CHAPTER XI

CATEGORIES

1. Category theory

We have used the terms "category" and "functor" and talked about universal mapping properties. We are now ready to discuss the general abstract theory behind these concepts.

A *category* C consists of the following:

(a) a class $obj(C)$ called the objects of C;

(b) for each ordered pair (a, b) with a and b in $obj(C)$, a set $hom_C(a, b)$ called the morphisms from a to b;

(c) for each ordered triple (a, b, c) from $obj(C)$, a binary operation

$$hom_C(a, b) \times hom_C(b, c) \to hom_C(a, c)$$

called composition and denoted $(f, g) \to gf$.

If $f \in hom_C(a, b)$, then a is called the *domain* of f and b is called the *codomain* of f, and we write $f : a \to b$ to summarize these facts.

In addition, we require the following rules to hold:

(i) The sets $hom_C(a, b)$ and $hom_C(a', b')$ are disjoint unless $(a, b) = (a', b')$.

(ii) For $f \in hom_C(a, b), g \in hom_C(b, c), h \in hom_C(c, d)$ we have

$$h(gf) = (hg)f.$$

(iii) For each a in $obj(C)$ there is an element $id_a \in hom_C(a, a)$ such that $fid_a = f$ for each f with codomain a and $id_a g = g$ for each g with domain a.

We have seen many examples of categories: the category of sets and mappings of sets, the category of groups and group homomorphisms, the category of rings and ring homomorphisms, the category of left (or right) modules over a given ring A and module homomorphisms, etc. The general concept of category generalizes all of these examples, but it is in fact much more powerful, as we shall see below.

The class $obj(C)$ is useful for discussing the category, but the *union* of the sets $hom_C(a, b)$ —the collection of all morphisms in the category—is really the basic structure under consideration. We could have defined a category as a class of elements called morphisms with a partially defined law of composition which is associative where defined and for which certain special kinds of left and right identities exist. $obj(C)$ could then be chosen to be the class of these identities.

Note that we have used the terms "class" and "set" in the above discussion. The reason for this is that classical set theory is not quite adequate to do general category theory since one wants to consider things like the category of "all" groups, and such things are not acceptable sets in the classical theory. In classical set theory, it is easy to produce paradoxes by considering "large sets", so the designers of that theory put in various restrictions to avoid those paradoxes. This was done in a way which which was a bit too restrictive for category theory. We shall leave such questions to category theorists and not generally concern ourselves with such issues in this course. Note, however, once two objects a and b are fixed, $hom_C(a, b)$ is supposed to be a set in the classical sense.

In each of the above examples of categories, the objects were themselves sets and the morphisms were maps between these sets. Although this is often the case, it need not be true in every case. We give two examples.

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1. Let C be a monoid. Let $obj(C)$ consist of a single object, say the identity 1 of C. Then every domain and codomain is this one object and composition must always be defined. Let $hom_C(1,1) = C$ and let composition in the category C be the given binary operation in the monoid C. Since there is only one object, only one identity is needed and clearly $id_1 = 1$. In this example, the object does not have an internal structure as a set with the morphisms being set maps.

2. Let Ω be a partially ordered set with an order relation $a \leq b$. We make Ω into a category by letting Ω the set of objects and by letting $hom_\Omega(a,b)$ consist of a single element (say the pair (a,b) to be definite) if $a \leq b$ and letting it be empty otherwise. If $hom(a,b)$ and $hom(b,c)$ are nonempty, that means that $a \leq b$ and $b \leq c$ whence $a \leq c$ by transitivity. If f is the unique element of $hom(a,b)$ and g is the unique element of $hom(b,c)$, we define gf to be the unique element of $hom(a,c)$. It is easy to see that this operation is associative. The unique element of $hom(a,a)$ (which is nonempty since $a \leq a$) is id_a.

