CHAPTER XII

MULTILINEAR ALGEBRA

1. Tensor and Symmetric Algebra

Let k be a commutative ring. By a k-algebra, we mean a ring homomorphism $\phi : k \to A$ such that each element of $\operatorname{Im}\phi$ commutes with each element of A. (If A is a ring, we define its center to be the subring $Z(A) = \{a \in A \,|\, ax = xa, \text{for all } x \in A\}$. So this can also be abbreviated $\operatorname{Im}\phi \subseteq Z(A)$.))

Such a k-algebra may be viewed as a k-module by defining $ax = \phi(a)x$ for $a \in k$ and $x \in A$. If $\phi : k \to A$ and $\psi : k \to B$ are k-algebras, a ring homomorphism $f : A \to B$ is called a k-algebra morphism if it is also a k-module homomorphism. We shall often speak loosely of a "k-algebra A" without mentioning the homomorphism ϕ explicitly. We may consider the *forgetful* functor which attaches to a k-algebra A the underlying k-module A. But to simplify the notation, we shall write A both for the ring and the associated k-module, with the context making clear which is intended. We want to construct a left adjoint to this functor. That will amount to the following: given a k-module M, we want to construct a k-algebra $T(M)$ which contains M and such that every k-module homomorphism $M \to A$ where A is a k-algebra may be extended uniquely to a k-algebra morphism $T(M) \to A$. In categorical terms, we will have an isomorphism of functors

$$\operatorname{Hom}_{k-Alg}(T(M), A) \cong \operatorname{Hom}_k(M, A).$$

To construct $T(M)$, we form the so called *tensor algebra*. Recall that since k is commutative, if M and N are k-modules, $M \otimes N$ also has a natural k-module structure. (All tensor products will be over k so we shall omit the subscript in $\otimes_k$.) The associative law for tensor products tells us that

$$(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$$

so that we may omit parentheses. With these facts in mind, define

$$T^0(M) = k$$

and for $n > 0$,

$$T^n(M) = \overbrace{M \otimes M \otimes \cdots \otimes M}^{n \text{ times}}.$$

Finally let

$$T(M) = \bigoplus_{n \geq 0} T^n(M).$$

We may make $T(M)$ into a ring as follows. Define a binary operation

$$T^p(M) \times T^q(M) \to T^{p+q}(M)$$

by sending

$$(u_1 \otimes \cdots \otimes u_p, v_1 \otimes \ldots v_q) \mapsto u_1 \otimes \cdots \otimes u_p \otimes v_1 \cdots \otimes v_q.$$

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(Note that for this to make sense and define a map $T^p(M) \times T^q(M) \to T^{p+q}(M)$ it is necessary to check that the right hand side is k-multilinear in each u_i and each v_j.) For p or $q = 0$, this rule has to be suitably interpreted with the empty tensor product of elements of M interpreted as the identity $1 \in k$. It is not hard to check that this operation yields an associative ring structure for $T(M)$. $T(M)$ may be made into a k-algebra by defining $\phi : k \to T(M)$ by

$$\phi(x) = x \in k = T^0(M).$$

In addition, we may view M as imbedded in $T(M)$ as the k-submodule $T^1(M) = M$. Note that any element of $T(M)$ may be written as a k-linear combination of elements of the form $x_1 \otimes \cdots \otimes x_n$ with $x_1, x_2, \dots, x_n \in M$. Such an element is clearly the product in $T(M)$ of the elements $x_1, \dots, x_n$, so $T(M)$ is generated as a ring by the elements of $k = T^0(M)$ and by $M = T^1(M)$. In such a case we shall say that it is generated as a k-algebra by the specified elements.

Let $\psi : k \to A$ be any k-algebra. If $f : M \to A$ is a k-module homomorphism and $F : T(M) \to A$ is an algebra homomorphism extending f, then since M generates $T(M)$ as a k-algebra, it follows that

$$F(x_1 \otimes \cdots \otimes x_n) = f(x_1) \dots f(x_n) \in A.$$

Since F is by definition a k-module homomorphism, it follows that this formula completely determines F so it is unique if it exists. On the other hand, it is easy to check that the right hand side is multilinear in $x_1, \dots, x_n$ so that there is a k-module homomorphism satisfying that formula. Referrring to the definition of the product in $T(M)$, it is easy to check that it is a k-algebra homomorphism extending f.

Proposition. *If M is free over k with basis $\{x_1, x_2, \dots, x_r\}$, then $T^n(M)$ is free with basis consisting of all elements of the form*

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$$

where $1 \le i_1, i_2, \dots, i_n \le r$.

