RECENT DEVELOPMENTS IN THE STUDY OF SINGULARITIES IN POSITIVE, ZERO, AND MIXED CHARACTERISTIC

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1. Introduction

The study of singularities is a fascinating topic in mathematics, with deep connections to topology, Hodge theory, birational geometry, analysis, and arithmetic. In its purest form, it is the study of some of the most fundamental objects of commutative algebra: local rings.

At the same time, singularities are essential in algebraic geometry, even if one is primarily interested in smooth projective varieties. Indeed, a morphism between smooth projective varieties will often have singular fibres. More broadly, even when starting with smooth projective varieties, the Minimal Model Program – whose main objective is the classification of all projective varieties – inevitably introduces singularities as it proceeds. The interplay between the nature of singularities and the geometry of

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projective varieties has been at the heart of some of the most profound breakthroughs in algebraic geometry.

The goal of this survey is to discuss some recent developments in the study of singularities in positive, zero, and mixed characteristic. As motivation, we begin by considering the case of hypersurfaces, which serves to introduce the general setting and guiding ideas for the results that follow. However, the scope of this survey extends far beyond hypersurface or complete intersection singularities, and more general notions will be introduced in subsequent chapters.

We also emphasise that the selection of topics reflects the authors' mathematical interests and is necessarily biased. The theory of singularities is vast and extends well beyond what can be covered in this survey.

Singularities via integrals. The most elementary way to introduce the singularities of interest to this survey is via a basic integral. Although this is neither the standard nor the most general approach to singularities, we find it to be a particularly intuitive and engaging point of entry.

We begin with a simple calculation that is likely familiar from undergraduate calculus courses, arising in the context of radial changes of coordinates:

$$\begin{aligned}\int_{B(0,1)} \frac{1}{(u^2+v^2)^\lambda} dudv &= \int_0^{2\pi}\int_0^1 \frac{1}{r^{2\lambda}} r dr d\theta \\ &= \begin{cases} 2\pi \ln(r)|_0^1 & \text{if } \lambda = 1 \\ -2\pi \frac{1}{(2\lambda-2)r^{2\lambda-2}}|_0^1 & \text{if } \lambda \neq 1. \end{cases}\end{aligned}$$

Here $B(0,1)$ is a ball in $\mathbb{R}^2$ of radius 1 centred at the origin. Note that the integrand can also be written as $\frac{1}{|x|^{2\lambda}}$ for $x = u + iv \in \mathbb{C}$. In particular, we get that

$$\int_{B(0,1)} \frac{1}{|x|^{2\lambda}} dudv = \begin{cases} \infty & \text{if } \lambda \geq 1 \\ < \infty & \text{if } \lambda < 1. \end{cases} \tag{1.0.1}$$

This motivates the following definition.

Definition 1.1. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial such that $f(0) = 0$. We say that f is *log canonical at the origin* if for all $t < 1$ there exists an open neigbhourhood U containing the origin 0 such that

$$\int_U \frac{1}{|f(x_1, \ldots, x_n)|^{2t}} < \infty.$$

In general, the supremum over all t such that $\int_U \frac{1}{|f(x_1,\ldots,x_n)|^{2t}} < \infty$ for some U as above, is called the *log canonical threshold at the origin* and denoted $\mathrm{LCT}_0(f)$.

By (1.0.1) we see that the polynomial $x \in \mathbb{C}[x]$ is log canonical at every point.

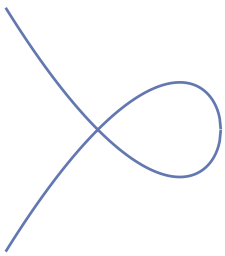
Exercise 1.2. Let $a_1, \dots, a_n$ be non-negative rational numbers. Using Fubini's theorem, verify that for an open neigbhourhood $U \subseteq \mathbb{C}^n$ of the origin

$$\int_U \frac{1}{|x_1|^{a_1} \cdots |x_n|^{a_n}} < \infty$$

if and only if $a_i < 2$ for every i. In particular, $f := x_1 \cdots x_n$ is log canonical.

Using basic inequalities involving absolute values and reciprocals, it is easy to deduce from the above exercise that $\mathrm{LCT}_0(f) \leq 1$ for every $f \in \mathbb{C}[x_1, \dots, x_n]$ vanishing at 0.

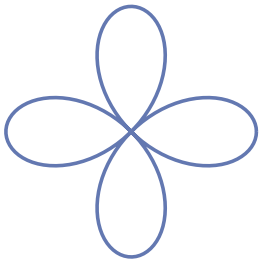
Example 1.3. Consider the following three polynomials. One can check that the



$y^2 + x^3 - x^2 = 0$



$y^2 - x^3 = 0$



$(x^2 + y^2)^3 - (x^2 - y^2)^2 = 0.$

first example is log canonical at the origin, whereas the second and third are not. A well-known exercise, commonly assigned in courses on birational geometry, asks students to compute that (see Example 2.8):

$$\mathrm{LCT}_0(y^2 - x^3) = \frac{5}{6}. \tag{1.3.1}$$

In what follows, we briefly discuss four methods for checking log canonicity and for computing the log canonical threshold, and elaborate on each of these approaches in later sections.

Analysis. Using various inequalities, one may attempt to bound the relevant integrals. We will not discuss this approach in detail in this survey. Let us note, however, that the integral viewpoint and the use of functional analysis – particularly L^2 methods – are of fundamental importance and have led to many breakthroughs in birational geometry and the study of singularities.

Birational geometry and blow-ups. One can read off log canonicity and the log canonical threshold from a resolution of singularities of the pair (X, D), which leads to the standard definition of log canonicity (see Section 2).

