MATH 465, LECTURE 3: THOM'S THEOREM

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Today, we will begin the proof the the following theorem, first proved by René Thom in what is possibly the best math PhD thesis of all time. See [1].

Theorem 0.1. *The cobordism group of unoriented n-dimensional manifolds is naturally isomorphic to the nth homotopy group of the Thom spectrum MO. That is, there is a natural isomorphism $\Omega_*^{\mathrm{un}} \cong \pi_* MO := \varinjlim_{k\to\infty} \pi_{n+k} MO(k)$.*

In this lecture, we'll construct a map, $\Theta : \Omega_n^{\mathrm{un}} \to \pi_n MO$, using the the Pontryagin-Thom construction.

Given a class $[M^n] \in \Omega_n^{\mathrm{un}}$ we can choose a representative M and a closed embedding ν of M into a Euclidean space $\mathbb{R}^{n+k}$ of sufficiently large dimension. By the tubular neighborhood theorem, [2], we can factor ν as the embedding of the zero section into the normal bundle N_ν, followed by an open embedding of N_ν into $\mathbb{R}^{n+k}$.

$$\begin{array}{ccc} M & \xhookrightarrow{\ \nu\ } & \mathbb{R}^{n+k} \\ \downarrow & \nearrow & \\ N_\nu & & \end{array}$$

We will use the Pontryagin-Thom construction to produce a homotopy class, first in the Thom space of N_ν and later in MO. We compose the embedding $i : N_\nu \to \mathbb{R}^{n+k}$ with the embedding of $\mathbb{R}^{n+k}$ into its one-point compactification S^{n+k}. Recalling that the Thom space $\mathrm{Th}(N_\nu)$ consists of the vectors in N_ν of length less than or equal to 1 modulo the vectors of length 1 (that is, the disk bundle of N_ν modulo the sphere bundle of N_ν), we can then map S^{n+k} to $\mathrm{Th}(N_\nu)$ by sending points in the image of the disk bundle of N_ν under i to their preimages in the disk bundle, and other points in S^{n+k} to the point made by collapsing the sphere bundle.

$$\begin{array}{ccc} N_\nu & \xhookrightarrow{\quad i \quad} & \mathbb{R}^{n+k} \\ & & \downarrow \\ \mathrm{Th}(N_\nu) \simeq \mathrm{Disk}(N_\nu)/\mathrm{Sph}(N_\nu) & \xleftarrow[\text{collapse}]{t} & S^{n+k} \simeq (\mathbb{R}^{n+k})^+ \end{array}$$

Using the point at infinity for the basepoint of S^{n+k} and the point made by collapsing the sphere bundle as the basepoint of $\mathrm{Th}(N_\nu)$, this is a pointed map and so defines an element of $\pi_{n+k}\mathrm{Th}(N_\nu)$.

N_ν is vector bundle of rank k and so is classified by a map
$$\begin{array}{ccc} M & \xrightarrow{\ \nu\ } & BO(k) \simeq \mathrm{Gr}_k(\mathbb{R}^\infty) \\ \uparrow & & \uparrow \\ N_\nu \simeq \nu^*\gamma^k & \longrightarrow & \gamma^k \end{array} .$$
Thus we have a pointed map $\mathrm{Th}(N_\nu) = \mathrm{Th}(\nu^*\gamma^k) \to \mathrm{Th}(\gamma^k)$. We can compose this with the collapse map to get

$$S^{n+k} \to \mathrm{Th}(N_\nu) \to \mathrm{Th}(\gamma^k) = MO(k)$$

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This pointed map defines an element of $\pi_{n+k}MO(k)$. There's also a map $\pi_{n+k}MO(k) \to \operatorname{colim}_k \pi_{n+k}MO(k) = \pi_n MO$, so by choosing an embedding of M we got a homotopy class of MO. We might worry that this depends on the representative M of $[M]$ or the embedding ν of M into Euclidean space. We will see that it does not.

First we check that the homotopy class does not depend on the choice of embedding, once we are given the manifold M. Given two embeddings ν and ν' into $\mathbb{R}^{n+k}$ and $\mathbb{R}^{n+k'}$ respectively, we can choose a large enough k'' so that we may produce a diagram as below that commutes, since the space of embeddings of M into $\mathbb{R}^{n+k''}$ is connected (in fact, highly connected) for k'' sufficiently large.

$$\begin{array}{ccc} M & \xhookrightarrow{\nu} & \mathbb{R}^{n+k} \\ {\scriptstyle \nu'}\downarrow & & \downarrow \\ \mathbb{R}^{n+k'} & \hookrightarrow & \mathbb{R}^{n+k''} \end{array}$$

This gives us elements of $\pi_{n+k}MO(k)$, $\pi_{n+k'}MO(k')$, and $\pi_{n+k''}MO(k'')$. Since the diagram commutes, these classes all map to the same element of $\pi + n + k''MO(k'')$, and hence define the same element of $\pi_n MO$.