Proof.

We prove more generally the following.

Lemma. *If k is a commutative ring and M is free with basis $\{x_1, \dots, x_s\}$ and N is free with basis $\{y_1, \dots, y_t\}$, then $M \otimes N$ is free with basis $\{x_i \otimes y_j \,|\, 1 \le i \le s, 1 \le j \le t\}$.*

Proof of the Lemma. This follows from the fact that M is the direct sum of submodules kx_i, N is the direct sum of submodules ky_j, tensor products commute with direct sums, and after identification as a k-submodule of $M \otimes N$, $kx_i \otimes ky_j = k(x_i \otimes y_j)$.

Remark. The argument works just as well if M is free with an infinite basis X. In that case the basis of $T^n(M)$ consists of elements of the form $x_1 \otimes \cdots \otimes x_n$ where $x_1, \dots, x_n \in X$. A similar remark applies to the Lemma.

Suppose now that M is the free k-module on a set X. As above, we shall suppose $X = \{X_1, \dots, X_r\}$ is finite, but the arguments work just as well if it is infinite. The above proposition shows that the elements of $T(M)$ are uniquely expressible as linear combinations over k of *non-commuting* monomials

$$X_{i_1} X_{i_2} \dots X_{i_n}$$

where $1 \le i_1, i_2, \dots, i_n \le r$. Multiplication of such monomials is just done by juxtaposition. Thus in this case we may identify $T(M)$ with the analogue of a polynomial ring for non-commuting indeterminates. Any set theoretic map of X into a (possibly non-commutative) k-algebra A may be uniquely extended to a k-algebra homomorphism of $T(M)$ into A.

We may reconstruct the usual polynomial ring from the above construction (M free on X) by "forcing" the indeterminates to commute. More generally, suppose M is any k-module, and consider the ideal I in $T(M)$ generated by all elements of the form

$$[x, y] = xy - yx = x \otimes y - y \otimes x$$

where $x, y \in M$. It is not hard to see that I consists of all linear combinations over k of elements of the form $a[x, y]b$ where $a, b \in T(M)$. Since $T(M)$ is the direct sum of its *homogeneous* components $T^n(M)$, it follows that I is generated by elements of the form $a[x, y]b$ where $a \in T^p(M), x, y \in M$, and $b \in T^q(M)$ for some p and q. Such elements are in $I \cap T^{p+q+2}(M)$. It follows that $I \cap T^0(M) = I \cap T^1(M) = \{0\}$, and

$$I = \bigoplus_{n \geq 2} I^n \qquad \text{where } I^n = I \cap T^n(M).$$

Define the *symmetric* algebra on M to be the k-algebra

$$S(M) = T(M)/I = \bigoplus_{n \geq 0} S^n \qquad \text{where } S^n = T^n(M)/I^n.$$

By the above remarks, $S^0(M) = k, S^1(M) = M$, and $S(M)$ is generated by k and M. Also, in $S^n(M)$ for $n \geq 2$, we have

$$x_1x_2 \dots x_{p-1}xyy_{p+2} \dots y_n = x_1x_2 \dots x_{p-1}yxy_{p+2} \dots y_n$$

for $x_1, x_2, \dots, x_{p-1}, x, y, y_{p+2}, \dots, y_n \in M$. (The difference of the corresponding elements in $T^n(M)$ is in I^n.) In other words, one can interchange adjacent terms in a monomial product. It follows that one can interchange any two terms in a monomial and indeed the product in $S(M)$ is commutative.

Proposition. *Let k be a commutative ring and assume that M is a free k-module with basis X. Then the monomials in X of degree n form a basis for $S^n(M)$. Hence, $S(M) \cong k[X]$.*

Proof.
We leave this as an exercise for the student who has the time and inclination. One way to proceed is to state carefully the appropriate universal mapping property for $S(M)$, and then to use that property and the corresponding property for $k[X]$ to define ring homomorphisms between the two rings which can be shown (again by the proper universal mapping properties) to be inverse to one another.

Note that by the argument given above, I^n is in fact spanned by all

$$x_1 \otimes \dots \otimes x \otimes \dots \otimes y \otimes \dots x_n - x_1 \otimes \dots \otimes y \otimes \dots \otimes x \otimes \dots x_n$$

where $x_1, \dots, x, \dots, y \dots, x_n \in M$.

Exercises.