Thus it would seem that the proofs in general category theory cannot rely on any interpretation of objects in a cateogory as sets and morphisms in the category as ordinary mappings between sets. In fact, there are representation theorems in category theory which say that subject to suitable hypotheses, we may reinterpret a category as a subcategory of the category of sets and act as though the objects had elements. Be this as it may, it is really counter to the philosophy of category theory to work with elements.

Let C be a category. A morphism f in C is called an *isomorphism* if it is invertible, i.e., if there is a g in C with $dom(g) = cod(f) = b, cod(g) = dom(f) = a$ and $gf = id_a, fg = id_b$. A morphism f in C is called a *monomorphism* (or injection) if

$$fg_1 = fg_2 \Rightarrow g_1 = g_2$$

whenever $dom(f) = cod(g_1) = cod(g_2)$ and $dom(g_1) = dom(g_2)$. f is called an *epimorphism* (or surjection) if

$$g_1 f = g_2 f \Rightarrow g_1 = g_2$$

whenever $cod(f) = dom(g_1) = dom(g_2)$ and $cod(g_1) = cod(g_2)$.

It is easy to see that in any category C every isomorphism is a *bijection* (.e., both a monomorphism and epimorphism). In the category of sets and many other categories, every bijection is an isomorphism. However, that is not true in every category. For example, let C be the category with one object the closed interval $[0,1]$ and with the morphisms being continuous bijections (in the set theoretic sense) $f : [0,1] \to [0,1]$ which are differentiable on the open interval $(0,1)$. Every morphism f in C is both a monomorphism and an epimorphism in the categorical sense—see below—but there are certainly such maps without differentialbe inverses, and those would not be isomorphisms.

If C is a category, the *dual category* C^0 is the category with the same objects and morphisms as C, but for each morphism f, the domain of f in C^0 is defined to be the codomain of f in C, the codomain of f in C_0 is defined to be the domain of f in C, and the composition gf in C^0 is defined to be the result of the composition fg in C. In general, for any diagram in the category C, there is a corresponding diagram in the category C^0 obtained by reversing all the arrows. Clearly, $(C^0)^0 = C$.

Let C and D be categories. A *covariant functor* $F : C \to D$ is a function which associates to each object a in $obj(C)$ an object $F(a)$ in $obj(D)$ and to each morphism $f : a \to b$ in C a morphism $F(f) : F(a) \to F(b)$ such that

(i) $F(gf) = F(g)F(f)$ whenever the composition gf is defined and,
(ii) $F(id_a) = id_{F(a)}$ for each object a in $obj(C)$.

Recalling that the definition of category can be recast without the objects, one is better off thinking of a functor of being a function which preserves the law of composition when defined and also preserves identities. A *contravariant* functor $F : C \to D$ is defined to be a covariant functor $F : C^0 \to D$. (We use the same notation!) Then $F(gf) = F(f)F(g)$ whenever gf is defined in C.

A simple but useful functor is the *forgetful functor* which, for each category whose objects are sets and morphisms are maps between sets, associates with an object a in the cateogry its underlying set $|a|$ and with a morphism f the underlying set map $|f| : |a| \to |b|$.

PROPOSITION. *In the category of sets, a morphism f is mono in the categorical sense if and only if it is one-to-one, and it is epi in the categorical sense if and only if it is onto. For the category of groups, the* *category of abelian groups, and the category of left (right) A-modules over a fixed ring A, a morphism f is mono (epi) if and only if* $|f|$ *is mono (epi).*

Proof.

The first part is not too hard. If f is one-to-one then

$$fg_1 = fg_2 \Rightarrow g_1 = g_2$$

since

$$f(g_1(x)) = f(g_2(x)) \Rightarrow g_1(x) = g_2(x)$$

for every $x \in dom(g_1) = dom(g_2)$. Conversely, let f be a monomorphism and suppose $f(x) = f(y)$. Let $\{\cdot\}$ be any set with a single element and define $g_1(\cdot) = x$ and $g_2(\cdot) = y$. Since $fg_1 = fg_2$, we may conclude that $g_1 = g_2$, i.e., $x = y$. Similar arguments work for epimorphisms.