Arithmetic geometry. One may try to reduce the equation modulo a prime $p > 0$ and use arithmetic methods to study log canonicity. For example, one can show that for a polynomial $f \in \mathbb{Z}[x_1, \dots, x_n]$, if

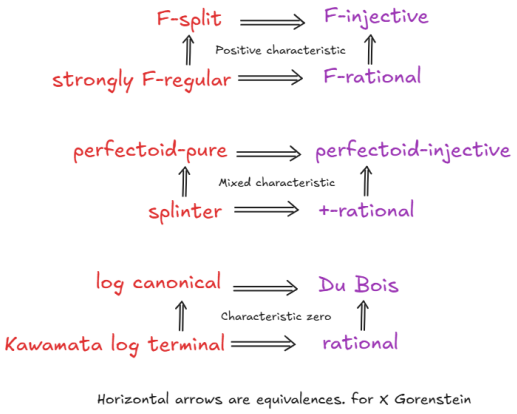
$$f^{p-1} \bmod p \notin (x_1^p, \dots, x_n^p)$$

for some sufficiently large prime p, then f is log canonical. We will discuss this method in detail later in connection with Frobenius singularities (Section 3).

Differential equations, topology, and Hodge theory. In this approach, one constructs a family of differential operators parametrised by s such that applying the operator to f^{s+1} yields a scalar multiple of f^s, where the scalar depends only on s. These scalars vary polynomially in s, producing the *Bernstein–Sato polynomial*; in turn, its roots determine the log canonical threshold. This construction will lead us naturally to the topic of higher singularities and Hodge theory (Section 5), and relates log canonicity and the log canonical threshold to topological invariants.

How does the mixed characteristic setting fit into this picture? Briefly speaking, it serves as a bridge between the characteristic zero and positive characteristic worlds. As one of the highlights of this survey, we will explain how recent developments in mixed characteristic commutative algebra, following Scholze's work on perfectoid spaces, can be used to significantly strengthen the arithmetic methods for checking log canonicity (see Section 4).

As the discussion above indicates, there is much to unravel. In addressing the problem of understanding and calculating the invariants introduced above – and, more importantly, in providing an overview of recent developments in the study of singularities – a secondary goal emerges: to introduce the whole zoo of invariants in zero, positive, and mixed characteristic. Although their sheer number may seem overwhelming, we aim to present a coherent picture highlighting the deep connections among them.



Remark 1.4. Throughout this survey, we restrict attention to normal singularities to avoid introducing additional technicalities. Moreover, since this survey is aimed at graduate students in algebraic geometry, we present the material using a geometric language, focusing on affine varieties for simplicity and clarity of exposition, although most definitions and results extend to general excellent rings. While the discussed topics have deep implications and strong connections to global geometry, limitations of space and scope prevent us from pursuing them in this survey.

2. Classical singularities in characteristic zero

In this section, the base field k is algebraically closed of characteristic 0. The existence of resolutions of singularities is known in this setting thanks to Hironaka.

We begin this section with a discussion of Kawamata log terminal and log canonical singularities, which arise naturally in the framework of the Minimal Model Program.

2.1. **Kawamata log terminal and log canonical.** In this subsection, X is a normal variety and is $\mathbb{Q}$-*Gorenstein*, meaning that some multiple mK_X of a canonical divisor is Cartier. Although the $\mathbb{Q}$-Gorenstein assumption may appear ad hoc, it is in fact quite natural: many singularities satisfy this condition, and it is preserved under the Minimal Model Program. A special case is given by *quasi-Gorenstein* singularities, where K_X itself is Cartier. Quasi-Gorenstein singularities that are Cohen–Macaulay are called *Gorenstein*; examples include hypersurfaces and complete intersections.

Definition 2.1. With assumptions as above, let $f\colon Y \to X$ be a *log resolution* of X, this means that Y is smooth and the exceptional locus $\operatorname{Exc}(f)$ is simple normal crossing. Let $E_1, \dots, E_n$ be the irreducible components of $\operatorname{Exc}(f)$ and write

$$f^*K_X = K_Y + \sum_{i=1}^{n} a_i E_i \quad \text{for } a_i \in \mathbb{Q}. \tag{2.1.1}$$

We say that

(1) X is *Kawamata log terminal* if $a_i < 1$ for every $1 \le i \le n$, and
(2) X is *log canonical* if $a_i \le 1$ for every $1 \le i \le n$.

We call such X *klt* or *lc*, for short. The numbers a_i are known as *discrepancies*.

Remark 2.2. One can define the above notions for singularities over fields of positive characteristic even in the absence of resolutions of singularities, by verifying the discrepancy inequalities for every projective birational morphism over X.

Example 2.3. Consider a smooth curve C in a smooth projective surface S such that $C \cong \mathbb{P}^1$ and $C^2 = -a$ for some integer $a > 0$. Suppose that there exists a contraction $f\colon S \to Z$ of the curve C. We omit the verification that Z is $\mathbb{Q}$-Gorenstein; assuming this, we may write for some $t \in \mathbb{Q}$:

$$f^*K_Z = K_S + tC.$$

Using the adjunction formula $(K_S + C) \cdot C = \deg K_C = -2$, we compute

$$0 = f^*K_Z \cdot C = (K_S + tC) \cdot C = (K_S + C) \cdot C + (t-1)C^2 = -2 - (t-1)a.$$

It follows that $t = 1 - \frac{2}{a}$, and hence

$$f^*K_Z = K_S + \left(1 - \frac{2}{a}\right) C.$$

Therefore, Z has klt singularities.

