MATH 465, LECTURE 6: ORDERING HANDLES

J. FRANCIS, NOTES BY A. BEAUDRY

Our aim in the next two lectures is to connect the theory of handlebody decompositions with that of CW-complexes, and to present several basic features of handle attachments in parallel with the corresponding facts of cell attachments.

Recall the following definition of attaching a handle of index q.

Definition 0.1. Let W be an n-manifold with boundary partitioned into two disjoint components, $\partial W \cong \partial_0 W \sqcup \partial_1 W$. Given a smooth embedding $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W$, define the attachment of a q-handle along ϕ as the union

$$W + \phi = W \cup_{S^{q-1}\times D^{n-q}} D^q \times D^{n-q}.$$

The outgoing boundary component of $W + \phi$ is given by

$$\partial_1(W+\phi) = \partial_1 W - \overset{\circ}{\mathrm{Im}}\,(\phi^q) \cup_{S^{q-1}\times S^{n-q-1}} D^q \times S^{n-q-1}.$$

The attachment of a q-handle is, up to homotopy, equivalent to the attachment of a q-cell, since the inclusion and collapse maps

$$W \cup_{S^{q-1}\times\{0\}} D^q \times \{0\} \longrightarrow W \cup_{S^{q-1}\times D^{n-q}} D^q \times D^{n-q} \longrightarrow W \cup_{S^{q-1}} D^q$$

are homotopy equivalences.

Recall that a CW-pair (X, A) consists of a space A and a space X obtained by sequentially adding cells to A. That is, we have a sequence of spaces $A \to X_0 \to X_1 \to \ldots$, where $X = \varinjlim X_q$, and the $(q+1)$th space is obtained from the qth space by attaching $(q+1)$-disks along attaching maps ϕ_i^q

$$\begin{tikzcd} \coprod_i S_i^{q-1} \arrow[r, "\amalg_i \phi_i^q"] \arrow[d] & X_{q-1} \arrow[d] \\ \coprod_i D_i^q \arrow[r] & X_q \end{tikzcd}$$

in such a way that the above diagram is a pushout.

A priori, you might think that a different notion might emerge from allowing the cells to be added in arbitrary order, rather than by increasing order of dimension. But no, you can always add the cells in increasing order of dimension. That is, suppose $q \le r$ and that X is obtained from A by adding two cells in the "wrong" order: first adding

$$\begin{tikzcd} S^{r-1} \arrow[r, "\psi^r"] \arrow[d] & A \arrow[d] \\ D^r \arrow[r] & A \cup_{S^{r-1}} D^r \end{tikzcd}$$

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and then adding

$$\begin{tikzcd} S^{q-1} \arrow[r, "\psi^q"] \arrow[d] & A \cup_{S^{r-1}} D^r \arrow[d] \\ D^q \arrow[r] & (A \cup_{S^{r-1}} D^r) \cup_{S^{q-1}} D^q \end{tikzcd}$$

The map $A \to A \cup_{S^{r-1}} D^r$ is $(r-1)$-connected. Consequently, you can, up to homotopy, factor ψ^q through a map $\tilde{\psi}^q : S^{q-1} \to A$. We can then add the q-cell to A first, along the attaching map $\tilde{\psi}^q$, and then r-cell afterward, again along the map ψ^r. Because the homotopy type of the cell attachment only depends on the homotopy class of the attaching maps, we obtain a homotopy equivalence

$$(A \cup_{S^{r-1}} D^r) \cup_{S^{q-1}} D^q \simeq (A \cup_{S^{q-1}} D^q) \cup_{S^{r-1}} D^r.$$

Thus, up to homotopy, we are always able to arrange the handles in order. Fundamentally, this is a consequence of the fact that the homotopy groups $\pi_m(S^n)$ vanish for $m < n$. This fact has a differential topology proof, involving transversality: Replacing $\psi : S^m \to S^n$ by a smooth map. By Sard's theorem, this map has a regular value. Since the dimension of S^m is less than that of S^n, a value is regular if and only if it is not in the image of ψ. Thus, the map factors through the complement of a point. Since $S^n - \mathrm{pt}$ is contractible, the map is nullhomotopic and the result follows.

Our goal is to now to formulate a corresponding result for handlebodies. Since the fundamental fact that allows this cell rearrangement, that $\pi_m S^n = 0$ for $m < n$, is true by differential topology and transversality, we might optimistically expect it to hold for handlebodies and also be provable by transversality.