1. Let k be a commutative ring, and let A be any k-algebra. Let $A \times A \to A$ be the binary operation yielding multiplication, and suppose as usual that it is denoted $(x, y) \mapsto xy$. Show that this mapping is bilinear and hence yields a k-module homomorphism $A \otimes A \to A$. Conversely, suppose you are given a k-module homomorphism $A \otimes A \to A$. Show that it may be used to define a binary operation on A which satisfies each distributive law. Exhibit a commutative diagram which will have to apply in order that the associative law applies. Similarly, construct a commutative diagram which will have to apply for their to be a multiplicative identity. (Hint. Recall that if A is a k-algebra, there is given implicitly at least a ring homomorphism $k \to A$ with image in the center of A.)

2. Let k be a commutative ring and let M be a free k-module of rank > 1. Show that the center of the tensor algebra $Z(T(M)) = k$.

3. (Optional). Let M be free on $\{X_1, X_2, \dots, X_n\}$. Show that $S(M) \cong k[X_1, X_2, \dots, X_n]$.

2. Exterior Algebra

Let $M^n = \overbrace{M \times \cdots \times M}^{n \text{ times}}$. The component $T^n(M)$ of the tensor algebra has the property that any multilinear map $f : M^n \to N$ into a k-module N defines a unique k-module homomorphism $F : T^n(M) \to N$, i.e., there is an isomorphism of functors

$$\mathrm{Hom}_k(T^n(M), N) \cong \mathrm{Mult}(M^n, N).$$

Since $S^n(M) = T^n/I \cap T^n$ and since $I \cap T^n$ is spanned (over k) by all

$$x_1 \otimes \cdots \otimes x \otimes \cdots \otimes y \otimes \ldots x_n - x_1 \otimes \cdots \otimes y \otimes \cdots \otimes x \otimes \ldots x_n,$$

it follows that a multilinear map $f : M^n \to N$ defines a unique k-module homomorphism $F : S^n(M) \to N$ if and only if f is *symmetric*, that is

$$f(x_1, \ldots, x, \ldots, y, \ldots x_n) = f(x_1, \ldots, y, \ldots, x, \ldots x_n)$$

whenever two arguments are interchanged. Since the symmetric group is generated by transpositions, a multilinear map f is symmetric if and only if it does not change whenever any permutation is perfomed on its arguments. In fact, as mentioned previously, it suffices to consider only adjacent transpositions. In any event, there is an isomorphism of functors

$$\mathrm{Hom}_k(S^n(M), N) \cong \mathrm{SymMult}(M^n, N).$$

Having considered multilinear symmetric maps, it is natural to also consider multilinear, antisymmetric maps, i.e., maps for which the sign changes when two arguments are interchanged. Unfortunately, if $1 = -1$ in k (as would be the case for $\mathbf{F}_2$) antisymmetric maps are also symmetric. On the other hand, if f is antisymmetric, then

$$f(x_1, \ldots, x, \ldots, x, \ldots x_n) = -f(x_1, \ldots, x, \ldots, x, \ldots x_n)$$

so that

$$2f(x_1, \ldots, x, \ldots, x, \ldots x_n) = 0.$$

Hence, if 2 is a unit, it follows that $f(x_1, \ldots, x, \ldots, x, \ldots x_n) = 0$. Conversely, if f satisfies the condition $f(x_1, \ldots, x, \ldots, x, \ldots x_n) = 0$ for all $x \in M$, then it is not hard to see that f must be antisymmetric whether 2 is a unit or not. (Just replace x by $x+y$ and expand by multilinearity.). With this as motivation, we make the following definition. A multilinear map $f : M^n \to N$ is said to be *alternating* if it vanishes whenever two arguments are equal. It is not hard to see that it suffices to consider the case in which two adjacent arguments are equal.

As above, we would like to exhibit an isomorphism of functors

$$\mathrm{Hom}_k(\wedge^n(M), N) \cong \mathrm{AltMult}(M^n, N)$$

where $\wedge^n(M)$ is a suitably defined object. To do this, let J^n be the k-submodule of $T^n(M)$ spanned by all $x_1 \otimes \cdots \otimes x \otimes \cdots \otimes x \otimes \cdots \otimes x_n$. (As above, it will suffice to take all $x_1 \otimes \cdots \otimes x \otimes x \otimes \cdots \otimes x_n$.) For $n = 0$ or 1, let $J^n = \{0\}$. Define $\wedge^n(M) = T^n(M)/J^n$. Clearly, any alternating multilinear form carries J^n to 0 so it defines a unique k-module homomorphism $F : \wedge^n(M) \to N$ as required. $\wedge^n(M)$ is called the nth *exterior* power of M.

