CHAPTER VI

HOM AND TENSOR

1. The functor Hom

Let A be a ring (not necessarily commutative). Consider the collection of *all* left A-modules M and *all* module homomorphisms $f : M \to N$ of left A-modules. (The phrase "set of all ..." must be taken with a grain of logical salt to avoid the well known paradoxes of set theory. There are various ways to accomplish this. We refer the reader to a text on logic or set theory (such as Bourbaki's Set Theory volume)). The composition $f \circ g$ of two module homomorphisms—when defined—is also a module homomorphism so in keeping with our previous discussion of the category of sets, we shall refer to the above collection of modules and module homomorphisms as the *category of left A-modules*. In the case $A = \mathbf{Z}$, we call it simply the category of abelian groups.

Given two left A-modules M and N, we denote by $\mathrm{Hom}_A(M, N)$ the set of all modules homomorphisms $f : M \to N$. If f, $g \in \mathrm{Hom}_A(M, N)$, then as mentioned earlier $f + g$ defined by

$$(f + g)(x) = f(x) + g(x) \qquad x \in M$$

is a homomorphism of M to N as abelian groups. It is also a module homomorphism since

$$\begin{aligned}(f + g)(ax) = f(ax) + g(ax) = af(x) + ag(x) &= a(f(x) + g(x)) \\ &= a(f + g)(x).\end{aligned}$$

It is not hard to see that the binary operation $+$ makes $\mathrm{Hom}_A(M, N)$ into an abelian group. (In fact it is a subgroup of $\mathrm{Hom}(M, N)$ the group of all homomorphisms of M into N treated simply as abelian groups.)

Recall that for $M = N$, $\mathrm{Hom}(M, M)$ is in fact a ring which we called previously the endomorphism ring of M. It is not hard to see that $\mathrm{Hom}_A(M, M)$ is a subring.

Suppose $h : N \to N'$ is an A-module homomorphism. Define

$$\mathrm{Hom}_A(M, h) : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N')$$

by

$$\mathrm{Hom}_A(M, h)(f) = h \circ f.$$

$\mathrm{Hom}_A(M, h)(f)$ is a homomorphism of abelian groups since

$$\begin{aligned}(\mathrm{Hom}_A(M, h)(f + g))(x) &= (h \circ (f + g))(x) = h((f + g)(x)) \\ &= h(f(x) + g(x)) = h(f(x)) + h(g(x)) = (h \circ f)(x) + (h \circ g)(x) \\ &= \mathrm{Hom}_A(M, h)(f)(x) + \mathrm{Hom}_A(M, h)(g)(x) \\ &= (\mathrm{Hom}_A(M, h)(f) + \mathrm{Hom}_A(M, h)(g))(x).\end{aligned}$$

$\mathrm{Hom}_A(M, -)$ preserves composition since

$$\mathrm{Hom}_A(M, g \circ h) = \mathrm{Hom}_A(M, g) \circ \mathrm{Hom}_A(M, h)$$

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whenever $g \circ h$ is defined. (Check this for yourself!)

Note that we have associated with each object N in the category of A-modules an object $\mathrm{Hom}_A(M, N)$ in the category of abelian groups and with each homomorphism $f : N \to N'$ in the first category a homomorphism $\mathrm{Hom}_A(M, h)$ in the second category. Moreover, we have done this in such a way that composition is preserved. We say that $\mathrm{Hom}_A(M, -)$ is a *covariant functor*.

Similarly, suppose $g : M' \to M$ is an A-module homomorphism. (Note that the *direction* of the arrow is reversed relative to the case for N.) Define $\mathrm{Hom}_A(g, N) : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N)$ by

$$\mathrm{Hom}_A(g, N)(f) = f \circ g.$$

(Note that $\mathrm{Hom}_A(M, N)$ *reverses* the direction of the arrow.) As above, $\mathrm{Hom}_A(g, N)$ is a homomorphism of abelian groups. However

$$\mathrm{Hom}_A(g \circ h, N) = \mathrm{Hom}_A(h, N) \circ \mathrm{Hom}_A(g, N)$$

so that composition is reversed. For,

$$\begin{aligned}\mathrm{Hom}_A(g \circ h, N)(f) = f \circ (g \circ h) = (f \circ g) \circ h &= \mathrm{Hom}_A(h, N)(f \circ g)\\ &= \mathrm{Hom}_A(h, N)(\mathrm{Hom}_A(g, N)(f))\\ &= \mathrm{Hom}_A(h, N) \circ \mathrm{Hom}_A(g, N)(f).\end{aligned}$$

We say that $\mathrm{Hom}_A(-, N)$ is a *contravariant* functor.