Exercise 2.4. Verify that if C is an elliptic curve and Z is $\mathbb{Q}$-Gorenstein, then it is log canonical but not klt. Check that if $g(C) \ge 2$, then Z is not log canonical.

Next, we generalise the above setting to pairs. In what follows, given a proper birational map $f\colon Y \to X$ which is an isomorphism over an open locus $U \subseteq X$ with $\operatorname{codim}_X(X \setminus U) \geq 2$, the *strict transform* $f_*^{-1}D$ of a prime divisor D is defined to be the closure of $D|_U$ in Y.

Moreover, a $\mathbb{Q}$-*divisor* D is a formal sum $\sum b_i D_i$ where $a_i \in \mathbb{Q}$ and D_i are distinct prime divisors. It is *effective* if $a_i \geq 0$ for every i, and it is $\mathbb{Q}$-*Cartier* if some multiple mD is a Cartier divisor.

Definition 2.5. With assumptions and notations as before, let $D = \sum a_i D_i$ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We take $f\colon Y \to X$ to be the log resolution of $(X, \operatorname{Supp} D)$, this means that Y is smooth and $\operatorname{Exc}(f) \cup f^*(D)$ is simple normal crossing. Let $E_1, \dots, E_n$ be the irreducible components of $\operatorname{Exc}(f)$ and write

$$f^*(K_X + D) = K_Y + \sum_i a_i f_*^{-1} D_i + \sum_j b_j E_j. \tag{2.5.1}$$

We say that (X, D) is *klt* (resp. *lc*) if $a_i, b_j < 1$ (resp. $a_i, b_j \leq 1$) for every i and j.

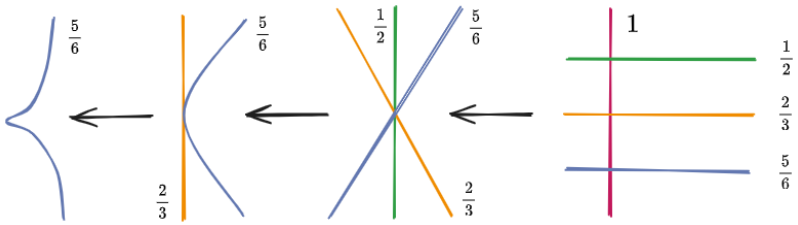
Remark 2.6. It is not necessary to assume that both K_X and D are $\mathbb{Q}$-Cartier. It is enough to work with D such that $K_X + D$ is $\mathbb{Q}$-Cartier to define when (X, D) is log canonical or klt. For simplicity, we do not strive to make the most general definitions.

We say that (X, D) is klt or lc at a point $P \in X$, if it is so on some open Zariski neighbourhood containing P. For X log canonical and a point $P \in X$, the *log canonical threshold* $\operatorname{lct}_P(X; D)$ is the supremum over rational numbers $t \geq 0$ such that (X, tD) is log canonical at P.

Theorem 2.7 ([Laz04])**.** *Let $X := \mathbb{A}^n_{\mathbb{C}}$ and let $D := \operatorname{div}(f)$ for $f \in \mathbb{C}[x_1, \dots, x_n]$. The the definitions of log canonicity and the log canonical thresholds from Definition 1.1 and Definition 2.5 agree.*

Proof. This follows from Exercise 1.2 and integration by substitution. Namely, by definition, discrepancies measure the difference between volume forms on X and volume forms on the log resolution Y. We leave it as an exercise to the reader to sort out the details. □

Example 2.8. Using the above theorem it is relatively easy to find the log canonical threshold $\frac{5}{6}$ of a cuspidal curve $C \subseteq \mathbb{A}^2_{\mathbb{C}}$ as in (1.3.1). The picture below describes the



simplest log resolution of $(\mathbb{A}^2, C)$. The numbers on the right indicate the discrepancies of $(\mathbb{A}^2, \frac{5}{6}C)$ at each curve. We leave it to the reader to compute all these numbers using intersection theory as in Example 2.3.

Example 2.9. A particularly important class of singularities in birational geometry is given by *rational double points*, listed below. These are exactly the Gorenstein klt surface singularities in characteristic 0.

$$\begin{aligned} A_n &: \quad f(x) = z^2 + x^2 + y^{n+1} \\ D_n &: \quad f(x) = z^2 + y(x^2 + y^{n-2}) \quad (n \geq 4) \\ E_6 &: \quad f(x) = z^2 + x^3 + y^4 \\ E_7 &: \quad f(x) = z^2 + x(x^2 + y^3) \\ E_8 &: \quad f(x) = z^2 + x^3 + y^5. \end{aligned}$$



$x^2 + y^2 - z^2 = 0$

Next, we discuss *inversion of adjunction*.

Theorem 2.10 ([Kaw07])**.** *With assumptions as earlier, suppose that D is a normal Cartier divisor. Then (X, D) is log canonical along D if and only if D is log canonical.*

In particular, if D is log canonical, then X is log canonical along D. A similar statement holds for klt singularities [KM98]. Providing full details, however, would require a discussion of further classes of singularities arising in the Minimal Model Program, such as *plt* and *dlt*, which we shall not pursue here.

The key **slogan** in singularity theory is that if a Cartier divisor $D \subseteq X$ has *mild singularities*, then the ambient space X should also have *mild singularities* along D. Ideally, every reasonable class of singularities satisfies this principle, known as *deformation*. For singularities satisfying *deformation* and a *Bertini-type* property[1], if a fibre of a proper morphism has mild singularities, then so do all nearby fibres.