First, let us see that $(W, \partial_0 W)$ has the homotopy type of a CW-pair, where W is handlebody, i.e., W obtained by successive addition of handles to a trivial cobordism: $W \cong \partial_0 W \times [0,1] + \phi_1^{q_1} + \ldots + \phi_m^{q_m}$. This true is because each handle ϕ^q is attached through a fattened up sphere. So by getting rid of the fat, you get space with added cells, rather than handles. Now, by the argument above we can rearrange the cells in order of dimension, and obtain a CW-pair homotopy equivalent to $(W, \partial_0 W)$.

Lemma 0.2. *Let W be a handlebody of dimension n. Then W is diffeomorphic relative $\partial_0 W$ to a decomposition $\partial_0 W \times [0,1] + \sum_{I_0} \phi_i^1 + \sum_{I_1} \phi_i^1 + \ldots + \sum_{I_n} \phi_i^n$, where I_q is the indexing set for handles of index equal q.*

In order to prove this result, we require an analogue of the following fact for CW-complexes: The homotopy type of a cell attachment depends only on the homotopy type of the attaching map. The notion corresponding to homotopy of attaching maps in our differential topology setting is an that of an *isotopy*. Recall:

Definition 0.3. Let f and g be two smooth embeddings of M into N. An isotopy from f to g is a smooth map $F : [0,1] \times M \to N$ which is a homotopy from f to g, and such that each F_t is an embedding. F is a diffeotopy if, additionally, each F_t is a diffeomorphism of $M \times \{t\}$ to N.

The aptness of this analogy is demonstrated by the following lemma.

Lemma 0.4. *Let $\Phi : [0,1] \times S^{q-1} \times D^{n-q} \to \partial_1 W$ be an isotopy from ϕ_0^q to ϕ_1^q. Then $W + \phi_0^q$ is diffeomorphic to $W + \phi_1^q$.*

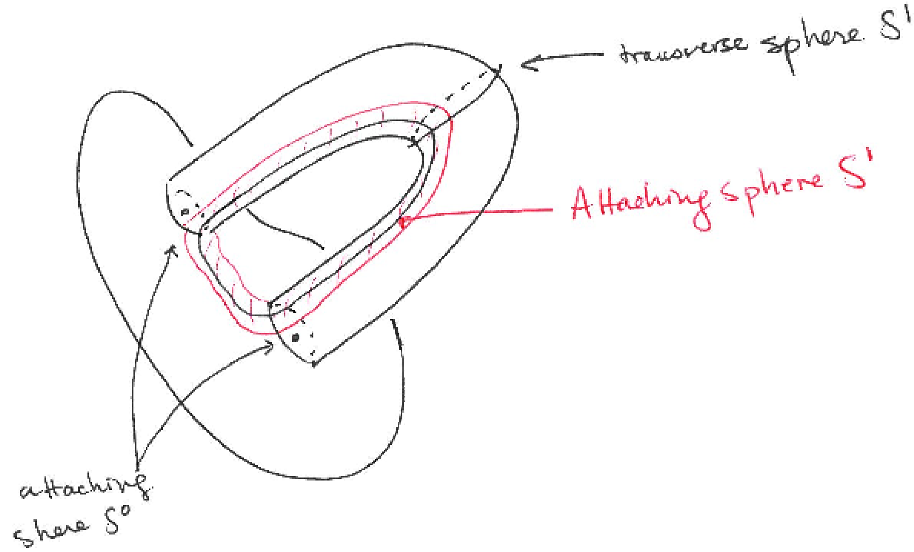
Proof. By the isotopy extension theorem, [1], we may select a diffeotopy $\tilde{\Phi} : [0,1] \times W \to W$ extending the isotopy Φ over W (i.e., $\tilde{\Phi}_t \circ \phi_0^q = \Phi_t$), and such that $\tilde{\Phi}$ is constant outside of a closed neighborhood of the image of Φ. The diffeomorphism $\tilde{\Phi}_1$ extends over the identity map on the handles ϕ_0 and ϕ_1, thereby producing a diffeomorphism $W + \phi_0 \cong W + \phi_1$.

□

Thus, we can move our handles. Before proceeding with our proof of our main lemma, it helpful to introduce a little terminology, to pin down two key geometric features of our situation.

