MATH 465, LECTURE 8: HANDLE CANCELLATION

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We continue utilizing the analogy between handlebodies and CW-complexes. In the previous lecture, we showed that a handlebody can always be obtained by the addition of handles in increasing order of index, just as the cells of cellular space can be attached in increasing order of dimension. We now turn to handle cancellation, the analogue of the following construction:

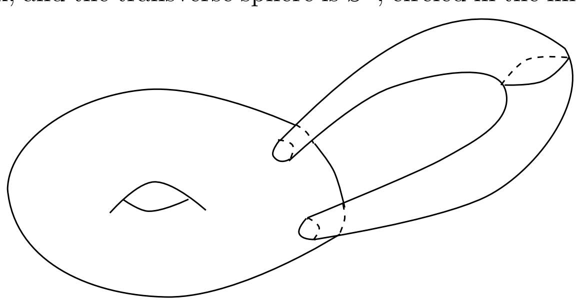
Let X be space, and attach a q-cell to X via a map $f : S^{q-1} \to X$. If the map f is nullhomotopic, we may add a $(q+1)$-cell to the space $X \cup_f D^q$ to obtain a space homotopic to X. For instance, we have a homotopy equivalence $X \cup_f D^q \simeq X \vee S^q$. Attaching a $(q+1)$-cell via the natural inclusion $S^q \to X \vee S^q$ we obtain the space $X \vee D^n \simeq X$. We want to do something similar for handles, so as to be able to cancel a trivially attached q-handle by adding a $(q+1)$-handle. This requires first formulating a notion of triviality:

Definition 0.1. We say that a handle φ^q is trivial if there exists a smooth factorization of the attaching map $\varphi^q : S^{q-1} \times D^{n-q} \to D^{n-1} \to \partial_1 W$ through a standard embedding of $S^{q-1} \times D^{n-q} \to D^{n-1}$, where all of these maps are smooth embeddings.

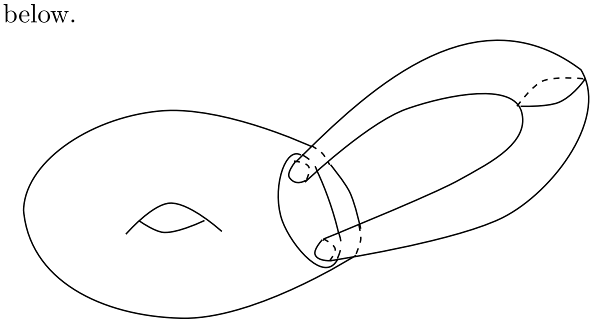
We may now aim to establish the following:

Lemma 0.2. *If the handle φ^q is trivial, there exists a handle φ^{q+1} so that W is diffeomorphic to $W + \varphi^q + \varphi^{q+1}$ relative $\partial_0 W$.*

Before giving the formal proof, we first do a graphical walk-through in dimension 3. The handle φ^q is attached to W along a copy of $S^{q-1} \times D^{n-q}$. Its attaching sphere is a copy of S^{q-1} and its transverse sphere a copy of S^{n-q-1}. In the following picture, the attaching sphere is S^0, the centers of the two embedded disk, and the transverse sphere is S^1, circled in the middle of the added handle.



By the assumption of the triviality of ϕ^1, we have a factorization of ϕ^1 through a standard embedding $S^0 \times D^2 \to D^2$, drawn below.



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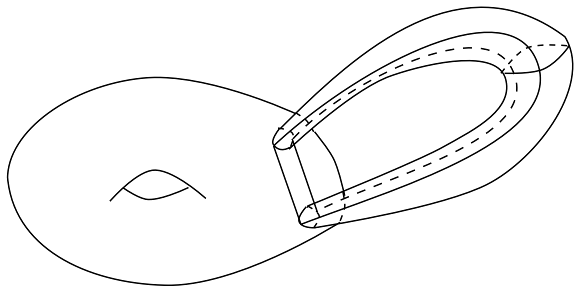
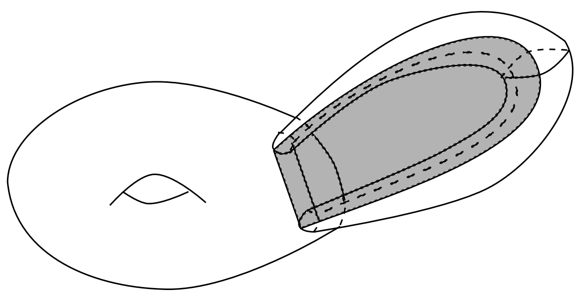
Then we can attach φ^2 by a copy of $S^1 \times D^1$ which runs half along the handle formed by φ^1 and half through the copy of D^2 we mapped into $\partial_1 W$ above.



Now the two handles together have formed a copy of D^n, so $W \cong W + \varphi^q + \varphi^{q+1}$.