We can subsume the above notions in a single concept. If $g : M' \to M$ and $h : N \to N'$ are module homomorphisms define

$$\mathrm{Hom}_A(g, h) : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N')$$

by

$$\mathrm{Hom}_A(g, h)(f) = h \circ f \circ g.$$

Then we may say that $\mathrm{Hom}_A(-, -)$ is a functor of two variables, contravariant in the first and covariant in the second.

PROPOSITION. *Let A be a ring. If*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is an exact sequence of left A-modules, then the induced sequence

$$0 \to \mathrm{Hom}_A(M'', N) \xrightarrow{\mathrm{Hom}(g,N)} \mathrm{Hom}_A(M, N) \xrightarrow{\mathrm{Hom}(f,N)} \mathrm{Hom}_A(M', N)$$

is exact. Similarly, if

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N''$$

is an exact sequence of left A-modules, then the induced sequence

$$0 \to \mathrm{Hom}_A(M, N') \xrightarrow{\mathrm{Hom}(N,f)} \mathrm{Hom}_A(M, N) \xrightarrow{\mathrm{Hom}(N,g)} \mathrm{Hom}_A(M, N'')$$

is exact.

PROOF. We prove the first statement. To avoid writing, we suppress the subscript A.

Suppose $\mathrm{Hom}(g, N)(j'') = j'' \circ g = 0$. Since g is an epimorphism, it follows that $j'' = 0$. Hence, $\mathrm{Ker}(\mathrm{Hom}(g, N)) = \{0\}$ and $\mathrm{Hom}(g, N)$ is a monomorphism.

By functorality, $\mathrm{Hom}(f, N) \circ \mathrm{Hom}(g, N) = \mathrm{Hom}(f \circ g, N) = \mathrm{Hom}(0, N)$. However, it is not hard to see that the latter is 0. Suppose on the other hand that $\mathrm{Hom}(f, N)(j) = j \circ f = 0$. Since $\mathrm{Ker}\, g = \mathrm{Im}\, f$, it follows that j vanishes on $\mathrm{Ker}\, g$. Define $j'' : M'' \to N$ by $j''(g(x'')) = j(x)$. Since j vanishes on $\mathrm{Ker}\, g$ and g is an epimorphism, j'' is well defined, and it is easy to see that it is an A-module homomorphism. Then $\mathrm{Hom}(g, N)(j'') = j'' \circ g = j$ as required.

The second statement is proved in a similar way. $\square$

We call the properties of $\mathrm{Hom}(-, -)$ described in the above proposition *left exactness*.

PROPOSITION. *Let M', M, and N be A-modules and suppose $f, g \in \mathrm{Hom}_A(M', M)$. Then*

$$\mathrm{Hom}_A(f+g, N) = \mathrm{Hom}_A(f, N) + \mathrm{Hom}_A(g, N) : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N).$$

Similarly, if M, N, and N' are A-modules with $f, g \in \mathrm{Hom}_A(N, N')$ then

$$\mathrm{Hom}_A(M, f+g) = \mathrm{Hom}_A(M, f) + \mathrm{Hom}_A(M, g).$$

PROOF. Again leaving off the subscript A, we have for $j \in \mathrm{Hom}_A(M, N)$,

$$\begin{aligned}\mathrm{Hom}(f+g, N)(j) = j \circ (f+g) = j \circ f + j \circ g &= \mathrm{Hom}(f, N)(j) + \mathrm{Hom}(g, N)(j) \\ &= (\mathrm{Hom}(f, N) + \mathrm{Hom}(g, N))(j).\end{aligned}$$

The other case is similar. □

COROLLARY. *Let M', M'' , and N be A-modules. Then*

$$\mathrm{Hom}_A(M' \oplus M'', N) \cong \mathrm{Hom}_A(M', N) \oplus \mathrm{Hom}_A(M'', N).$$

Similarly,

$$\mathrm{Hom}_A(N, M' \oplus M'') \cong \mathrm{Hom}_A(N, M') \oplus \mathrm{Hom}_A(N, M'').$$

PROOF. We leave this for the student to investigate. The crucial idea is that the assertion $M \cong M' \oplus M''$ can be reduced to the existence of homomorphisms $i' : M' \to M$, $j' : M \to M'$, $i'' : M'' \to M$, and $j'' : M \to M''$ such that

$$j'i' = id_{M'}, j''i'' = id_{M}'',$$

and

$$i'j' + i''j'' = id_M.$$

Then use the previous proposition, and the fact that Hom preserves or reverses compositions as required. □