As in the case of the symmetric algebra, we may also approach the problem from a ring theoretic point of view. Let

$$J = \bigoplus_{n \geq 0} J^n.$$

It is easy to see that J is an ideal in $T(M)$ and in fact it is the ideal generated by all $x^2 = x \otimes x$ for $x \in M$. It follows that $\wedge(M) = T(M)/J = \bigoplus \wedge^n(M)$ is a k-algebra with $k = \wedge^0(M)$ and $M = \wedge^1(M)$. Clearly,

$\wedge(M)$ is generated as a k-algebra by the k-submodule M, and $x^2 = 0$ for all $x \in M$. Moreover, if $f : M \to A$ is a k-module homomorphism into a k-algebra A with the property that $f(x)^2 = 0$ in A for $x \in M$, then f may be extended uniquely to a k-algebra homomorphism $F : \wedge(M) \to A$. $\wedge(M)$ is called the exterior algebra of M. The image of the element $x_1 \otimes \cdots \otimes x_n \in T^n(M)$ in $\wedge^n(M)$ is denoted $x_1 \wedge \cdots \wedge x_n$. If $f : M \to N$ is a k-module homomorphism, then the universal mapping property described above defines a k-algebra homomorphism $\wedge(f) : \wedge(M) \to \wedge(N)$, and it is easy to check that

$$\wedge(f)(x_1 \wedge \cdots \wedge x_n) = f(x_1) \wedge \cdots \wedge f(x_n).$$

Proposition. *Let M be a free k-module with basis $X = \{x_1, \ldots x_r\}$. Then $\wedge^n(M)$ is free with basis*

$$\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} \mid 1 \le i_1 < i_2 < \cdots < i_n \le r\}.$$

In particular,

$$rank(\wedge^n(M)) = \binom{r}{n}.$$

Proof.
It is not hard to see that J^n is spanned by all

$$y_1 \otimes \cdots \otimes y_i \otimes \cdots \otimes y_j \otimes \cdots \otimes y_n + y_1 \otimes \cdots \otimes y_j \otimes \cdots \otimes y_i \otimes \cdots \otimes y_n$$

and all

$$y_1 \otimes \cdots \otimes y_i \otimes \cdots \otimes y_i \otimes \cdots \otimes y_n$$

where $y_1, \ldots, y_i, \ldots, y_j, \ldots, y_n$ are distinct elements of the *basis* X. It is not hard to check that these elements are linearly independent, so they form a basis for J^n. The elements described in the proposition clearly span $\wedge^n(M)$. If there were a dependence relation among those elements, then some linear combination of their preimages in $T^n(M)$ would be in J^n so that linear combination would be uniquely expressible as a linear combination of the elements listed above. It is easy to see that no such relation can hold in $T^n(M)$. (Details are left to the student.)

Note that if M is free of rank r, then $\wedge^n(M) = \{0\}$ if $n > r$. Can you prove the same thing if M is generated over k by r elements?

Exercises.

1. Show that a multilinear map $f : M^n \to M$ is alternating if and only if it vanishes whenever two adjacent arguments are equal.
2. Let k be a commutative ring and let M be a k-module. Show that if M is generated as a module by a set with r or fewer elements then $\wedge^k(M) = 0$ for $k > r$.

3. Determinants

Let k be a commutative ring and let M be a free k-module of rank r. Then $\wedge^r(M)$ is free of rank 1. In fact, if $\{x_1, \ldots, x_r\}$ is a basis for M over k, then $x_1 \wedge \cdots \wedge x_r$ consitutes a basis for $\wedge^r(M)$. Let $f : M \to M$ be a module homomorphism. By functorality, it induces a module homomorphism $\wedge^r(f) : \wedge^r(M) \to \wedge^r(M)$ and if u is any generator of $\wedge^r(M)$, we have $\wedge^r(f)(u) = au$ for some $a \in k$. It is not hard to see that since $\wedge^r(f)$ is a module homomorphism, a depends only on f and not on the generator u. (In fact, *any* k-module homomorphism of a free module of rank 1 into itself is of the form λ_a where $\lambda_a(x) = ax$.) We call a the *determinant* of the homomorphism f and we denote it $\det(f)$.