Definition 0.5. Let $\phi^q : S^{q-1} \times D^{n-q} \to \partial W$ be an embedding along which a q-handle is attached. The attaching sphere of the handle is $\phi(S^{q-1} \times \{0\})$. The transverse sphere of the handle is the image of $\{0\} \times S^{n-q-1}$ in the handle $\partial(W + \phi) = \partial W - \phi(S^{q-1} \times D^{n-q} \cup_{S^{q-1} \times S^{n-q-1}} D^q \times S^{n-q-1}$.

When attaching multiple handles in succession, the topology of the resulting manifold will be controlled by the intersections of first transverse sphere with the second attaching sphere. We illustrate this with the following picture, in which we first attach a 1-handle along an attaching 0-sphere, to increase the genus of a surface by 1, and then we attach a 2-handle along an attaching 1-sphere to fill in the hole, the result of which is diffeomorphic to the original surface:



In this picture, the transverse sphere of the first handle intersects the attaching sphere of the second handle transversely in a single point. Observe that there would be no way to fill in the hole and "cancel" the 1-handle without the attaching 1-sphere intersecting the transverse sphere in some way.

Lemma 0.2 follows from repeated application of the following maneuver:

Lemma 0.6. *Given W with handles ϕ^r and ϕ^q,*

$$\phi^r : S^{r-1} \times D^{n-r} \to \partial_1 W$$

and

$$\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 (W + \phi^r)$$

such that $q \leq r$, there exists $\tilde{\phi}^q$ isotopic to ϕ^q missing the handle attached by ϕ^r. Hence we have a diffeomorphism $W + \phi^r + \phi^q \cong W + \tilde{\phi}^q + \phi^r$.

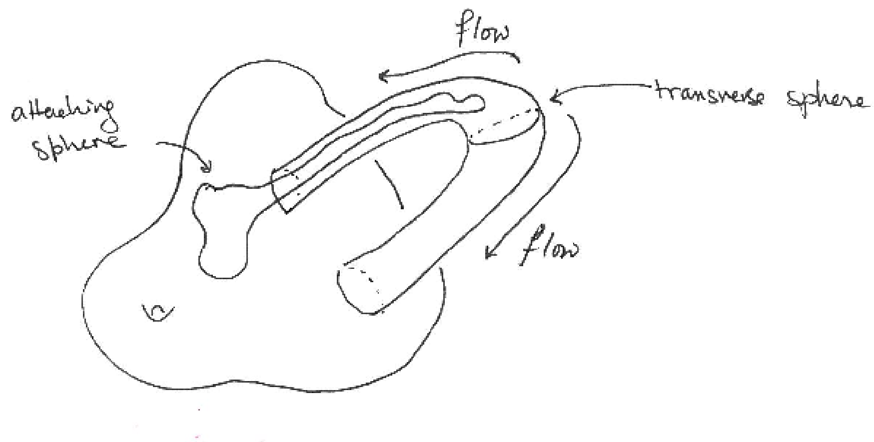
Proof. Following the intuition of our previous picture, let us analyze the intersection of the transverse sphere of ϕ^r and the attaching sphere of ϕ^q.

The transverse sphere of $W + \phi^r$ is $\{0\} \times S^{n-r-1} \in \partial(W + \phi^r)$ and the attaching sphere is $S^{q-1} \times \{0\}$. By the transversality theorem , we may move the handle ϕ^q through an isotopy so that the intersection of the attaching $(q-1)$-sphere and transverse $(n-r-1)$-sphere are transversal. Since $q \leq r$ the dimension of the intersection is less than

$$(n - r - 1) + (q - 1) \leq (n - r - 1) + (r - 1) < n - 1,$$

and the interesection is thus empty.

We complete the proof with a picture.



We can flow the attaching sphere down the handle so that ϕ^q has image disjoint with the handle attached by ϕ^r. Since isopotic handles give diffeomorphic manifolds, we get

$$W + \phi^r + \phi^q \cong W + \tilde{\phi}^q + \phi^r$$

$\square$

This completes our discussion of hangle ordering. Our next topic is handle cancelation. For continuity, the notes from the end of this lecture are appended at the beginning of the notes for lecture 8.

References

[1] Hirsch, Morris. Differential topology. Corrected reprint of the 1976 original. Graduate Texts in Mathematics, 33. Springer-Verlag, New York, 1994. x+222 pp.

[2] Lück, Wolfgang. A basic introduction to surgery. Available from `http://www.math.uni-muenster.de/u/lueck/`.