

APPLICATIONS

1.1. The eigenvalues are $\lambda = 5, -5$. An orthonormal basis of eigenvectors consists of

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

1.2. The eigenvalues are $\lambda = 5, -5$. A basis of eigenvectors consists of

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which are not perpendicular. However, the matrix is not symmetric, so there is no special reason to expect that the eigenvectors will be perpendicular.

1.3. The eigenvalues are $0, 1, 2$. An orthonormal basis is

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

1.4. The columns of the matrix

$$\begin{bmatrix} -\frac{5}{\sqrt{2}} & 0 & \frac{5}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} & 3 & \frac{4}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & -4 & \frac{3}{\sqrt{2}} \end{bmatrix}$$

form an orthonormal basis of eigenvectors corresponding to the eigenvalues $-4, 1, 6$.

1.5. $(P^t AP)^t = P^t A^t (P^t)^t = P^t AP$.

2.1. (a)

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(b)

$$\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{70}} \begin{bmatrix} 4 \\ 5 \\ -2 \\ 5 \end{bmatrix}, \frac{1}{\sqrt{994}} \begin{bmatrix} -8 \\ 25 \\ 4 \\ -17 \end{bmatrix} \right\}$$

2.2. The eigenvalues are 0 with multiplicity 2 and 3 with multiplicity 1. A basis for the eigenspace corresponding to the eigenvalue 0 is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Applying Gram Schmidt to this yields

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

an eigenvector of length 1 for the eigenvalue 3 is

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

An orthonormal basis of eigenvectors is

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

2.3. The characteristic polynomial is $-(\lambda+1)^3 - 12(\lambda+1) - 16$. Put $X = \lambda+1$. Then the equation becomes $X^3 - 12X - 16 = 0$, and this factors as $(X-4)(X+2)^2 = 0$, so the roots are $X = 4, -2, -2$. That means the eigenvalues are $\lambda = 3$ with multiplicity 1 and $\lambda = -3$ with multiplicity 2. For $\lambda = 4$, a normalized eigenvector is

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -3$,

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of eigenvectors. Applying Gram-Schmidt yields

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

However, it would also make sense to reverse the order and apply Gram-Schmidt to obtain

$$\mathbf{u}'_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}'_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

2.4. (a) $\mathbf{v}'_1 = \mathbf{v}_1$, $\mathbf{v}'_2 = \mathbf{v}_2 - c\mathbf{v}'_1 = \mathbf{v}_2 - c\mathbf{v}_1$ for an appropriate scalar c , so it is clear that \mathbf{v}'_2 can't be zero, unless \mathbf{v}_2 is a multiple of \mathbf{v}_1 . (b) $\mathbf{v}'_3 = \mathbf{v}_3 - y\mathbf{v}'_2 - z\mathbf{v}'_1$ for appropriate scalars y, z . If it were zero, that would say that \mathbf{v}_3 is a linear combination of \mathbf{v}'_1 and \mathbf{v}'_2 , which we know are linear combinations of \mathbf{v}_1 and \mathbf{v}_2 . Hence, \mathbf{v}_2 would be a linear combination of \mathbf{v}_1 and \mathbf{v}_2 , which by assumption is not the case.

(The same argument works in general for any number of vectors, as long as they form a linearly independent set. If any $\mathbf{v}'_i = \mathbf{0}$, then that would, after a lot of algebra, give a way to express \mathbf{v}_i in terms of $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$.)

3.1. (a) $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$. (b) $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$.

3.2. Replacing θ by $-\theta$ doesn't change the cosine entries and changes the signs of the sine entries.

3.3. The 'P' matrix is

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Its inverse is its transpose, so the components of $-g\mathbf{j}$ in the new coordinate system are given by

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} = -g \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}.$$

3.4. Such a matrix is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The diagonal entries are $-1, 3$.

3.5. Such a matrix is

$$\begin{bmatrix} -\frac{5}{5\sqrt{2}} & 0 & \frac{5}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} & \frac{-4}{5} & \frac{3}{5\sqrt{2}} \end{bmatrix}$$

The diagonal entries are $-4, 1, 6$.