Proposition. *Let M be free of rank r over the commutative ring k. (a)* $\det(\mathrm{Id}_M) = 1$. *(b) If* $f, g \in \mathrm{Hom}_k(M, M)$*, then* $\det(gf) = \det(g)\det(f)$. *(c) If* $f \in \mathrm{Hom}_k(M, M)$ *is invertible, then* $\det(f)$ *is a unit in* k.

Proof.
(a) and (b) follow easily from the functorality of $\wedge^r$. (c) follows from (a) and (b).

As above, suppose $\{x_1, \dots, x_r\}$ is a basis for the free k-module M. If $f \in \mathrm{Hom}_k(M, M)$, then we have

$$f(x_i) = \sum_{j=1}^{r} a_{ji} x_j, \qquad i = 1, 2, \dots, r$$

and the *matrix* $A = (a_{ji})$ completely determines f since it is uniquely characterized by its values on a basis. A is called the matrix of the homomorphism f. On the other hand, given an $r \times r$ matrix A with entries in k, we may define a k-module homomorphism $f : M \to M$ by the above formula. Hence, there is a one to one correspondence between the endomorphism ring $\mathrm{Hom}_k(M, M)$ and the set $M_r(k)$ of $r \times r$ matrices with entries in k. As in the linear algebra of vector spaces over fields, it is not hard to check that under this correspondence, if f corresponds to A and g corresponds to B then $f + g$ corresponds to the matrix sum $A + B$ and gf corresponds to the matrix product BA. Hence the correspondence provides and isomorphism of rings $\mathrm{Hom}_k(M, M) \cong M_r(k)$.

Note that we can apply the above theory to the module $M = k^r$ of r-tuples with entries in k. That module has the usual standard basis $\{e_1, \dots, e_r\}$ where e_i has all entries 0 except the ith entry which is 1. If we identify an r-tuple as a *column* vector as is usual in linear algebra, then every k-endomorphism f of M is given by $f(x) = Ax$ where the product on the right is the matrix product of an $r \times r$ matrix with a column vector. Here A is the matrix corresponding to f with respect to the standard basis. In effect we can identify f with its matrix A in this case. Whenever we write $\det(A)$ for A an $r \times r$ matrix, it may be assumed that this is the context if no other module M and basis are specified.

Let M be free of rank r and let $f : M \to M$ be a k-module homomorphism. For each i with $0 \le i \le r$, we have the induced homomorphism $\wedge^i(f)$. Consider in particular $f^\# = \wedge^{r-1}(f)$. By the above proposition, $\wedge^{r-1}(M)$ has rank r. In fact, the elements

$$y_i = x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_r, \qquad i = 1, \dots, r,$$

form a basis for $\wedge^{r-1}(M)$. Also, $x_i y_i = (-1)^{i-1} u$ where $u = x_1 \wedge \cdots \wedge x_r$, and $y_i x_j = 0$ for $i \ne j$. (Why?) By definition,

$$\begin{aligned} \det(f)u &= f(x_1) \dots f(x_{i-1}) f(x_i) f(x_{i+1}) \dots f(x_r) \\ &= (-1)^{i-1} f(x_i) \wedge^{r-1} (f)(y_i) = (-1)^{i-1} f(x_i) f^\#(y_i). \end{aligned}$$

Let $f^\#(y_i) = \sum_i b_{ki} y_k$ and as above $f(x_i) = \sum_j a_{ji} x_j$. Then it follows that

$$\begin{aligned} f(x_i) f^\#(y_i) &= \sum_{j,k} a_{ji} b_{ki} x_j y_k = \sum_j a_{ji} b_{ji} x_j y_j \\ &= (\sum_j a_{ji} b_{ji} (-1)^{j-1}) u. \end{aligned}$$

Hence, $\det(f) = \sum_j c_{ij} a_{ji}$ where $c_{ij} = (-1)^{i+j} b_{ji}$. (The two -1 's add up to something even.) This is the so called Laplace expansion of $\det(f)$. The quantity b_{ji} can be shown to be the determinant of the matrix obtained from A by deleting the jth row and the ith column (or vice-versa?) It is called the j, i-minor of the matrix A.

Let A^+ be the matrix with entries c_{ij}. Then the above formula calculates the diagonal entries of the product $A^+ A$. The off diagonal entries are easily seen to be zero by repeating the above calculations for $f(x_i) f^\#(y_j)$ with $i \ne j$ and recalling that $x_i y_j = 0$ in that case.