Consider the categories referred to in the statement of the Proposition. If $|f|$ is a mono (epi) in the category of sets, it follows easily that f is a mono (epi) in the desired category by functorality and the fact that $|g_1| = |g_2| \Rightarrow g_1 = g_2$ for those categories.

Conversely, we need to show for these categories that the forgetful functor preserves monos (epis). That is fairly easy for the category of abelian groups or modules over a ring, but the category of (possibly nonabelian) groups presents some problems.

Let $f : G' \to G$ be a group homomorphism which is a monomorphism in the category of groups. It is easy to see that $|f|$ must be one-to-one. For, define $g_1, g_2 : \operatorname{Ker} f \to G'$ by $g_1(x) = x$ and $g_2(x) = 1$ for $x \in \operatorname{Ker} f$. $fg_1 = fg_2$ so $g_1 = g_2$ whence $\operatorname{Ker} f = \{1\}$ and f is a monomorphism in the usual sense, i.e. it is one-to-one.

Suppose $f : G' \to G$ is an epimorphism in the category of groups. To show $|f|$ is onto as a set map is quite a bit harder than you might think. Let $K = \operatorname{Im} f$ and assume $K \neq G$. Clearly, it suffices to find two homomorphisms g, g' from G into a third group which differ outside K but agree on K. (Their compositions with f would have to be the same.) We do this as follows. Define $g : G \to \mathcal{S}(G)$ to be the representation of G as permutations of itself by left multiplication, i.e., $g(x)(y) = xy$. If $x \in K$, then $g(x)$ clearly carries each right coset Kt into itself. Suppose first that we can choose t not in K such that $t^2 \neq 1$. Define $\tau : G \to G$ by $\tau(x) = x$ if x is not in $K \cup Kt$ and $\tau(x) = xt, \tau(xt) = x$ for $x \in K$. Define $g' : G \to S(G)$ by $g'(x) = \tau \circ g(x) \circ \tau$. (Note $\tau^{-1} = \tau$.) Then it is easy to see that g' is a homomorphism and that g and g' agree on K. However, $g(t)(t) = t^2$ and $g'(t) = \tau(g(t)(\tau(t)) = \tau(g(t)(1)) = \tau(t) = 1$. Suppose instead every element of G outside of K is of order 2. Then it is not hard to see that K is normal in G. In that case, let $g : G \to G/K$ be the canonical projection and $g' : G \to G/K$ be the trivial homomorphism. These also agree on K and differ outside it.

Note that in the cateogry of rings, it is *not* true that every epimorphism in the categorical sense in onto as a set map. For example, let $i : \mathbf{Z} \to \mathbf{Q}$ be the natural inclusion of the ring of integers in the rational number field. Suppose $g_1 i = g_2 i$ for two ring homomorphisms $g_1, g_2 : \mathbf{Q} \to A$ where A is a third ring. Let $x = q/p \in \mathbf{Q}$. Using $px = q$, we get $g_1(p)g_1(x) = g_1(q)$ and similary $g_2(p)g_2(x) = g_2(q)$. Since the two g's agree on $\mathbf{Z}$, it follows that $g_1(p)g_1(x) = g_1(p)g_2(x)$. Since $g_1(\mathbf{Q}) \cong \mathbf{Q}$ is a field, it follows that $g_1(p)$ is invertible in A, so $g_1(x) = g_2(x)$. It follows that $i : \mathbf{Z} \to \mathbf{Q}$ is an epimorphism in the categorical sense. Note also that i is clearly also a monomorphism in the category of rings, so it is a bijection which certainly is not an isomorphism in the category of rings.

If C is a category, a *subcategory* C' consists of a subclass $obj(C')$ of $obj(C)$ and for each pair of objects a, b in $obj(C')$ a subset $hom_{C'}(a, b)$ of $hom_C(a, b)$ such that $id_a \in hom_{C'}(a, a)$ for each a in $obj(C')$ and also if $f : a \to b, g : b \to c$ are morphisms in C', then gf (in $hom_C(a, c)$) is in $hom_{C'}(a, c)$. If C' is a subcategory of C, we may define the *inclusion* functor of C' in C in the obvious manner. A subcategory C' of C is said to be *full* if $hom_{C'}(a, b) = hom_C(a, b)$ for each pair of objects a, b in C'.