In favourable situations, the above principle can even be promoted to an if-and-only-if statement, known as *inversion of adjunction*, by introducing pairs:

$$\begin{aligned} D \text{ has mild singularities} &\iff (X, D) \text{ has mild singularities along } D \\ &\implies X \text{ has mild singularities along } D. \end{aligned}$$

Remark 2.11. The notion of log canonicity can be extended beyond the normal setting, leading to *semi-log-canonical* (*slc*) singularities. These play a fundamental role in moduli theory, since natural compactifications of moduli spaces of smooth varieties typically parametrise varieties with slc singularities along the boundary.

There are many further directions closely connected to the study of singularities of the Minimal Model Program. These include the investigation of the fundamental group of such singularities [Bra21] and the dual complex of the exceptional divisor of a resolution [dFKX17]. The log canonical threshold occupies a central position in the Minimal Model Program and plays a key role in the proof of boundedness

[1]In particular, if X has mild singularities and $H \subseteq X$ is a general member of a base-point-free linear system, then H also has mild singularities. We will not discuss this property in detail, but almost all singularities considered in this survey will satisfy it.

of singular Fano varieties [HMX14, Bir21]; it also relates to the study of arc spaces [Mus02]. In recent years, strong links have emerged with K-stability, notably through the normalised volume – an invariant measuring klt singularities – which has deep connections to moduli theory and questions of termination [LXZ22, HLZ25].

2.2. **Rational and Du-Bois.** While klt and lc singularities are of Minimal Model Program flavour, rational and Du Bois ones are cohomological in nature. Later on, we will see how they fit into the broader framework of Hodge-theoretic singularities.

Definition 2.12. Let X be a normal variety over k. We say that X has *rational singularities* if $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$ and a resolution $f\colon Y \to X$ of X.

One can check that this notion is independent of the choice of resolution.

Remark 2.13. Verifying that a singularity is rational is important in the study of the cohomology of varieties, since it allows one to reduce many questions to the smooth case. Indeed, if a variety X has rational singularities, $\mathcal{F}$ is a vector bundle on X, and $f\colon Y \to X$ is a resolution, then $H^i(X, \mathcal{F}) \cong H^i(Y, f^*\mathcal{F})$ for all i.

Fortunately, the notion of rationality can be related to that of klt-ness, the latter being much easier to verify in practice in the geometric setting, since it reduces to computing discrepancies via intersection theory as in Example 2.8.

Proposition 2.14. *Let X be a $\mathbb{Q}$-Gorenstein normal variety over k. If X is klt, then it has rational singularities. Moreover, the converse holds true when X is Gorenstein.*

Proof assuming X is CM. Let $f\colon Y \to X$ be a log resolution of singularities. The key component of the proof is Grauert-Riemenschneider vanishing stating that $R^i f_* \omega_Y = 0$ for $i > 0$. Then:

$$X \text{ is klt} \overset{(\star)}{\Longrightarrow} f_*\omega_Y = \omega_X \iff Rf_*\omega_Y^\bullet = \omega_X^\bullet \overset{(\star\star)}{\Longleftrightarrow} Rf_*\mathcal{O}_Y = \mathcal{O}_X,$$

where $\omega_X^\bullet$ is the dualising complex of X. The last condition is equivalent to X having rational singularities. Here, $(\star)$ admits a converse if X is quasi-Gorenstein, and $(\star\star)$ follows by Grothendieck duality. □

Next, we move to the study of deformations of rational singularities.

Theorem 2.15 ([Elk78])**.** *Let X be a variety over k and let D be a normal Cartier divisor. If D has rational singularities, then X has rational singularities along D.*

Finally, we fast-forward from Elkik's 1978 work to 2021 to discuss one of the most recent breakthrough results on rational singularities, due to Kebekus-Schnell. For klt singularities, an analogous statement was previously obtained in [GKKP11].

Theorem 2.16 ([KS21])**.** *Let X be a normal variety over k with rational singularities. Then X satisfies the* extension property*: namely $f_*\Omega_Y^i = \Omega_X^{[i]}$ for every resolution of singularities $f\colon Y \to X$.*

Here $\Omega_X^{[i]} := (\Omega_X^i)^{**}$. Equivalently, this result says that for a regular differential form on the smooth locus $X_{\mathrm{sm}} \subseteq X$, its pull-back to $f^{-1}(X_{\mathrm{sm}}) \subseteq Y$ admits no poles along the exceptional locus, and in fact extends to a regular form on Y.

What is particularly important to note, is that the method of proof – most notably the use of intersection differential forms $I\underline{\Omega}_X^i$ arising from the theory of Hodge modules – has become a fundamental ingredient in very recent developments on higher singularities, which we will discuss at the end of this survey.

The next class of singularities which we shall study are *Du-Bois singularities*. To this end, given a variety X over k we consider a simplicial resolution

$$X \xleftarrow[f]{} Y_1 \underset{d_0}{\overset{d_1}{\leftleftarrows}} Y_2 \underset{d_0}{\overset{d_2}{\leftleftarrows}} Y_3 \cdots$$

with d_i being projective morphisms and Y_k being smooth. We obtain natural projections $f_k \colon Y_k \to X$. Therewith, we define the *Du-Bois* complex:

$$\underline{\Omega}_X^0 := \mathrm{Tot}\left((Rf_1)_*\mathcal{O}_{Y_1} \xrightarrow{d_0^* - d_1^*} (Rf_2)_*\mathcal{O}_{Y_2} \xrightarrow{d_0^* - d_1^* + d_2^*} (Rf_3)_*\mathcal{O}_{Y_3} \to \dots \right) \tag{2.16.1}$$

as an element of $D^b_{\mathrm{coh}}(X)$. This definition is independent of the choice of a simplicial resolution, and each $(Rf_n)_*\mathcal{O}_{Y_n}$ is represented by an explicit Godement resolution. While the above definition may appear ad hoc at first glance, we will elaborate on its conceptual context in Subsection 5.2.