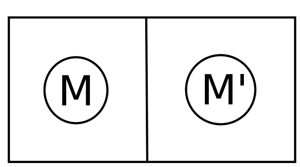
We now verify that the homotopy class does not depend on the choice of cobordism representative M of $[M]$. So let W be a cobordism between M and M'. Choose k and an embedding $W \to \mathbb{R}^{n+k} \times [0,1]$ so that we have the diagram below.

$$\begin{array}{ccc} M & \xhookrightarrow{\nu} & \mathbb{R}^{n+k} \times \{0\} \\ \downarrow & & \downarrow \\ W & \xhookrightarrow{\tilde{\nu}} & \mathbb{R}^{n+k} \times [0,1] \\ \uparrow & & \uparrow \\ M' & \xhookrightarrow{\nu'} & \mathbb{R}^{n+k} \times \{1\} \end{array}$$

By performing the Pontryagin-Thom construction on W fiberwise along the interval $[0,1]$, we get a homotopy $S^{n+k} \times [0,1] \xrightarrow{\tilde{t}} \mathrm{Th}(W) \longrightarrow MO(k)$ so that the restrictions to $S^{n+k} \times \{0\}$ and $S^{n+k} \times \{1\}$ are the classes produced by the Pontryagin-Thom construction for M and M' respectively. As a consequence, we obtain that Θ is indeed well-defined.

We next show that Θ is a homomorphism of groups. Recall that the group structure on Ω_n^{un} is given by disjoint union, the group structure on $\pi_n MO$ is given by the fold map, and that both groups are abelian.

Given elements $[M]$ and $[M']$ of Ω_n^{un} we choose k so that we have embeddings of M and M' into $\mathbb{R}^{n+k}$, and in such a way as to disjointly embed M into the left half space of $\mathbb{R}^{n+k}$ and M' into the right half space. There are now two ways to collapse the space.



By collapsing around the outside and performing the Pontryagin-Thom construction, we get a map $S^{n+k} \to MO(k)$ defined by the element $\Theta([M \amalg M'])$. However, if we collapse both along the outside as well along the middle hyperplane, we obtain a map $S^{n+k} \vee S^{n+k} \to MO(k)$ defined by wedge of the two maps $\Theta([M]) \vee \Theta([M'])$. Precomposing with the fold map fold : $S^{n+k} \to S^{n+k} \vee S^{n+k}$, we obtain again obtain the previous collapse map defined by disjoint union $M \amalg M'$:

That is, we have an equality $\Theta([M \amalg M']) = (\Theta([M]) \vee \Theta([M'])) \circ \mathrm{fold}$. Since disjoint union defines the abelian group structure on Ω_n^{un} and precomposing with the fold map defines the abelian group structure of the homotopy group, we thereby obtain the equality $\Theta([M]+[M']) = \Theta([M])+\Theta([M'])$.

In order to demonstrate that Θ is an isomorphism, we will construct an inverse map: Given an element $[f] \in \pi_{n+k}MO(k)$ defining an element of $\pi_n MO$, we will use certain general position arguments to produce an n-dimensional manifold.

First, observe that $\mathrm{Th}(\gamma^k)$ is the colimit of the Thom spaces of the universal bundles $\gamma_s^k \to \mathrm{Gr}_k(\mathbb{R}^s)$ as k tends to infinity: $\mathrm{colim}_s \, \mathrm{Th}(\gamma_s^k) \cong \mathrm{Th}(\gamma^k)$. Since S^{n+k} is compact, any map $f : S^{n+k} \to \mathrm{Th}(\gamma^k) = MO(k)$ must factor through some compact subspace of $\mathrm{Th}(\gamma^k)$, which is necessarily contained in some $\mathrm{Th}(\gamma_s^k)$. We will then define M^n to be the pullback in the diagram below, where i is chosen to be an inclusion transverse to f. We will discuss Thom's transversality theorem see next time, which will allow us to choose such an i.

$$\begin{array}{ccc} S^{n+k} & \xrightarrow{\ f\ } & \mathrm{Th}(\gamma_s^k) \\ \uparrow & & \uparrow{\scriptstyle i} \\ f^{-1}\mathrm{Gr}_k(\mathbb{R}^s) & \longrightarrow & \mathrm{Gr}_k(\mathbb{R}^s) \end{array}$$

Using this construction, we will then complete the proof of Thom's theorem.

References

[1] Thom, René. Quelques propriétés globales des variétés différentiables. Comment. Math. Helv. 28, (1954). 17–86.

[2] Milnor, John; Stasheff, James. Characteristic classes. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. vii+331 pp.

[3] Stong, Robert. Notes on cobordism theory. Mathematical notes. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968 v+354+lvi pp.