is homotpic to a $CW$ complex with no cells above dimension $n$, then $i \circ T_W$ is trivial if and only if $_W1$ is trivial. ☐

With this lemma, here is the sketch of how to prove the proposition:

**Proposition 1.4.** *We use framed surgery. We choose any $W$ by surgery, and kill the lowest nonzero homotopy of $W$, $\pi_i W$.*

$$[f] \in \pi_i W$$

*then we choose any embedding*

$$S^i \hookrightarrow W^0$$

*into the interior of $W$, by the map $f$, and perform surgery along this map. Then*

$$W + f \cong W \coprod_{S^i} D^{i+1}$$

*so while this may introduce higher homotopy groups, it kills the lowest ones.*

*The only difficult detail is that we need to carry along the trivialization of $T_W$ to $T_{W+f}$. For the moment let's just assume we can do that.*

*Now we just repeat this process. But to choose* embeddings *into $W$, the dimension $i$ cannot be too large. This works only up to the middle dimension. After that, by Whitney Embedding theorem, we may not be able to embed the high-dimensional spheres.*

*This completes the proof sketch.*

Now, if the dimension of $W$ is odd, we see by Poincare duality that more than the middle dimension is zero.

**Corollary 1.5.** *Let $n$ be even. Then $W^{n+1}$, with $\partial W = M^n$, can be modified so that $W$ is contractible. That is, any exotic sphere bounding a parallelizable manifold is the boundary of a contractible manifold. (If $M \in bP_{2n+1}$, then $M$ bounds a contractible manifold.)*

*Proof.* By the proposition, fix $\pi_{2k} W = 0$. By Hurewicz, this means that $H_{2k} W = 0$. By Poincare duality, $H_{2k+1} W = 0$. So $H_* W = 0$ for all $* > 0$. □

But in the proof of the h-cobordism theorem, we saw that there are no exotic disks. That is, if $W$ is a contractible compact manifold, thatn $W$ is diffeomorphic to a disk. This was for dimensions 5 or bigger. (The question for $n = 4$ is an open question, John thinks.)

So if $M$ bounds a contractible manifold, then $M$ is diffeomorphic to *the* sphere. So this implies that $bP_{2k+1} = 0$.

We're done with half of the $bP$ analysis. We also need to apologize the $\Theta_n$ stuff, so we're about a fourth of the way done.

What we see then is that

$$\Theta_n / bP_{n+1} = \begin{cases} 0 & n = 5 \\ 0 \text{ or } \mathbb{Z}/2 & n = 6 \\ 0 & n = 7 \end{cases}$$

and $bP_{2k+1} = 0$.

So $\Theta_6 = 0$ or $\mathbb{Z}/2$.

1.2. $bP_{2k}$**.** What about $bP_{2k}$? It turns out this further splits into whether $k$ is even or odd. We'll get a topological invariant by studying the intersection form, and the kind of intersection form we get will depend a lot on whether $k$ is even or odd.

For $M \in bP_{2k}$, we can obtain a $(k-1)$-connected framed manifold $W$ with $\partial W = M$. This has an associated intersection form which is an algebraic "invariant" of $M$. How much is $W$ specified by this intersection form? And for what intersection forms do we get a manifold which is in fact trivial?

So next time, in the case where $k = 2m$ is even, we'll construct a homomoprhism

$$\mathbb{Z} \to bP_{4m}$$

by a process known as plumbing, by plumbing disk bundles. It will turn out that this is a surjective homomorphism with a kernel given by a simple formula.

(We will be taking an even unimodular lattice, and mapping it to a $W$ whose boundary is an exotic sphere.)

In the case where $k = 2m + 1$ is odd, $bP_{4m+2}$ is subtle, and depends on some sophisticated homotopy theory. It has only been closed up recently by Hopkins et al., in their work relating to the Arf invariant.

We'll show next time, in fact, that $bP_8$ is $\mathbb{Z}/28$, and that $\Theta_6 / bP_{n+1}$ is actually 0.