Definition 2.17. Let X be a variety over k. We say that X has *Du Bois* singularities if the natural map $\mathcal{O}_X \to \underline{\Omega}_X^0$ is an isomorphism.

It is known that rational singularities are always Du Bois ([Kov99, Sai00]).

Theorem 2.18. *Let X be a normal $\mathbb{Q}$-Gorenstein variety over k. If X has lc singularities, then it has Du Bois singularities. Moreover, the implication in the opposite direction holds true when X is quasi-Gorenstein.*

The "moreover" part is due to [KSS10]. The first part is a deep theorem of Kovács and Kollár [KK10].

Du Bois singularities play a vital role in moduli theory as the following result shows.

Theorem 2.19 ([DB81, DBJ74])**.** *Let $f \colon Y \to S$ be a flat proper family of complex algebraic varieties. Assume that all fibres of f have Du Bois singularities. Then for all $q \geq 0$, the sheaves $R^q f_* \mathcal{O}_Y$ are locally free and compatible with base change.*

The deformation property for Du Bois singularities was established in [KS16]. Thus, in the above theorem, if one fibre of f is Du Bois, then so are all nearby fibres.

Remark 2.20. Recently, Murayama [Mur25] extended the vanishing of Grauert-Riemenschneider, and many other vanishing results, from finitely generated algebras over fields of characteristic zero to all excellent rings of characteristic zero. Consequently, the results of this section apply in that generality as well.

3. Classical singularities in characteristic $p > 0$

Let R be a ring of characteristic $p > 0$. The *Frobenius map* $F\colon R \to R$ given by $F(r) = r^p$ is a ring homomorphism, but it is not R-linear. To remedy this, we introduce a new R-module $F_*R := \{ F_*r \mid r \in R \}$, where F_*r is a formal symbol representing r, and with the R-module structure defined by

$$xF_*r := F_*(x^p r) \qquad \text{for } x \in R.$$

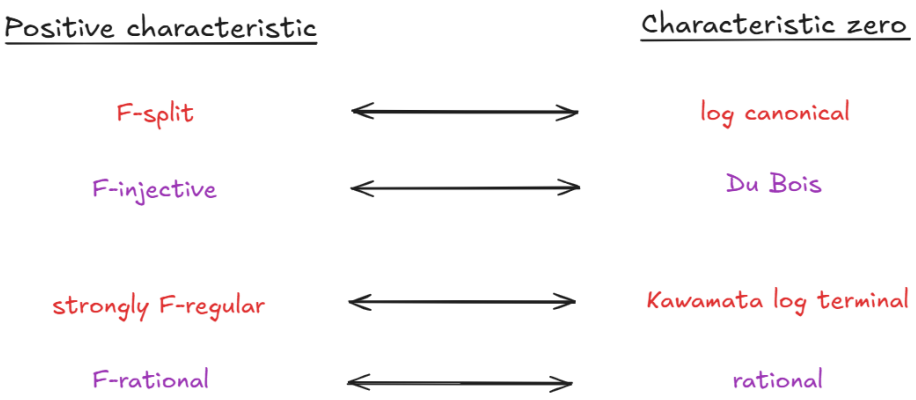
In other words, F_*R agrees with R as an abelian group, but its R-module structure is twisted by p-th powers. With this definition, the map

$$F\colon R \to F_*R \qquad r \mapsto F_*r^p$$

is a homomorphism of R-modules. Similarly, we define $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X$ for a scheme X of characteristic $p > 0$.

Throughout this section, we fix a perfect field k of characteristic $p > 0$.

Remark 3.1. A ring R of positive characteristic is F-finite if the Frobenius map $F\colon R \to R$ is finite. The results of this section extend naturally to finitely generated algebras over F-finite fields, and, more generally, to all excellent F-finite rings. Extending these results beyond the F-finite setting is an important and difficult problem in commutative algebra, which we do not discuss here due to space constraints.



We refer to the lecture notes [MP21] and the forthcoming book [SS26] for a more detailed exposition of the theory of F-singularities. Let us point out that F-split and strongly F-regular singularities introduced below, will be the characteristic 0 analogues of log canonical and Kawamata log terminal ones, respectively. The F-injective and F-rational singularities correspond to Du Bois and rational ones.

Remark 3.2. In our discussion of klt and lc singularities, the $\mathbb{Q}$-Gorenstein assumption was always imposed. This will not be the case for F-split and strongly F-regular ones. Nevertheless, these notions behave most naturally in the $\mathbb{Q}$-Gorenstein setting, and without this assumption many fundamental properties fail – such as deformation and inversion of adjunction – while many others remain poorly understood.

The singularities of cohomological nature, such as rational, Du Bois, F-rational, and F-injective, neither require nor benefit from the $\mathbb{Q}$-Gorenstein assumption.

3.1. **F-split.** The starting point for the study of characteristic $p > 0$ singularities is the theorem of Kunz characterising regular varieties by way of Frobenius.

Theorem 3.3 ([Kun69])**.** *Let X be a variety over k. Then X is regular if and only if the Frobenius morphism $F\colon X \to X$ is flat.*

The flatness of F is equivalent to $F_*\mathcal{O}_X$ being a locally free coherent sheaf on X.

Kunz's theorem suggests that characteristic $p > 0$ singularities can be studied using Frobenius. Our key definition is as follows. We restrict ourselves to affine varieties, as these notions behave very differently in the projective setting.

Definition 3.4. Let X be an affine variety over k. We say that X is *F-split* if the the natural map $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits.

