MATH 465, LECTURE 6: HANDLES

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Before beginning our study of the handlebody decompositions of manifolds, I want to give a few of the very striking consequences of Thom's calculation of $\pi_* MO$ sketched last time.

1. Thom's theorem, cont.

Recall that, in the last lecture, we sketched a proof of the following theorem:

Theorem 1.1 (Thom). *There is an equivalence $\Theta : \Omega_*^{\mathrm{un}} \to \pi_* MO \cong \mathbb{F}_2[x_i | i \neq 2^n - 1]$. That is, the unoriented cobordism ring is a polynomial algebra on generators of each positive degree not equal to $2^n - 1$, for any n. Furthermore, the spectrum MO is equivalent to a product of mod 2 Eilenberg-MacLane spectra.*

This immediately implies the following:

Corollary 1.2. *Two smooth manifolds M and N are unoriented cobordant if and only if their Stiefel-Whitney numbers agree, i.e., $\langle w(T_M), [M]\rangle = \langle w(T_N), [N]\rangle$ for every class $w \in H^*(BO, \mathbb{F}_2)$.*

Proof. The Hurewicz homomorphism is injective for any mod-2 Eilenberg-MacLane spectrum $H\mathbb{F}_2[n]$. Now recall that MO is a product of such spectra. The Hurewicz homomorphism $h : \pi_* MO \to H_*(MO; \mathbb{F}_2)$ is injective. Using the Thom isomorphism $H_*(MO, \mathbb{F}_2) \cong H_*(BO, \mathbb{F}_2)$, we have an injection $h \circ \Theta : \Omega_*^{\mathrm{un}} \to H_*(BO, \mathbb{F}_2)$. For a manifold M, the Stiefel-Whitney number of the stable normal bundle is $\langle w(N_M), [M]\rangle = \langle w, h\Theta[M]\rangle$. Since coefficients are in a field, n-dimension homology classes are distinguished by pairing them again n-dimensional cohomology classes, so equal Stiefel-Whitney numbers imply the equality of $h\Theta[M]$ and $h\Theta[N]$, hence of $[M]$ and $[N]$. □

We will next couple this finding with the following surprising result of Wu. Recall that the Wu class $v_k \in H^k(M, \mathbb{F}_2)$ is defined by the property that for any cohomology class $x_{n-k} \in H^{n-k}(M; \mathbb{F}_2)$, then there is an equality $v_k x_{n-k} = \mathrm{Sq}^k(x_{n-k})$. By Poincaré duality, such a v_k exists and is unique.

In the following theorem, M is a manifold, w is the total Stiefel-Whitney class of M, $w = \sum_i w_i(M)$, v is the total Wu class of M, and Sq is the total Steenrod operation $\mathrm{Sq} = \sum_j \mathrm{Sq}^j$.

Theorem 1.3 (Wu). *There is equality $w = \mathrm{Sq}(v)$. In particular, the Stiefel-Whitney classes of an n-dimensional manifold M are determined by the homotopy type of M.*

See [1] for a proof of Wu's theorem. Combining these results, we obtain:

Corollary 1.4. *If two smooth compact manifolds M and N are homotopy equivalent, then they are unoriented cobordant. I.e., there exists a smooth compact manifold with boundary W and a diffeomorphism $M \sqcup N \cong \partial W$.*

Proof. Since the Wu class v is determined by the homotopy type of M, as is the action of the Steenrod squares, thus the Stiefel-Whitney classes of M and N agree. Consequently, their Stiefel-Whitney numbers agree, and Thom's theorem implies that they are thereby cobordant.

□

This is a completely inobvious result, and a happy coincidence between classifying manifolds within a homotopy type, the eventual goal of surgery theory, and of classifying manifolds up to cobordism.

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2. Surgery

We now proceed to the main topic of this lecture: Surgery. In particular, we will begin our study of handlebody decompositions. Thom (and Dold) solved the classification of manifolds up to cobordism: There is a list of classes (given by combinations of real projective spaces and hypersurfaces in products of real projective spaces), and any manifold M is cobordant to one in this list. Exactly which one can be determined by the computation of all the Stiefel-Whitney numbers of M. Done. We could now ask for a complementary technique for building manifolds for which:

- Given any representative M^{n-1} of a cobordism class in Ω^{un}_{n-1}, this technique enables the construction of all the other manifolds cobordant to M;
- This technique serves as a analogue in the setting of manifolds of the theory of building homotopy types (of topological spaces) as CW-complexes.

Surgery is this technique. This begins with the following modest observation: $S^{q-1} \times D^{n-q}$ and $D^q \times S^{n-q-1}$ both have the same boundary, $\partial = S^{q-1} \times S^{n-q-1}$. Given an embedding of one of these manifolds into an $(n-1)$-manifold M, we could remove its interior and glue in the other one along the common boundary. These two manifolds will be cobordant, as is made apparent following construction.

Definition 2.1 (Adding a q-handle)**.** Let $(W, \partial W)$ be an n-manifold with boundary, and let $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial W$ be a smooth embedding. Attaching a handle along the map ϕ^q produces a new manifold, $W + \phi^q$, defined as the pushout:

$$\begin{array}{ccccc} S^{q-1} \times D^{n-q} & \hookrightarrow & \partial W & \hookrightarrow & W \\ \downarrow & & & & \downarrow \\ D^q \times D^{n-q} & & \longrightarrow & & W \cup_{S^{q-1} \times D^{n-q}} D^q \times D^{n-q} =: W + \phi^q \end{array}$$

Remark 2.2. Such an embedding ϕ^q can equivalently be thought of as just the embedding $\phi^q|_{S^{q-1}\times\{0\}} : S^{q-1} \times \{0\} \hookrightarrow \partial W$ together with a trivialization of the normal bundle of this embedding.