Let C and D be categories. The collection of all functors $F : C \to D$ can be made into the class of objects of a category which we shall denote $HOM(C, D)$. The objects of this category are the functors as mentioned above. A morphism $\phi : F \to G$ where F and G are functors from C to D is defined to be a *collection*

$$\{\phi_a \,|\, a \text{ in } obj(C)\}$$

where $\phi_a : F(a) \to G(a)$ is a morphism in D and for each morphism $f : a \to b$ in C, the diagram below commutes

$$\begin{array}{ccc} F(a) & \xrightarrow{\phi_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ F(b) & \xrightarrow[\phi_b]{} & G(b) \end{array}$$

Such a collection of morphisms in D is also called a *natural transformation* from the functor F to the functor G. If $\phi = \{\phi_a\}$ is as morphism from F to G and $\sigma = \{\sigma_a\}$ is a morphism from G to H, then we define $\sigma\phi = \{\sigma_a\phi_a\}$. It is easy to check that this is in fact a morphism from F to H (i.e., the appropriate diagram commutes.) With this definition, $HOM(C, D)$ becomes a category. (What is id_F for $F : C \to D$ a functor?)

Example 1: Let C be the category of left A-modules for a fixed ring A, and let $D = C$. Let $F(M) = \mathrm{Hom}_A(A, M)$ where the latter is viewed as a left A-module as in our earlier discussion of modules. If $f : M \to M'$ is a module homomorphism, let $F(f) = \mathrm{Hom}_A(A, f)$ as defined earlier. We know by our discussion of modules that F is a functor. Let I denote the *identity* functor which attaches to each object or morphism itself. For each module M define $\phi_M : \mathrm{Hom}_A(A, M) \to M$ by $\phi_M(\alpha) = \alpha(1)$. It is not hard to see that if $f : M \to M'$ is a module homomorphism, then $I(f)\phi_M = f\phi_M = \phi_{M'}\,\mathrm{Hom}_A(A, f)$. (Check this for yourself.) Hence, $\phi = \{\phi_M\}$ is a morphism (natural transformation) from F to I. Moreover, if you go back to our earlier discussion of modules, you will see how to define a morphism $\sigma : I \to F$ which inverts ϕ so that ϕ (also σ) is an isomorphism in the category of functors from C to C. Often, we say more loosely that $\mathrm{Hom}_A(A, M)$ is *naturally isomorphic* to M. (See the Exercises for a detailed set of steps taking you through this argument.)

So far we have discussed functors of one variable. We may extend to functors of many variables as you would expect from your knowledge of ordinary functions in set theory. If C' and C'' are categories, we define the *product category* $C' \times C''$ as follows. The objects of the product will consist of all ordered pairs (a', a'') where a' is an object in C' and a'' is an object in C''. A morphism in the product will similarly consist of a pair (f', f'') where f' is a morphism in C' and f'' is amorphism in C''. $dom(f', f'') = (dom(f'), dom(f''))$ and similarly for the codomain. Composition is defined by $(g', g'')(f', f'') = (g'f', g''f'')$. It is easy to check that this yields a category. If C', C'' and D are categories, a functor of two variables is a functor $F : C' \times C'' \to D$. As with ordinary functional notation, we write $F(a', a'')$ and $F(f', f'')$ for the results of applying the functor to objects and morphisms.

We may account for various combinations of covariance and contravariance for functors of several variables by taking products of categories with other categories or their duals.