Remark 3.5. In commutative algebra, it is customary to discuss F-purity in the context of singularities and F-splittings in that of global geometry. For affine varieties over F-finite fields, these two notions coincide, so we shall not define F-purity here.

Remark 3.6. Given a closed point $x \in X$, we say that X is *F-split at x* if there exists an open neighbourhood $U \subseteq X$ containing x such that $F\colon \mathcal{O}_U \to F_*\mathcal{O}_U$ splits. Equivalently, we require that $F\colon \mathcal{O}_{X,x} \to F_*\mathcal{O}_{X,x}$ splits. One can check that an affine variety X is F-split if and only if it is F-split at every closed point.

One can verify that F-split varieties are automatically reduced; however, they need not be normal or even integral. For simplicity, we mostly focus on the normal case.

Example 3.7. Take $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. Then

$$\begin{aligned} F_*k[x] &= F_*\big(k[x^p] \oplus xk[x^p] \oplus \ldots \oplus x^{p-1}k[x^p]\big) \\ &= k[x]F_*1 \oplus k[x]F_*x \oplus \ldots \oplus k[x]F_*x^{p-1} \end{aligned}$$

is a free $k[x]$-module with generators $F_*1, F_*x, \ldots, F_*x^{p-1}$. In particular we can construct a splitting of the Frobenius map $F\colon k[x] \to F_*k[x]$ by, for example, projecting onto the first component. In particular, X is F-split.

By an analogous argument, $F_*k[x_1, \ldots, x_n]$ is a free $k[x_1, \ldots, x_n]$-module with generators $F_*x_1^{a_1} \cdots x_n^{a_n}$ for $0 \leq a_1, \ldots, a_n \leq p-1$. Thus $\mathbb{A}^n_k$ is F-split as well. In general, every regular affine variety over k is F-split.

Next, we can compare F-split singularities with log canonical ones.

Proposition 3.8 ([HW02, Theorem])**.** *Let X be an F-split normal affine variety over k. Assume that X is $\mathbb{Q}$-Gorenstein. Then X is log canonical.*

The converse of the above proposition is false in general. For example, the hypersurface $x^2 + y^3 + z^5$ from Example 2.9 is log canonical, but it is F-split only when $p > 5$. Nevertheless, a form of a converse statement should hold for reductions modulo suitably chosen primes $p > 0$ as indicated by the following theorem.

Theorem 3.9 (cf. [MS11])**.** *Let X be a log canonical variety over $\mathbb{Q}$. Assume that the weak ordinarity conjecture holds for smooth projective varieties in every dimension. Then for infinitely many prime numbers p, the reduction $X \bmod p$ is F-split.*

Remark 3.10. In its general from, the theorem predicts that for every log canonical singularity X over a field K of characteristic zero and its model $\mathcal{X}$ over a finitely-generated $\mathbb{Z}$-subalgebra $A \subseteq K$, there exists a Zariski-dense set of closed points $S \subseteq \operatorname{Spec} A$ such that $\mathcal{X}_s$ is F-split for every $s \in S$.

We briefly recall the statement of the weak ordinarity conjecture. Below, a smooth projective variety Y of dimension d over a field k of characteristic $p > 0$ is said to be *weakly ordinary* if Frobenius $F\colon H^d(Y, \mathcal{O}_Y) \to H^d(Y, \mathcal{O}_Y)$ acts bijectively. For elliptic curves this notion agrees with the usual notion of ordinarity.

Conjecture 3.11 (Weak ordinarity conjecture)**.** *For a smooth projective variety Y over $\mathbb{Q}$, there are infinitely many primes $p > 0$ such that $Y \bmod p$ is weakly ordinary.*

In its general form, the conjecture says that a model of Y over a finitely generated $\mathbb{Z}$-algebra has weakly ordinary fibres for a Zariski-dense set of points.

As in the log canonical setting, one defines F-splittings for pairs. For simplicity, we restrict to integer coefficients. Let $X = \operatorname{Spec} R$ be an affine variety over k and let $D = V(f)$, for some $f \in R$, be a Cartier divisor. We say that (X, D) is *F-split* if

$$\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\cdot F_* f^{p-1}} F_*\mathcal{O}_X$$

splits. Clearly, if (X, D) is F-split, then X is F-split.

Theorem 3.12 ([PST25])**.** *Let X be a normal affine $\mathbb{Q}$-Gorenstein variety and let D be a normal Cartier divisor. Then (X, D) is F-split along D if and only if D is F-split.*

The inversion of adjunction and deformation alike break for F-split singularities without the $\mathbb{Q}$-Gorenstein assumption [Fed83, Sin99a].

The best way to verify if a ring is F-split is via the following Fedder's criterion.

Theorem 3.13 (Fedder's criterion [Fed83])**.** *The affine variety*

$$D = \operatorname{Spec} k[x_1, \dots, x_n]/(f)$$

is F-split at the origin if and only if $f^{p-1} \notin (x_1^p, \dots, x_n^p)$.

This condition requires that, in the expansion of f^{p-1} as a sum of distinct monomials, there exists a monomial whose exponents of each x_i are strictly less than p. In general, a ring $S = R/I$, with R regular, is F-split at $\mathfrak{m}$ if $[I^{[p]} : I] \not\subseteq \mathfrak{m}^{[p]}$.

We begin with some examples and refer to Appendix 1 for the proof.

Example 3.14. Let $f = x^2 + y^2 + z^2 \in k[x, y, z]$. We have the following cases:

(1) For $p = 2$, we have $f^{p-1} = f \in (x^2, y^2, z^2)$, so X is *not* F-split.
(2) For $p > 2$, we have

$$f^{p-1} = \binom{p-1}{\frac{p-1}{2}} (x^2 y^2)^{\frac{p-1}{2}} + \dots \notin (x^p, y^p, z^p),$$

so X is F-split as the coefficient $\binom{p-1}{\frac{p-1}{2}}$ is not divisible by p.