Proof. We write $S^q = D^q_+ \bigcup_{S^{q-1}} D^q_-$. Now we form the embedding $S^q \times D^{n-q-1} \to W + \varphi^q$ out of pieces $\varphi^{q+1}_+ : D^q_+ \times D^{n-q-1} \to \partial_1 W - (S^{q-1} \times \mathring{D}^{n-q})$ and $\varphi^{q+1}_- : D^q_- \times D^{n-q-1} \to D^q \times S^{n-q-1}$ which agree on the map $S^{q-1} \times D^{n-q-1} \to S^{q-1} \times S^{n-q-1}$ from the equator into the common boundary of W and the handle.

$$\begin{tikzcd}
D^q \times D^{n-q-1} \arrow[rr] \arrow[dr] & & \partial_1 W - (S^{q-1} \times \mathring{D}^{n-q}) \bigcup_{S^{q-1} \times S^{n-q-1}} D^q \times S^{n-q-1} \\
S^{q-1} \times D^{n-q-1} \arrow[u] \arrow[d] \arrow[r, "\text{equator}"'] & S^q \times D^{n-q-1} \arrow[ur, dashed] & \\
D^q \times D^{n-q-1} \arrow[ur] \arrow[uurr, bend right] & &
\end{tikzcd}$$

We define these maps as follows:

$$\varphi^{q+1} |_{\text{equator}} : S^{q-1} \times D^{n-q-1} \to S^{q-1} \times S^{n-q-1}$$

is the product of identity map on S^{q-1} and the inclusion of D^{n-q-1} into S^{n-q-1} as the lower hemisphere.

$$\varphi^{q+1}_- : D^q_- \times D^{n-q-1} \to D^q \times S^{n-q-1}$$

is the product of the identity on D^q and the inclusion of D^{n-q-1} into S^{n-q-1} as the lower hemisphere.

$$\varphi^{q+1}_+ : D^q_+ \times D^{n-q-1} \to \partial_1 W - (S^{q-1} \times \mathring{D}^{n-q})$$

extends the map $\varphi^{q+1} |_{\text{equator}}$, using the fact that $\varphi^{q+1} |_{\text{equator}}$ is contractible:

$$\begin{tikzcd}
S^{q-1} \times D^{n-q-1} \arrow[r] \arrow[d] & S^{q-1} \times S^{n-q-1} \arrow[r] & S^{q-1} \times D^{n-q} \arrow[d] \\
D^q \times D^{n-q-1} \arrow[rr] & & D^{n-1} \arrow[d] \\
& & W
\end{tikzcd}$$

Since all these maps agree on $S^{q-1} \times D^{n-q-1}$, we get the required map $S^q \times D^{n-q-1} \to \partial_1 W + \varphi^q$ and can attach the handle. You can check that the union of the two handles is a disk, so by contracting it there is a diffeomorphism $W \cong W + \varphi^q + \varphi^{q+1}$. $\square$

Notice that the attaching sphere of φ^{q+1} and the transverse sphere of φ^q intersect transversely in a point. We claim that is the only important geometric feature of this situation.

Exercise 0.3. Let $W_q = W + \varphi^q$ and $W_{q+1} = W + \varphi^q + \varphi^{q+1}$. Given that the attaching sphere of φ^{q+1} intersects the transverse sphere of φ^q transversely and in only one point, show that the cellular differential

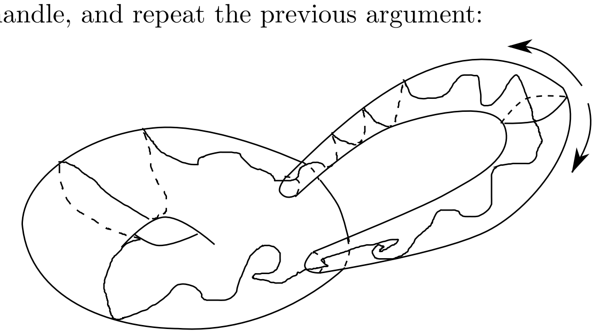
$$\mathbb{Z} = H_{q+1}(W_{q+1}, W_q) \to H_q(W_q) \to H_q(W_q, W) = \mathbb{Z}$$

is an isomorphism and so $H_*(W_{q+1}, W) = 0$ and W is homotopic to W_{q+1}.

This is a consequence of:

Lemma 0.4. *Let W be an n-manifold with $\partial W = \partial_0 W \coprod \partial_1 W$ and given handles $\varphi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ and $\varphi^{q+1} : S^q \times D^{n-q-1} \to \partial_1 W + \varphi^q$. If the attaching sphere of φ^{q+1} (that is, $\varphi^{q+1}(S^q \times 0)$) intersects the transverse sphere of φ^q transversely and in only one point, then there is a diffeomorphism $W \cong W + \varphi^q + \varphi^{q+1}$ fixing $\partial_0 W$.*

In a neighborhood of the transverse sphere things are very simple. We will flow by isotopy the complications off the handle, and repeat the previous argument:



Our next goal is to find a homological condition for when we can cancel handles.

References

[1] Lück, Wolfgang. A basic introduction to surgery. Available from `http://www.math.uni-muenster.de/u/lueck/`.