3.6. Let A, B be orthogonal, i.e., they are invertible and $A^t = A^{-1}, B^t = B^{-1}$. Then AB is invertible and

$$(AB)^t = B^t A^t = B^{-1} A^{-1} = (AB)^{-1}.$$

The inverse of an orthogonal matrix is orthogonal. For, if A^t is the inverse of A , then A is the inverse of A^t . But $(A^t)^t = A$, so A^t has the property that its transpose is its inverse.

3.7. The rows are also mutually perpendicular unit vectors. The reason is that another way to characterize an orthogonal matrix P is to say that the P^t is the inverse of P , i.e., $P^t P = P P^t = I$. However, it is easy to see from this that P^t is also orthogonal. (Its transpose $(P^t)^t = P$ is also its inverse.) Hence, the columns of P^t are mutually perpendicular unit vectors. But these are the rows of P .

4.1. The principal axes are given by the basis vectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In the new coordinates system the equation is

$$(x')^2 + 3(y')^2 = 2$$

which is the equation of an ellipse.

4.2. The eigenvalues are $-1, 3$. The conic is a hyperbola. The principal axes may be specified by the unit vectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The equation in the new coordinates is $-(x')^2 + 3(y')^2 = 4$. The points closest to the origin are at $x' = 0, y' = \pm \frac{2}{\sqrt{3}}$. In the original coordinates, these points are $\pm(\sqrt{2/3}, \sqrt{2/3})$.

4.3. The principal axes are those along the unit vectors

$$\mathbf{u}_1 = \frac{1}{5} \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

The equation in the new coordinate system is

$$-25(x')^2 + 50(y')^2 = 50.$$

The curve is a hyperbola. The points closest to the origin are given by $x' = 0, y' = \pm 1$. In the original coordinates these are the points $\pm \frac{1}{5}(4, 3)$. There is no upper bound on the distance of points to the origin for a hyperbola.

4.4. The principal axes are along the unit vectors given by

$$\mathbf{u}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

The equation in the new coordinates system is

$$-3(x')^2 + (y')^2 + 3(z')^2 = 1.$$

This is an elliptic hyperboloid of one sheet centered on the x' -axis.

5.1. Compare this with Exercise 1 in Section 7.

The equations are

$$\begin{aligned}2x &= \lambda(2x + y) \\2y &= \lambda(x + 2y) \\x^2 + xy + y^2 &= 1\end{aligned}$$

This in effect says that (x, y) give the components of an eigenvector for the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

with $\frac{2}{\lambda}$ as eigenvalue. However, since we already classified the conic in the aforementioned exercise, it is easier to use that information here. In the new coordinates the equation is $(x')^2 + 3(y')^2 = 2$. The maximum distance to the origin is at $x' = \pm\sqrt{2}, y' = 0$ and the minimum distance to the origin is at $x' = 0, y' = \pm\sqrt{2/3}$. Since the change of coordinates is orthogonal, we may still measure distance by $\sqrt{(x')^2 + (y')^2}$. Hence, the maximum square distance to the origin is 2 and the minimum square distance to the origin is $2/3$.

Note that the problem did not ask for the locations in the original coordinates where the maximum and minimum are attained.

5.2. (Look in the previous section for an analogous problem.) The equations are

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad x^2 + y^2 + z^2 = 1.$$

Thus we need to find eigenvectors of length 1 for the given matrix. However, we already did this in the aforementioned exercise. The answers are

$$\pm \frac{1}{\sqrt{6}}(1, 2, 1), \quad \pm \frac{1}{\sqrt{2}}(-1, 0, 1), \quad \pm \frac{1}{\sqrt{3}}(1, -1, 1).$$

The values of the function at these three points are respectively $-3, 1, 3$. Since a continuous function on a closed bounded set must attain both a maximum and minimum values, the maximum is 3 at the third point and the minimum is -3 at the first point.

5.3. The Lagrange multiplier condition yields the equations

$$\begin{aligned}x &= \lambda x \\y &= \lambda y \\z &= \lambda z \\x^2 + y^2 &= z^2.\end{aligned}$$

If $\lambda \neq 1$, then the first two equations show that $x = y = 0$. From this, the last equation shows that $z = 0$. Hence, $(0, 0, 0)$ is one possible maximum point. If

$\lambda = 1$, then the third equation shows that $z = 0$, and then the last equation shows that $x = y = 0$. Hence, that gives the same point. Finally, we have to consider all points where $\nabla g = \langle 2x, 2y, -2z \rangle = \mathbf{0}$. Again, $(0, 0, 0)$ is the only such point. Since $x^2 + y^2 + z^2 \geq 0$ and it does attain the value 0 on the given surface, it follows that 0 is its minimum value.