F-splittings can sometimes depend on more subtle arithmetic properties, as illustrated by the example below, where X is F-split if and only if $p \equiv 1 \pmod 3$.

Example 3.15. Let $f = x^3+y^3+z^3 \in k[x,y,z]$. Since the degree of the homogeneous polynomial f^{p-1} is $3(p-1)$, the only monomial in f^{p-1} that could be outside of (x^p, y^p, z^p) is

$$x^{p-1}y^{p-1}z^{p-1}.$$

When $3 \nmid (p-1)$, this term does not appear and X is not F-split. When $3 \mid (p-1)$, this term appears with coefficient

$$\binom{p-1}{\frac{p-1}{3}, \frac{p-1}{3}, \frac{p-1}{3}},$$

which is not divisible by p. Hence, in this case X is F-split.

3.2. **Strongly F-regular.** In this subsection, we discuss the positive characteristic analogue of klt singularities. We start with the following easy exercise.

Exercise 3.16. Let X be an affine variety. Show that $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X$ splits if and only if $F^e\colon \mathcal{O}_X \to F^e_*\mathcal{O}_X$ splits for all (equivalently, for some) $e > 0$.

An affine variety is strongly F-regular if it is F-split "up to perturbation".

Definition 3.17. Let $X = \operatorname{Spec} R$ be an affine variety over k. We say that X is *strongly F-regular* if for every $h \in R$, which is not in any minimal prime of R, there exists $e > 0$ such that the following map splits:

$$\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \xrightarrow{F^e_*h} F^e_*\mathcal{O}_X. \tag{3.17.1}$$

One can verify that strongly F-regular varieties are automatically normal. Moreover, it is enough to check (3.17.1) for one single $h \in R$ as long as $V(h) \supseteq \operatorname{Sing}(X)$. This fact is related to a deep theorem on the existence of a *test element* (cf. [HH94a]).

Example 3.18. The affine variety $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$ is strongly F-regular. To this end, pick non-zero $h \in k[x_1, \dots, x_n]$ and choose $e \gg 0$ so that every monomial $x_1^{a_1} \cdots x_n^{a_n}$ of h satisfies $0 \le a_i < p^e$ for all i. Pick any such monomial, say $x_1^{b_1} \cdots x_n^{b_n}$, and let c be its coefficient:

$$h = cx_1^{b_1} \cdots x_n^{b_n} + \dots$$

Now, we define the splitting $\theta\colon F^e_*R \to R$ of (3.17.1) on the generators of F^e_*R via:

$$\phi(F^e_* x_1^{a_1} \cdots x_n^{a_n}) = \begin{cases} \frac{1}{c} & \text{if } (a_1, \dots, a_n) = (b_1, \dots, b_n) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\phi(F^e_* f) = 1$, which concludes the proof. One can enhance this argument to show that every regular ring is strongly F-regular.

Next, we compare strongly F-regular singularities with klt singularities.

Proposition 3.19 ([HW02])**.** *Let X be a strongly F-regular affine variety over k. Assume that X is $\mathbb{Q}$-Gorenstein. Then X is klt.*

Theorem 3.20 ([Smi00, Har01])**.** *Let X be a klt variety over $\mathbb{Q}$. Then for all but a finite number of primes p, the reduction $X \bmod p$ is strongly F-regular.*

The more general version as described in Remark 3.10 also holds. Note that in contrast to the lc case, the klt variant does not require assuming the ordinarity conjecture, and the statement is valid for all sufficiently large primes.

The analogue of Fedder's criterion is given by Glassbrenner's criterion.

Theorem 3.21 ([Gla96])**.** *The affine variety $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f)$ is strongly F-regular at the origin if and only if*

$$\exists_{e>0} \quad \text{such that} \quad f^{p^e-1}\operatorname{Jac}(f) \not\subseteq (x_1^{p^e}, \dots, x_n^{p^e}).$$

Here $\operatorname{Jac}(f) \subseteq k[x_1, \dots, x_n]$ is the ideal generated by all partial derivatives:

$$\operatorname{Jac}(f) = (\partial_{x_1} f, \dots, \partial_{x_n} f).$$

Remark 3.22. Using this criterion, one can verify that all rational double points in Example 2.9 are strongly F-regular when $p > 5$. In fact, the same holds for all klt surface singularities. This observation is a key ingredient in Hacon and Xu's proof [HX15] of the Minimal Model Program for threefolds in characteristic $p > 5$.

One can show that strong F-regularity deforms for normal $\mathbb{Q}$-Gorenstein varieties over k [AKM98], and that an appropriate inversion of adjunction statement can be established. On the other hand, these properties break outside of the $\mathbb{Q}$-Gorenstein case [Sin99b]. We omit the details.

3.3. **F-injective and F-rational.** In this section, we extensively use the notion of local cohomology and refer the reader to the appendix for a brief overview. Our goal is to define positive characteristic analogues of Du Bois and rational singularities.

Definition 3.23. Let $X = \operatorname{Spec} R$ be an affine variety over k. We say that X

- is *F-injective* if for every closed point $x \in X$ and every integer i the map
$$F\colon H^i_x(\mathcal{O}_X) \to H^i_x(F_*\mathcal{O}_X) \qquad \text{is injective,}$$
- is *F-rational* if for every closed point $x \in X$, every integer i, and every $f \in R$, which is not in any minimal prime of R, there exists $e > 0$ such that the following map is injective:
$$H^i_x(\mathcal{O}_X) \xrightarrow{F^e} H^i_x(F^e_*\mathcal{O}_X) \xrightarrow{\cdot F^e_* f} H^i_x(F^e_*\mathcal{O}_X).$$

One can verify that F-rational varieties are automatically normal.