Suppose A is a *commutative* ring. In that case, we may endow $\mathrm{Hom}_A(M, N)$ with a structure of an A-module. For $f \in \mathrm{Hom}_A(M, N)$, define $af \in \mathrm{Hom}_A(M, N)$ by

$$(af)(x) = f(ax) = af(x) \qquad x \in M.$$

af so defined is in fact an A-module homomorphism since

$$(af)(bx) = a(f(bx)) = a(bf(x)) = b(af(x)) = b((af)(x)).$$

Note that this argument depends on the commutativity of A. If A were not commutative, we could only conclude that $af \in \mathrm{Hom}_Z(M, N)$. It is routine to check that the operation af satisfies the requirements for an A-module.

As you might expect, if A is commutative, $\mathrm{Hom}_A(-, -)$ is in fact a functor into the category of A-modules. Thus, not only is $\mathrm{Hom}_A(M, N)$ an A-module, but given A-module homomorphisms g and h, $\mathrm{Hom}_A(g, h)$ is an A-module homomorphism. The proof is another tedious grinding through the definitions. It is left to you as an exercise.

We obtain interesting special cases of the functor Hom by taking M or N to be the ring A itself. Suppose first that $M = A$. Even in the non-commutative case we may endow $\mathrm{Hom}_A(A, N)$ with a left A-module structure by setting $(af)(x) = f(xa)$. For, it is easy to check that $af \in \mathrm{Hom}_A(A, N)$ (as above), and that we get a left A-module structure. The only tricky point is establishing associativity. Thus,

$$((ab)f)(x) = f(xab) = f((xa)b) = (bf)(xa) = a(bf)(x)$$

so $(ab)f = a(bf)$. (What about defining $(af)(x) = f(ax)$?)

Proposition. *Let A be a ring, and suppose that N is a left A-module. Then $f \rightsquigarrow f(1)$ defines an A-module isomorphism*

$$\mathrm{Hom}_A(A, N) \cong N.$$

Proof. Define a function $N \to \mathrm{Hom}_A(A, N)$ by $x \rightsquigarrow g_x$ where $g_x(c) = cx$. It is not hard to check that the function defined in the proposition and the function just defined are in fact A-module homomorphisms given the above definitions. (However, you have to concentrate to be sure you are proving the correct assertions using the correct definitions and reasons.) The two functions are inverses since

$$f \rightsquigarrow f(1) \rightsquigarrow g_{f(1)} \text{ where } g_{f(1)}(c) = cf(1) = f(c)$$

so $g_{f(1)} = f$ and $f \rightsquigarrow f$ under the relevant composition. On the other hand

$$x \rightsquigarrow g_x \rightsquigarrow g_x(1) = 1x = x$$

so the other composition is also the identity. □

The above isomorphism is an example of what is called a "natural" isomorphism. That is we can be sure that it will be consistent with other operations we may be called upon to perform. This notion can be made much more precise in the context of general category theory, and we shall do so later.

Consider next the case $N = A$. Then we may make $\mathrm{Hom}_A(M, A)$ into a *right* A-module by

$$(fa)(x) = f(x)a.$$

You should check that this works, and that for a general ring, setting $(af)(x) = af(x)$ does not make $af : M \to A$ into an A-module homorphism.

If we assume A is commutative, we don't have to worry about distinguishing left modules from right modules, and problems like that alluded to in the above paragraph do not arise. So we shall generally restrict attention to that case when discussing $\mathrm{Hom}_A(M, A)$, which is called the *dual* module.

The most familiar case is that for which $A = k$ is a field. Then $\mathrm{Hom}_k(M, k)$ is called the dual space and often denoted M^*. Much of what is done in the theory of dual spaces in linear algebra can be expressed in the present more abstract framework.

Exercises.

1. Let A be a ring and let F be a functor from the category of left A-modules to the category of abelian groups. That to each left A-module M, there is an abelian group $F(M)$, to each module homomorphism $f : M \to N$, there is a homomorphism $F(f) : F(M) \to F(N)$, $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined, and $F(\mathrm{Id}_M) = \mathrm{Id}_{F(M)}$. We say that F is additive if $F(f + g) = F(f) + F(g)$ for $f, g \in \mathrm{Hom}_A(M, N)$. Show that if F is additive, it carries finite direct sums into finite direct sums. This generalizes the corresponding statement for Hom_A which is discussed in the text.