Note that in this example $\nabla g = \mathbf{0}$ didn't give us any other candidates to examine, but in general that might not have been the case.

5.4. On the circle $x^2 + y^2 = 1$, $f(x, y) = 1 + 4xy$, so maximizing $f(x, y)$ is the same as maximizing xy . The level sets $xy = c$ are hyperbolas. Some of these intersect the circle $x^2 + y^2 = 1$ and some don't intersect. The boundary between the two classes are the level curves $xy = 1$ and $xy = -1$. The first is tangent to the circle in the first and third quadrants and the second is tangent in the second and fourth quadrants. As you move on the circle toward one of these points of tangency, you cross level curves with either successively higher values of c or successively lower values of c . Hence, the points of tangency are either maximum or minimum points for xy . The maximum points occur in the first and third quadrants with xy attaining the value 1 at those points.

Note also that the points of tangency are exactly where the normal to the circle and the normal to the level curve are parallel.

6.1. The characteristic polynomial of

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

is

$$\begin{aligned} \det \begin{bmatrix} -1 - \lambda & 1 & 0 \\ 1 & -2 - \lambda & 1 \\ 0 & 1 & -1 - \lambda \end{bmatrix} &= (-1 - \lambda)((-2 - \lambda)(-1 - \lambda) - 1) - 1(-1 - \lambda) \\ &= -(\lambda + 1)(\lambda^2 + 3\lambda) = -(\lambda + 1)(\lambda + 3)\lambda. \end{aligned}$$

Hence, the eigenvalues are $-\omega^2(m/k) = \lambda = -1, -3$, and 0. As indicated in the problem statement, the eigenvalue 0 corresponds to a non-oscillatory solution in which the system moves freely at constant velocity.

For $\lambda = -1$, we have $\omega = \sqrt{k/m}$ and

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving with equal displacements in opposite directions and the middle particle staying still.

For $\lambda = -3$, we have $\omega = \sqrt{3k/m}$ and

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving together with equal displacements in the same direction and the middle particle moving with twice that displacement in the opposite direction.

6.2. The characteristic polynomial of

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

is

$$\begin{aligned} \det \begin{bmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{bmatrix} &= (-2-\lambda)((-2-\lambda)(-2-\lambda) - 1) - 1(-2-\lambda) \\ &= -(\lambda+2)(\lambda^2 + 4\lambda + 2). \end{aligned}$$

Hence, the eigenvalues are $-\omega^2(m/k) = \lambda = -2, -2 + \sqrt{2}$, and $= -2 + \sqrt{2}$.

For $\lambda = -2$, we have $\omega = \sqrt{2k/m}$ and

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving with equal displacements in opposite directions and the middle particle staying still.

For $\lambda = -2 - \sqrt{2}$, we have $\omega = \sqrt{(2 + \sqrt{2})k/m}$ and

$$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving together with equal displacements in the same direction and the middle particle moving with $\sqrt{2}$ times that displacement in the opposite direction.

For $\lambda = -2 + \sqrt{2}$, we have $\omega = \sqrt{(2 - \sqrt{2})k/m}$ and

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \text{ yielding basic eigenvector } \mathbf{u} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

This corresponds to both end particles moving together with equal displacements in the same direction and the middle particle moving with $\sqrt{2}$ times that displacement in the same direction. Notice that the intuitive significance of this last normal mode is not so clear.

6.3. The given information about the first normal mode tells us that a corresponding basic eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Any basic eigenvector for the second normal mode

must be perpendicular to \mathbf{v}_1 , so we can take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Hence, the relation between the displacements for the second normal mode is $x_2 = -2x_1$.