Example: If $\mathbf{Ab}$ denotes the category of abelian groups, then $\mathrm{Hom}(-, -)$ is a functor from $\mathbf{Ab} \times \mathbf{Ab}$ to $\mathbf{Ab}$ which is contravariant in the first variable and covariant in the second variable; more precisely it is a functor $\mathbf{Ab}^0 \times \mathbf{Ab} \to \mathbf{Ab}$. If ${}_AMod$ denotes the category of left A-modules and Mod_A the category of right A-modules, then $- \otimes_A -$ is a functor from $Mod_A \times {}_AMod$ to $\mathbf{Ab}$ which is covariant in both variables. If A is commutative, this functor may be redefined to take its values in ${}_AMod = Mod_A$ again.

It would be worth your while at this time to go through these notes from the beginning and try to find as many categories, functors, and natural transformations of functors as you can.

Note that we could consider the structure whose objects are “all” categories and whose morphisms are functors between such categories. Functors $F : C \to D$ and $G : D \to E$ may be composed according to the rules $(GF)(a) = G(F(a))$ and $(GF)(f) = G(F(f))$, and the result is a functor from C to E. We have already mentioned the identity functor for a category. Hence, we could define the *category of all categories*. At the very least, this is a useful way to motivate some of the things we might do with categories. Unfortunately, the classical paradoxes of set theory are bound to interfere with such a concept.

One feature of category theory is that it gives us a good way to discuss most “universal mapping” properties. Let $F : U \to C$ be a functor. Suppose we are given an object p in C together with morphisms $\pi_u : p \to F(u)$ in C, one for each object u in U, and suppose the following rules hold:

(i) For each morhism $\alpha : u \to u'$ in U, we have $f(\alpha)\pi_u = \pi_{u'}$.

(ii) Suppose we are given another s in C together with morphisms $\sigma_u : s \to F(u)$ such that for each morphism $\alpha : u \to u'$ in U, $F(\alpha)\pi_u = \pi_{u'}$. Then there is a *unique* morphism $\tau : s \to p$ such that for each object u in U, $\pi_u \tau = \sigma_u$.

In this case, we say that p together with the collection $\{\pi_u \,|\, u \text{ in } obj(U)\}$ constitutes a *left* or *inverse* or *projective* limit of the functor F. An inverse limit of F is unique up to isomorphism. In particular, if p and s together with the appropriate morphims each constitute a left limit, the there are unique morpisms $\tau : u \to u'$ and $\tau' : u' \to u$ making the appropriate triangles commute. It follows that $\tau\tau'$ and $\tau'\tau$ also make the appropriate triangles commutes. Since the identities of u and u' also do so, it follows that $\tau\tau' = id_u$ and $\tau'\tau = id_{u'}$, so τ and τ' are isomorphisms in C. We shall often write

$$p = \lim_{\leftarrow} F$$

but of course this notation suppresses the morphisms $\pi_u : p \to F(u)$.

The above definition takes no stand on whether or not such a limit actually exists. That will depend both on the functor and the category D.

Example 1: Let the category U consist of a set with $hom_U(u, u')$ empty whenever $u \neq u'$, and let each $hom_U(u, u)$ consist only of the identity element. A functor $F : U \to C$ just consists of a collection of objects $F(u)$ in C indexed by the set U. In this case, an inverse limit of the functor is also called a *product* of the objects in the collection. Compare this with the previous treatment of the concept of product in the various explicit categories we have considered earlier in this course. We see that the uniqueness of the product in a category is a special case of the uniqueness of the inverse limit.

Example 2: Let U be a category with two objects u and u' and two morphisms $\alpha, \beta : u \to u'$ (other than the two identities.) In this case a functor $F : U \to C$ is given in essence by specifying two objects a and b in C and two morphisms $f, g : a \to b$. (Take $a = F(u), b = F(v), f = F(\alpha)$, and $g = F(\beta)$. We could have $a = b$.) To specify an inverse limit of this functor, we must give an object k in C and a single morphism $i :\to a$ such that $fi = gi$. Moreover, i must be universal in the sense that given any other $i' : k' \to a$ such that $fi' = gi'$, there is a unique $t : k' \to k$ such that $it = i'$. We do not need to worry about the morphism from k (or k') to $b = F(v)$ because these are determined by composition. In this case, the inverse limit is called the *kernel* of the pair $f, g : a \to b$. In particular, if C is the category of groups, we may take $f : G \to G'$ to be and homomorphism and $g : G \to G'$ to be the trvial homomorphism ($g(x) = 1$ for all $x \in G$). Then it is not hard to see that the kernel is the usual kernel of the homomorphism with $i : \operatorname{Ker} f \to G$ the inclusion monomorphism.