2. Let A be a ring and let M be a left A-module. Show that $\mathrm{Hom}_A(M, M)$ is a subring of $\mathrm{Hom}(M, M)$ where the latter is just the endomorphism ring of M viewed as an abelian group.

3. Let A be a ring. It is stated as a Proposition in the text that the functor $\mathrm{Hom}_A(\_, \_)$ is left exact. The proof is given that it is left exact in the first variable. Write out the proof that it is left exact in the second variable. (See the statement of the Proposition if you are in doubt as to what needs proving.)

4. Let A be a commutative ring, and suppose f and g are A-module homomorphisms. Show that $\mathrm{Hom}_A(f, g)$ is an A-module homomorphism. (We already know it is a homomorphism of abelian groups.)

5. Suppose A is a commutative ring, and F is a free A-module of finite rank r.

(a) Show that $F^* = \mathrm{Hom}_A(F, A)$ is also free of rank r. Hint. Any A-module homomorphism on a free module is completely determined by what it does on a basis $\{x_1, x_2, \dots, x_r\}$. Define x_i^* by $x_i^*(x_j) = \delta_{ij}$.

(b) What can you say about the relationship between F and $(F^*)^*$? Hint: there is a natural homomorphism $F \to F^{**}$. Review your linear algebra, define this homomorphism, and show it is an isomorphism.

2. The Tensor Product

In linear algebra one devotes a certain amount of attention to bilinear forms. We recall the basic definitions. Let F be a field and let V be a vector space over F. A function $h : V \times V \to F$ is called a bilinear form if

$$\begin{aligned} h(ax + by, z) &= ah(x, z) + bh(y, z) \\ h(x, ay + bz) &= ah(x, y) + bh(x, z) \end{aligned}$$

for all $a, b \in F$ and $x, y, z \in V$. Bilinear forms are important in many situations. More generally, if U, V, and W are vector spaces, we can consider bilinear functions $h : U \times V \to W$, and the theory of such objects can be developed by analogy with the theory of linear functions. It turns out, however, that by introducing an appropriate functor—called the tensor product—we may reduce much of the new theory to the old theory.

We proceed with such a program which we develop in a much more general context than that of vector spaces.

Let A be a ring. If M is a *right* A-module, N is a *left* A-module, and L is an abelian group, we say that a function $h : M \times N \to L$ is bilinear provided

$$\begin{aligned} h(x' + x'', y) &= h(x', y) + h(x'', y) \qquad && x', x'' \in M, y \in N \\ h(x, y' + y'') &= h(x, y') + h(x, y'') \qquad && x \in M, y', y \in N \\ h(xa, y) &= h(x, ay) \qquad && x \in M, y \in N, a \in A. \end{aligned}$$

Note that since we do not assume that A is commutative, we shall encounter some of the difficulties described earlier for Hom. In particular, we only consider functions into an abelian group so that the behavior of the function relative to action by ring elements a is more complicated to state than above.

The tensor product $M \otimes_A N$ will be characterized as the "smallest" abelian group L for which there is such a bilinear function. In particular, we shall show there is an abelian group T and a bilinear function $\tau : M \times N \to T$ with the property that given any other bilinear function $h : M \times N \to L$ there is a unique homomorphism of abelian groups $H : T \to L$ such that $h = H \circ \tau$.

$$\begin{tikzcd} M \times N \arrow[d, "\tau"'] \arrow[dr, "h"] & \\ T \arrow[r, "H"'] & L \end{tikzcd}$$

By the usual argument capitalizing on the uniqueness of H, such an object—if it exists—is unique up to unique isomorphisms preserving the relevant diagrams. (Refer back to previous arguments in which a universal mapping property played a role.) Hence, our only problem is to show that such an object does in fact exist.