$W + \phi^q$ is clearly a topological manifold with boundary. It additionally has a smooth structure, which we will deal with at the end of this lecture. The boundary of this new n-manifold, $\partial(W + \phi)$ is given by the union of the respective boundaries of W and $D^q \times D^{n-q}$ over the their intersection:

$$\partial(W + \phi^q) = \partial W - \phi(S^q - 1 \times \mathring{D}^{n-q}) \cup_{S^{q-1} \times S^{n-q-1}} D^q \times S^{n-q-1}.$$

Here, $\mathring{D}$ is the open interior of the disk D.

Example 2.3. We construct a cobordism between the 2-sphere S^2 and X_g, the surface of genus g. We could, of course, exhibit these surfaces separately as boundaries of distinct 3-manifolds and then take their disjoint union; for the sake of illustrating our technique, we will construct a connected cobordism, the existence of which is perhaps less immediately clear.

Begin with $W = S^2 \times [0, 1]$, the boundary of which consists of has two disjoint 2-spheres $\partial_0 W = S^2 \times \{0\}$ and $\partial_1 W = S^2 \times \{0\}$. We will alter $\partial_1 W$ by adding a q-handle, for $q = 1$. Choose an embedding $\phi^1 : S^0 \times D^2 \hookrightarrow \partial_1 W$.

Then, by our construction above, the outgoing boundary component of our 3-manifold obtained by adding the handle along ϕ is $\partial_1(W + \phi^1) = S^2 - \phi(S^0 \times \mathring{D}^2) \cup_{S^0 \times S^1} D^1 \times S^1$. Unpacking this, we realize we removed the interiors of two disjoint disks in the image of ϕ^1 inside the sphere, and then attached on a cylinder (the handle) connecting the two boundary circles. In this way, $\partial(W + \phi^1) = S^2 \sqcup X_1$. We can now repeat this process to construct a cobordism between the sphere and a surface of any genus.

Moreover, this process is reversible by adding another handle, but of a different index. Let us start instead with $W = X_g \times [0, 1]$ and select an embedding $\phi^2 : S^1 \times D^1 \hookrightarrow \partial_1 W$. To add a handle along ϕ, first removes the embedded cylinder, then cap off the ends with two 2-disks. In the case of $g = 1$, we obtain the sphere from the one-holed torus; and in the case of $g = 0$, we obtain two disjoint spheres from $X_0 = S^2$.

We now address the existence of a smooth structure on $W + \phi^q$. Intuitively, we can imagine smoothing out the "kinks" along the boundary of the attachment of $D^q \times D^{n-q}$ along the image of ϕ in ∂W. We will us make this precise. First, recall that a choice of smooth structure on a manifold M is equivalent to the choice of subsheaf $\mathcal{O}_M^{\mathrm{sm}} \subset \mathcal{O}_M$ of "smooth functions," which must satisfy a local condition: Each each point x in M has a neighborhood U homeomorphic to $\mathbb{R}^n$, such that this homeomorphism defines an isomorphisms of rings between $\mathcal{O}^{\mathrm{sm}}(U)$ and the ring of smooth functions on $\mathbb{R}^n$. In other words, specifying a smooth structure on a topological manifold is equivalent to specifying which functions are smooth.

To our case, first choose a trivialization of the tubular neighborhood of the inclusion of $i : \partial W \hookrightarrow W$.

$$\begin{array}{ccc} \partial W & \overset{i}{\hookrightarrow} & W \\ & {\scriptstyle \nu}\searrow \quad \nearrow & \\ & N_i \cong \partial W \times \mathbb{R}_{\geq 0} & \end{array}$$

The space of such choices of a trivialization is contractible. Similarly, choose a trivialization

$$\begin{array}{ccc} S^{q-1} \times D^{n-q} & \overset{j}{\hookrightarrow} & D^q \times D^{n-q} \\ & {\scriptstyle \tilde{\nu}}\searrow \quad \nearrow & \\ & S^{q-1} \times \mathbb{R}_{\leq 0} & \end{array}$$

of the tubular neighborhood of $S^{q-1} \times D^{n-q}$ in the n-disk. Now consider the following *smooth* embeddings,

$$S^{q-1} \times D^{n-q} \times \mathbb{R}_{\geq 0} \subset S^{q-1} \times D^{n-q} \times \mathbb{R} \supset S^{q-1} \times D^{n-q} \times \mathbb{R}_{\leq 0}$$

which defines an open embedding of $\subset S^{q-1} \times D^{n-q} \times \mathbb{R}$ into $W + \phi$. To specify the smooth structure of $W + \phi$, we can essentially declare to such that this embedding is smooth. More precisly, define function $f \in \mathcal{O}_{W+\phi^q}$ to be smooth if and only if its restrictions $f|_W$, $f|_{D^q \times D^{n-q}}$ and $f|_{S^{q-1} \times D^{n-q} \times \mathbb{R}}$ are smooth. This defines a subsheaf $\mathcal{O}_{W+\phi}^{\mathrm{sm}} \subset \mathcal{O}_{W+\phi}$ and, thus, a smooth structure on $W + \phi$.

References

[1] Milnor, John; Stasheff, James. Characteristic classes. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. vii+331 pp.

[2] Lück, Wolfgang. A basic introduction to surgery. Available from `http://www.math.uni-muenster.de/u/lueck/`.