PROPOSITION. . *Given $f, g : a \to b$ in a category C, a kernel*

$$i : k \to a$$

is a monomorphism.

PROOF.

Suppose $h', h'' : c \to k$ are morphisms in C such that $ih' = ih''$. Then $fih' = gih''$ so there is a unique morphism $j : c \to k$ such that $ij = ih' = ih''$ and by uniqueness $j = h' = h''$.

Example 3: Let U be the underlying category of a partially ordered set. This is the case in which classically the notion of inverse limit originally arose. Although one could do this for any partially ordered set whatsoever, it works best if the set has the property that any two elements in the set have an upper bound.

By considering functors $F : U \to C^0$ into the dual category of C, we get the dual notion of *right*, *direct*, or *inductive* limit in C. The student is advised to carefully write out all the definitions above by *reversing* all the arrows. if as in Example 1 the category is in essence just a set, the direct limit is called the direct sum or just the sum the category C. For U the 2 object category of Example 2 the direct limit is called the *cokernel* of the pair of morphisms $f, g : a \to b$.

There are two other categorical concepts related to the notion of limit and definition by universal mapping properties. We shall introduce these concepts here and illustrate them by important constructions—some familiar to you and some new. Let $F : C \to$ **Sets** be a functor to the category of sets. We say that an object a in C *represents* F if there is an isomorphism of functors $\phi : F \cong hom_C(a, -)$. In that case, we also say that the functor F is *representable*. For example, let X be a set and let M_X : **Groups** $\to$ **Sets** be defined as follows: for a group G, let $M_X(G) = Map(X, |G|)$ and for $f : G \to G'$ a homomorphism, let $M_X(f) : Map(X, |G|) \to Map(X, |G'|)$ be defined by composition. Then the free group $F(X)$ on the set X represents M_X. Indeed, as we showed earlier in this course, each map $X \to G$ may be extended uniquely to a group homomorphism $F(X) \to G$. That provides a map $M_X(G) \to \mathrm{Hom}(F(X), G)$ for each group G. It is easy to check that this collection of maps is a morphism of functors and in fact is an isomorphism of functors. (What is its inverse?)

Note that if F is a representable functor, then the representing object is unique up to isomorphism. For suppose

$$hom(a, -) \cong F \cong hom(b, -).$$

Then, for each object x in C, we have an isomorphism

$$\phi_x : hom(a, x) \cong hom(b, x).$$

If we put $x = a$, this gives us $\alpha = \phi_a(id_a) \in hom(b, a)$ and $\beta \in hom(a, b)$ such that $\phi_b(\beta) = id_b$. The commativity of the square

$$\begin{array}{ccc} hom(a,a) & \longrightarrow & hom(b,a) \\ {\scriptstyle hom(a,\beta)}\downarrow & & \downarrow{\scriptstyle hom(b,\beta)} \\ hom(a,b) & \longrightarrow & hom(b,b) \end{array}$$

shows that $\phi_b(hom(a, \beta)(id_a) = hom(b, \beta)(\phi_a(id_a)$ so that

$$id_b = \phi_b(\beta) = hom(b, \beta)(\alpha) = \beta\alpha.$$

Similarly, $\alpha\beta = id_a$.