We now show how to construct an abelian group T and a function $\tau : M \times N \to T$ with the relevant property for a tensor product. First let F be the free abelian group with Z-basis the *set* $M \times N$. Let R be the subgroup of F spanned by all elements of one of the following forms

$$\begin{aligned} (x' + x'', y) - (x', y) - (x'', y) \qquad && x', x'' \in M, y \in N, \\ (x, y' + y'') - (x, y') - (x, y'') \qquad && x \in M, y', y'' \in N, \\ (xa, y) - (x, ay) \qquad && x \in M, y \in N, a \in A. \end{aligned}$$

Let $T = M \otimes_A N = F/R$. Let $x \otimes y$ denote the coset in T of (x, y) in F, and define $\tau : M \times N \to T$ by $\tau(x, y) = x \otimes y$. It is clear from the definition of T that τ is bilinear since we have put in R just exactly the elements necessary for that to be so. Suppose $h : M \times N \to L$ is any other bilinear function into an abelian group L. Define $\tilde{H} : F \to L$ by $\tilde{H}((x, y)) = h(x, y)$. Since $\{(x, y) \mid x \in M, y \in N\}$ is a basis for F, $\tilde{H}$ is completely determined by this specification. $\tilde{H}$ takes the subgroup R into zero because the bilinearity of h forces $\tilde{H}$ to take each element of the spanning set listed above into 0. For example,

$$\begin{gathered}\tilde{H}((x'+x'',y)-(x',y)-(x'',y))=\\ \tilde{H}((x'+x'',y))-\tilde{H}((x',y))-\tilde{H}((x'',y))=\\ h(x'+x'',y)-h(x',y)-h(x'',y)=0.]\end{gathered}$$

It follows that $\tilde{H}$ induces a homomorphism $H: T \to L$ given by

$$H(x\otimes y)=h(x,y).$$

Since by definition, $x\otimes y=\tau(x,y)$, it follows that $H\circ\tau=h$ as required. Finally, H is uniquely determined by the requirement $H\circ\tau=h$, since that equation tells us that $H(x\otimes y)=h(x,y)$, and the set of all $x\otimes y$ with $x\in M$ and $y\in N$ clearly spans T.

We have succeeded in associating to each pair M (a right A-module), N (a left A-module) an abelian group $M\otimes_A N$. However, as for Hom, we really want a functor, so we have to consider what happens for homomorphisms. Suppose $f: M\to M'$ is a homomorphism of right A-modules and $g: N\to N'$ is a homomorphism of left A-modules. Define $f\times g: M\times N\to M'\otimes_A N'$ by $(f\times g)(x,y)=f(x)\otimes g(y)$. It is easy to see that this function is bilinear. It follows from the definition that there is a unique homomorphism $f\otimes g: M\otimes_A N\to M'\otimes_A N'$ such that

$$(f\otimes g)(x\otimes y)=(f\otimes g)(\tau(x,y))=(f\times g)(x,y)=f(x)\otimes g(y)$$

i.e. such that

$$(f\otimes g)(x\otimes y)=f(x)\otimes g(y).$$

Again, using this same universal mapping property, it is not hard to see that

$$f\circ f'\otimes g\circ g'=(f\otimes g)\circ(f'\otimes g')$$

when these compositions are defined. Thus, the tensor product is a covariant functor of two arguments.

Proposition. *Let $f, g: M\to M'$ be homomorphisms of right A-modules, and let $h: N\to N'$ be a homomorphism of left A-modules. Then*

$$(f+g)\otimes h=f\otimes h+g\otimes h.$$

Similarly, if the roles of the two variables are reversed.

Proof. By definition,

$$\begin{aligned}((f+g)\otimes h)(x\otimes y)&=(f+g)(x)\otimes h(y)=(f(x)+g(x))\otimes h(y)\\ &=f(x)\otimes h(y)+g(x)\otimes h(y)\\ &=(f\otimes h)(x\otimes y)+(g\otimes h)(x\otimes y)\\ &=(f\otimes h+g\otimes h)(x\otimes y).\end{aligned}$$

Since the set of all $x\otimes y$ spans $M\otimes_A N$, the desired formula follows. □

Corollary. *Let M, M', and M'' be right A-modules with $M\cong M'\oplus M''$, and let N be a left A-module. Then*

$$M\otimes_A N\cong M'\otimes_A N\oplus M''\otimes_A N$$

and similarly for the second variable.

Proof. As in the case of Hom, the result follows because the direct sum relation may be specified in terms of composition and addition of appropriate homomorphisms. □

As for Hom, the tensor product has appropriate exactness properties. As in the case of Hom, we write $f\otimes N$ for $f\otimes id_N$.

PROPOSITION. *Let*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact sequence of right A-modules and let N be a left A-module. Then

$$M' \otimes_A N \xrightarrow{f \otimes N} M \otimes_A N \xrightarrow{g \otimes N} M'' \otimes_A N \to 0$$

is an exact sequence of abelian groups. Similarly for the other variable.

PROOF. First, it is clear that $g \otimes M$ is an epimorphism since each generator $x'' \otimes y$ of $M'' \otimes N$ is clearly of the form $g(x) \otimes y$ for some $x \in M$.