Proposition. *Let A be an $r \times r$ matrix with entries in the commutative ring k. Then*

$$A^+A = AA^+ = \det(A)I.$$

In particular, A is invertible in $M_r(k)$ if and only if $\det(A)$ *is a unit in k.*

Proof. The above argument verifies the equation for A^+A. The argument in the other order is quite similar. We already know that if A is invertible, then $\det(A)$ is a unit. Conversely, if $\det(A)$ is a unit, the above equations show that $(\det(A))^{-1}A^+$ is an inverse for A.

Let A be an $r \times r$ matrix over a commutative ring k. If we define $f : k^r \to k^r$ by $f(x) = Ax$, then it is clear by expanding out

$$f(e_1) \wedge \cdots \wedge f(e_r)$$

in terms of the matrix entries a_{ij}, that $\det(A) = \det(f)$ is a polynomial of degree r in those entries a_{ij}. In fact, as is the case in linear algebra over a field, $\det(A)$ is the sum of the $r!$ terms obtained by forming all products gotten by taking one term from each row and from each column—with ± 1 in front depending on the sign of the permutation which gives the row index as a function of the column index for that product. It is often useful to know that equations of the form $\det(A) = 0$ result in polynomial equations.

Suppose now that M is a module over k generated by elements $y_1, \dots, y_m$. Suppose f is a k-module homomophism and

$$f(y_i) = \sum_j a_{ji}y_j \qquad \text{for } i = 1, 2, \dots, m.$$

Let $L = k[f]$ be the (commutative) subring of $\operatorname{Hom}_k(M, M)$ generated by k and f. It consists of all polynomials of the form

$$a_0 \operatorname{Id} + a_1 f + \cdots + a_t f^t \qquad \text{with } a_j \in k.$$

Let B be the $m \times m$ matrix in $L = k[f]$ with entries

$$\begin{aligned} b_{ii} &= f - a_{ii} \operatorname{Id}, \\ b_{ij} &= -a_{ij} \operatorname{Id} \qquad i \neq j. \end{aligned}$$

If we view M as an L-module in the obvious way, we have

$$\sum_j b_{ij}y_j = 0 \qquad \text{for } i = 1, 2, \dots, m.$$

Multiply this system of equations by the entries of $C = B^+$ in the suitable order and add to get

$$\sum_{i,j} c_{li}b_{ij}y_j = 0, \qquad l = 1, 2, \dots, m.$$

Using $B^+B = \det(B)I$ yields

$$\det(f)y_l = 0, \qquad\qquad l = 1, 2, \dots, m,$$

and since $y_1, \dots, y_m$ generate M, it follows that $\det(B) = 0$ as an element of $\operatorname{Hom}_k(M, M)$. Recalling what B is, we have

$$\det(fI - A) = 0$$

where the matrix in parentheses must be viewed as a matrix with entries in the ring $k[f]$. This result is sometimes called the *Hamilton-Cayley* Theorem.

Exercises.

1. Let k be a commutative ring, and let M be a k-module. Define $\Phi : \bigwedge^i(M) \to \mathrm{Hom}_k(\bigwedge^j(M), \bigwedge^{i+j}(M))$ by

$$\Phi(u_1 \wedge \cdots \wedge u_i)(v_1 \wedge \cdots \wedge v_j) = u_1 \wedge \cdots \wedge u_i \wedge v_1 \wedge \cdots \wedge v_j$$

for $u_1, \dots, u_i, v_1, \dots, v_j \in M$.

(a) Show that this condition does define an element of $\mathrm{Hom}_k(\bigwedge^j(M), \bigwedge^{i+j}(M))$. Hint: Use the appropriate universal mapping property.

(b) Assume M is k-free of rank r. Show that for $i + j = r$ the map Φ is an isomorphism.

Note that $\bigwedge^r(M) \cong k$ (but the isomorphism depends on a choice of basis) so (b) tells us that

$$\bigwedge{}^{r-j}(M) \cong \mathrm{Hom}_k(\bigwedge{}^{j}(M), k) \qquad \text{the dual of } \bigwedge{}^{j}(M).$$

(c) Why isn't this a natural transformation of functors? Does it have any reasonable naturality properties?

2. Let k be a commutative ring and let M be a free k-module with basis $\{x_1, x_2, \dots, x_r\}$. Define $z_i = (-1)^{i-1}y_i = (-1)^{i-1}x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_r$. Then $\{z_1, \dots z_r\}$ is a basis for $\bigwedge^{r-1}(M)$. Express the matrix entries for $\bigwedge^{r-1}(f)$ with respect to this basis in terms of the matrix entries a_{ij} for f with respect to the basis for M.