- 6.4.** For one normal mode, $\omega = \sqrt{\frac{k}{m}}$, and the relative motions of the particles satisfy $x_2 = 2x_1$. For the other normal mode, $\omega = \sqrt{\frac{6k}{m}}$ and the relative motions of the particles satisfy $x_1 = 2x_2$.
- 6.5.** For one normal mode, $\lambda = -3 + \sqrt{5}$ and $\omega = \sqrt{(3 - \sqrt{5})\frac{k}{m}}$. The eigenspace is obtained by reducing the matrix

$$\begin{bmatrix} -1 - \sqrt{5} & 2 \\ 2 & 1 - \sqrt{5} \end{bmatrix}.$$

Note that this matrix *must be singular* so that the first row must be a multiple of the second row. (The multiple is in fact $\frac{-1 - \sqrt{5}}{2}$. Check it!) Hence, the reduced matrix is

$$\begin{bmatrix} 1 & (1 - \sqrt{5})/2 \\ 0 & 0 \end{bmatrix}.$$

The relative motions of the particles satisfy $x_1 = \frac{\sqrt{5} - 1}{2}x_2$. A similar analysis show that for the other normal mode, $\omega = \sqrt{(3 + \sqrt{5})\frac{k}{m}}$, and the relative motions of the particles satisfy $x_1 = -\frac{\sqrt{5} + 1}{2}x_2$.

- 6.6.** The information given tells us that two of the eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Any basic eigenvector for the third normal mode must be perpendicular to this. If its components are $\begin{bmatrix} v_1 \\ v_2 \\ v_2 \end{bmatrix}$, then we must have

$$v_1 + v_2 + v_3 = 0 \quad \text{and} \quad v_2 - 2v_2 + v_3 = 0.$$

By the usual method, we find that a basis for the null space of this system is given by

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

whence we conclude that the relative motions satisfy $x_1 = -x_3, x_2 = 0$.

- 7.1.** The set is not linearly independent.

7.2. (a) For $\lambda = 2$, a basis for the corresponding eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda = -1$, a basis for the corresponding eigenspace is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) An orthonormal basis of eigenvectors is

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

(c) We have

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}, \quad P^t A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The answers would be different if the eigenvalues or eigenvectors were chosen in some other order.

7.3. The columns of P must *also* be unit vectors.

7.4. (a) For $l = 6$, $\frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ constitutes a basis for the corresponding eigenspace.

For $l = 1$, $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ constitutes a basis for the corresponding eigenspace.

(b) $P = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. The equation is $6u^2 + v^2 = 24$.

(c) The conic is an ellipse with principal axes along the axes determined by the two basic eigenvectors.

7.5. First find an orthonormal basis of eigenvectors for $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. The eigenvalues are $\lambda = 3, 2, -1$. The corresponding basis is

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Picking new coordinates with these as unit vectors on the new axes, the equation is $3(x')^2 + 2(y')^2 - (z')^2 = 6$. This is a hyperboloid of one sheet. Its axis is the z' -axis and this points along the third vector in the above list.

If the eigenvalues had been given in another order, you would have listed the basic eigenvectors in that order also. You might also have conceivably picked a negative of one of those eigenvectors for a basic eigenvector. The new axes would be the same except that they would be labeled differently and some might have directions reversed. The graph would be the same but its equation would look different because of the labeling of the coordinates.

Note that just to identify the graph, all you needed was the eigenvalues. The orthonormal basis of eigenvectors is only necessary if you want to sketch the surface relative to the *original* coordinate axes.

7.6. One normal mode $\mathbf{x} = \mathbf{v} \cos(\omega t)$ has $\lambda = -1$,

$$\omega = \sqrt{\frac{k}{m}}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In this mode, $x_1 = x_2$, so the particles move in the same direction with equal displacements.

A second normal mode $\mathbf{x} = \mathbf{v} \cos(\omega t)$ has $\lambda = -5$,

$$\omega = \sqrt{\frac{5k}{m}}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

In this mode, $x_1 = -x_2$, so the particles move in the opposite directions with equal displacements.

7.7. (a) Not diagonalizable. 1 is a triple root of the characteristic equation, but $A - I$ has rank 2, which shows that the corresponding eigenspace has dimension 1.

(b) If you subtract 2 from each diagonal entry, and compute the rank of the resulting matrix, you find it is one. So the eigenspace corresponding to $l = 2$ has dimension $3 - 1 = 2$. The other eigenspace necessarily has dimension one. Hence, the matrix is diagonalizable.

(c) The matrix is real and symmetric so it is diagonalizable.