A related concept is that of *adjoint* functors. Let C and D be categories and suppose $F : C \to D$ and $G : D \to C$ are functors. We say that G is a left adjoint of F and F is a right adjoint of G if there is an isomorphism of bifunctors

$$\phi : hom_C(G(-), -) \cong hom_D(-, F(-)).$$

According to the above definition, this means that for each object d in D, $G(d)$ represents the composite functor $hom_D(d, F(-))$. Thus, at least the objects $G(d)$ are uniquely determined up to isomorphism. It is not hard to check that any two left adjoints of a functor F are unique up to isomorphism of functors.

We now give some examples of adjoint functors.

As above, we have for each set X,

$$\mathrm{Hom}(F(X), G) \cong Map(X, |G|)$$

and it is easy to see that this provides an isomorphism of bifunctors. On the left we are using *hom* in the category of groups and on the right we are using *hom* in the category of sets. Hence, the free group functor is adjoint to the forgetful functor $G \mapsto |G|$.

Let k be a commutative ring and let X be a set. The polynomial ring $k[X]$ in the category of commutative rings is analagous to the free group $F(X)$ in the cateogry of groups. However, we must also have some way tor take account of the coefficient ring. To this end, recall that we called a ring homomorphism $\phi : k \to A$ into a commutative ring A a *commutative k-algebra*. We may form a category with these k-algebras: let a morphism f from $\phi : k \to A$ to $\psi : k \to B$ be a ring homomorphism $f : A \to B$ such that

$$\begin{array}{ccc} & k & \\ \phi\swarrow & & \searrow\psi \\ A & \underset{f}{\longrightarrow} & B \end{array}$$

commutes. Compose such morphisms in the obvious way. It is easy to check that these definitions satisfy the requirements of a category which we call the category of commutative k-algebras. Often we shall simplify the terminology by just calling A a k-algebra without mentioning the homomorphism $k \to A$. Then our previous discussion of polynomials showed us that any set map of X into a commutative k-algebra A may be extended uniquely to a homomorphism

$$k[X] \to A.$$

Some tedious checking shows that this provides an isomorphism of functors

$$hom_{k-algebras}(k[X], -) \cong Map(X, |-|)$$

so that the polynomial ring functor $k[X]$ is the left adjoint of the forgetful functor from k-algebras to sets. (Note that the forgetful functor strictly speaking attaches to a k-algebra $k \to A$, the set $|A|$.)

You should state the corresponding fact about the functor which attaches to a set X the free A-module with that set as basis where A is a ring.

Suppose again that A is a (not necessarily commutative) ring. The tensor product over A of modules may also be considered an adjoint functor except that the situation is a bit more complicated because it is a functor of two variables. Denote by $Bil(L, M; N)$ the set of bilinear maps $f : L \times M \to N$ where L is a right A-module, M is a left A-module and N is an abelian group; then it is clear how to make this a *functor* of the last argument N. (Define $Bil(L, M; g)$ for a homomorphism g of abelian groups.) Then, it is easy to check that our basic identification

$$\mathrm{Hom}(L \otimes_A M, N) \cong Bil(L, M; N)$$

provides an isomorphism of functors. (It is even an isomorphism of trifunctors!) Thus, $L \otimes_A M$ represents the functor of N on the right.

To exhibit the tensor product as an adjoint functor requires a bit more work, and we shall do it in the most generally useful form. Suppose that L is a right A-module and M is a left A-module. Suppose in addition that a second ring B acts on M on the right in such a way that

$$(ax)b = a(xb) \qquad \text{for } a \in A, x \in M, b \in B.$$

In that case we call M a *bimodule*. Then it is not hard to check that the definition

$$(x \otimes y)b = x \otimes (yb) \qquad x \in L, y \in M, b \in B$$

makes sense and turns $L \otimes_A M$ into a right B-module. Suppose finally that N is a right B-module. Then it is not hard to check that

$$(fa)(x) = f(ax) \qquad a \in A, f \in \mathrm{Hom}_B(M, N), x \in M$$

takes B-homomorphisms into B-homomorphisms and makes $\mathrm{Hom}_B(M, N)$ into a right A-module. We shall define an isomorphism of abelian groups