By functorality, since $g \circ f = 0$, it follows that $(g \otimes N) \circ (f \otimes N) = 0$ so that $\operatorname{Im}(f \otimes N) \subseteq \operatorname{Ker}(g \otimes N)$. To show the inclusion in the other direction, we argue as follows. Let $\pi : M \otimes N \to L = M \otimes N / \operatorname{Im}(f \otimes N)$ be the canonical projection. Define a function $p : M'' \times N \to L$ by

$$p(x'', y) = \pi(x \otimes y) \text{ where } g(x) = x''.$$

If $g(x_1) = g(x) = x''$, then $g(x_1 - x) = 0$, and $x_1 - x = f(x')$ for some $x' \in M'$. Hence, $g(x_1) \otimes y - g(x) \otimes y = (g(x_1) - g(x)) \otimes y = f(x') \otimes y \in \operatorname{Im}(f \otimes N)$. Thus, p is well defined. By similar calculations, it is not hard to see that p is bilinear. It follows that there is a homomorphism $P : M'' \otimes N \to L$ such that

$$P(x'' \otimes y) = p(x'', y) = \pi(x \otimes y) \text{ for } x \text{ with } g(x) = x''.$$

Thus, $P \circ (g \otimes N) = \pi$.

$$\begin{tikzcd}
M \otimes N \arrow[r, "g \otimes N"] \arrow[d, "\pi"'] & M'' \otimes N \arrow[dl, "P"] \\
L &
\end{tikzcd}$$

It follows that anything in $\operatorname{Ker}(g \otimes N)$ is also in $\operatorname{Ker} \pi = \operatorname{Im}(f \otimes N)$ as claimed. $\square$

Assume next that A is a commutative ring. Then we need not distinguish between left and right modules, and we can write all actions on the left for convenience. $xa \otimes y = x \otimes ay$ becomes simply $ax \otimes y = x \otimes ay$. Also, we may make $M \otimes_A N$ into an A-module by defining

$$a(x \otimes y) = ax \otimes y = x \otimes ay.$$

This way of defining the action makes some implicit assumptions which must be justified. Namely, in effect, we are claiming that the map $(x, y) \rightsquigarrow ax \otimes y$ is bilinear. (Prove it!) Given that, it induces a map on the tensor product such that $x \otimes y \rightsquigarrow ax \otimes y$ which defines the desired action. It is not hard to see that this action yields an A-module structure on $M \otimes_A N$ as claimed.

In the commutative case, it is not hard to see that if f and g are A-module homomorphisms, then so is $f \otimes g$. Thus, - $\otimes$ - is a functor of two variables from the category of A-modules to the category of A-modules.

PROPOSITION. *Let A be a commutative ring. If M and N are A-modules, then there is a natural isomorphism*

$$M \otimes_A N \cong N \otimes_A M.$$

If M, N, and L are A-modules, then there is a natural isomorphism

$$(M \otimes_A N) \otimes_A L \cong M \otimes_A (N \otimes_A L).$$

PROOF. We leave it to the student to invent the appropriate isomorphisms. Remember to show that they are well defined.

Note: There is an associativity law for tensor product in the non-commutative case but it requires a framework which somewhat complicated to describe.

Exercises.

1. Let A be a ring. It may be viewed both as a left A-module and as a right A-module. With the latter point of view, we may form $A \otimes_A M$ for any left A-module M. Show that

$$a(b \otimes x) = ab \otimes x \qquad a,\ b \in A,\ x \in M$$

gives $A \otimes_A M$ a well defined structure as a left A-module. (This may be a little bit subtler than you imagine.) Show that $A \otimes_A M \cong M$ as left A-modules by defining suitable module homomorphisms in both directions and showing they are inverse to one another.

2. For the following problems, all tensor products are over $\mathbf{Z}$.
   (a) Show for any abelian group N that $(\mathbf{Z}/n\mathbf{Z}) \otimes N \cong N/nN$.
   (b) Show that $(\mathbf{Z}/n\mathbf{Z}) \otimes (\mathbf{Z}/m\mathbf{Z}) \cong \mathbf{Z}/d\mathbf{Z}$ where $d = \gcd(n, m)$.
   (c) Show that $T \otimes \mathbf{Q} = 0$ if T is a torsion group. **Hint:** Show that $T \otimes \mathbf{Q}$ is a torsion group, and also show that multiplication by any $n \neq 0$ is an isomorphism.