$$\mathrm{Hom}_B(L \otimes_A M, N) \cong \mathrm{Hom}_A(L, Hom_B(M, N))$$

by

$$f \mapsto f^{\#} \qquad \text{where } f^{\#}(l)(m) = f(l \otimes m).$$

There is a lot to check here. Namely, you have to show first that $f^{\#}(l)$ is a B-homomorphism, then you have to show that $f^{\#}$ is an A-homomorphism of $L \to \mathrm{Hom}_B(M, N)$, and finally you have to show that $f \to f^{\#}$ is a homomorphism of abelian groups. That this is an isomorphism follows by defining its inverse

$$g \mapsto g^{\$} \qquad \text{where } g^{\$}(l \otimes m) = g(l)(m).$$

Again, all the appropriate things have to be proved, but at least it is fairly clear that this homomorphism is inverse to the one given above.

Finally, you should show that these maps define morphisms of functors (in fact of functors of all three variables.)

Exercises.

1. Let C be a category and let $f : a \to b$ be an isomorphism. Show that f is both a monomorphism and an epimorphism is C.

2. Show for the category of sets and the category of abelian groups that a morphism is an epimorphism in the category if and only if its underlying set theoretic map is onto.

3. An object a in a category C is called initial if for each object c in C there is a unique morphism $f_c : a \to c$ in C. Show that if a is an initial object, then the collection of morphisms $f_c : a \to c$ present a as a projective limit of the identity functor. Conversely, show that any projective limit of the identity functor is an initial object. Find the initial objects in the categories of groups and rings if any. Define the concept final object.

4. Let A be a ring. View A as a left module over itself, and for each left A-module M, make the abelian group $\mathrm{Hom}_A(A, M)$ into a left A-module by defining

$$(af)(b) = f(ba) \qquad \text{where } a, b \in A, f \in \mathrm{Hom}_A(A, M).$$

You may assume this all works out properly. Moreover, for $g : M \to N$ $\phi = \{\phi_M\}$ an A-module homomorphism, define $\mathrm{Hom}_A(A, g) : \mathrm{Hom}_A(A, M) \to \mathrm{Hom}_A(A, N)$ by

$$\mathrm{Hom}_A(A, g)(f) = g \circ f \qquad \text{for } f \in \mathrm{Hom}_A(A, M).$$

You may assume the result is an element of $\mathrm{Hom}_A(A, N)$. Moreover, you may assume that $\mathrm{Hom}_A(A, -)$ so defined is a covariant functor from the category of left A-modules to itself. Define $\phi_M : \mathrm{Hom}_A(A, M) \to M$ by

$$\mathrm{Hom}_A(A, g)(f) = g \circ f \qquad \text{for } f \in \mathrm{Hom}_A(A, M).$$

You may assume the result is an element of $\mathrm{Hom}_A(A, N)$. Moreover, you may assume that $\mathrm{Hom}_A(A, -)$ so defined is a covariant functor from the category of left A-modules to itself. Define $\phi_M : \mathrm{Hom}_A(A, M) \to M$ by

$$\phi_M(f) = f(1) \qquad \text{for } f \in \mathrm{Hom}_A(A, M).$$

(a) Show that the collection $\{\phi_A\}$ is a natural transformation of functors

$$\phi : \mathrm{Hom}_A(A, -) \to \mathrm{Id}$$

where Id denotes the identity functor.

(b) Define an inverse morphism $\psi : \mathrm{Id} \to \mathrm{Hom}_A(A, -)$ by giving a collection of module homomorphisms $\psi_M : M \to \mathrm{Hom}_A(A, M)$ with the right properties.

5. Study the discussion in the text in more detail and show that the isomorphisms

$$\mathrm{Hom}_B(L \otimes_A M, N) \cong \mathrm{Hom}_A(L, \mathrm{Hom}_B(M, N))$$

provide an isomorphism of functors between the appropriate categories. You may skip the details such as showing that the operations defined make $L \otimes_A M$ into a right B-module, $\mathrm{Hom}_B(M, N)$ into left A-module, etc.