Next we compare these notions to F-splitting and strong F-regularity.

Proposition 3.24. *Let X be an affine variety over k. Then:*

- *if X is F-split, then it is F-injective, and*
- *if X is strongly F-regular, then it is F-rational.*

Moreover, the converse holds true when X is quasi-Gorenstein.

The first part of the proposition is clear, since split maps are automatically injective on local cohomology. The quasi-Gorenstein case requires more work.

The next item on our list is the study of reduction modulo p. All of the statements below extend to general models as in Remark 3.10.

Theorem 3.25 ([Sch09, BST17])**.** *Let X be an affine variety over $\mathbb{Q}$. If $X \bmod p$ is F-injective for a sufficiently large $p > 0$, then X is Du Bois.*

Moreover, assuming the weak ordinarity conjecture in every dimension, if X is Du Bois, then for infinitely many primes p, the reduction $X \bmod p$ is F-injective.

Theorem 3.26 ([Smi97, MS97, Har98])**.** *Let X be an affine variety over $\mathbb{Q}$. If $X \bmod p$ is F-rational for a sufficiently large prime $p > 0$, then X has rational singularities.*

Moreover, if X has rational singularities, then for all but a finite number of primes p, the reduction $X \bmod p$ is F-rational.

Remark 3.27. In contrast to F-splitting and strong F-regularity (see Propositions 3.8 and 3.19), the implications from F-injectivity and F-rationality to Du Bois and rational singularities, respectively, are more delicate, since defining these notions in characteristic $p > 0$ is delicate.

We have already discussed the inversion of adjunction for singularities related to the Minimal Model Program: klt, lc, strongly F-regular, F-split. We have also discussed the deformation problem for characteristic zero singularities of cohomological nature: rational and Du Bois.[2]

What remains is the case of cohomological singularities in characteristic $p > 0$. Here, F-rational singularities always deform by [HH94b]. By contrast, the deformation problem for F-injectivity remains open outside of the Cohen–Macaulay case, even when D is normal (cf. [HMS14]).

Conjecture 3.28. *Let X be an affine variety over k and let D be a Cartier divisor. If D is F-injective, then X is F-injective along D.*

3.3.1. *Other.* The study of singularities in positive characteristic stems in large part from tight closure theory, a foundational subject that has driven decades of development in commutative algebra [HH90, Smi94]. More broadly, Frobenius singularities encompass a wide range of topics. One important direction is the study of the F-signature, which predates – and in characteristic $p > 0$ serves as an analogue of – the aforementioned normalised volume, with applications to the study of fundamental groups of F-singularities [Tuc12, CRST18].

Another closely related topic, and one very close to the author's heart, is the study of klt singularities in positive characteristic $p > 0$, particularly in low dimensions, using methods from both F-singularity theory and the Minimal Model Program [CTW18, HW19, ABL22]. This work is primarily concerned with dimension three, with partial results in dimension four [HW23]. We return to this theme in the final

[2]One can also define these cohomological singularities for pairs and study inversion of adjunction. Since this is somewhat less natural, we do not pursue it in this survey.

section when discussing quasi-F-singularities. These questions are also connected to $W\mathcal{O}_X$-rationality both in low and arbitrary dimension with applications involving congruences over finite fields [GNT19, Bau26].

4. Mixed characteristic singularities

From the 1960s onward, progress in commutative algebra has been strongly influenced by a family of questions collectively referred to as the *homological conjectures* [Hoc07]. One of the most important among these is Hochster's *direct summand conjecture*, both because of its intrinsic significance and because its statement is particularly elementary and easy to explain.

Definition 4.1. Let R be a Noetherian domain. We say that R is a *splinter* if every finite extension of domains $\phi\colon R \hookrightarrow S$ splits in the category of R-modules, that is, there there exists an R-linear map $\psi\colon S \to R$ for which the composite

$$R \xhookrightarrow{\phi} S \xrightarrow{\psi} R$$

is the identity.

The direct summand conjecture predicts that regular rings are splinters. This is now a theorem thanks to the recent breakthrough work of André.

Theorem 4.2 (André 18' [And18], Hochster 73')**.** *Every regular Noetherian domain is a splinter.*

Historically, the most serious difficulty in proving the direct summand conjecture arose in mixed characteristic, with the ring $\mathbb{Z}[x_1, \dots, x_n]$ serving as a central case.

Until the end of this subsection, we restrict ourselves to the following setting.

Setting 4.3. All rings are finitely generated over a Dedekind domain V. We call $X := \operatorname{Spec} R$ for such a ring R, an affine variety over V.

The key examples of Dedekind domains of interest for us are fields, discrete valuation rings (such as $\mathbb{Z}_p$), and more general Dedekind domains (such as $\mathbb{Z}$ or the rings of integers of number fields).

To place the direct summand conjecture in context, we first review what is known in characteristic zero and in positive characteristic.

Proposition 4.4. *With notation as above, suppose that V is a field of characteristic zero. Then R is a splinter if and only if it is normal.*

In particular, over a field of characteristic 0, the notion of a splinter imposes a very mild restriction on the ring and is therefore not especially interesting in this context.

Sketch of a proof. Suppose that R is normal and let $R \subseteq S$ be a finite extension of domains. By a standard argument, we reduce to the case where $K \subseteq L$ is Galois,

with $K := \operatorname{Frac}(R)$ and $L := \operatorname{Frac}(S)$. The splitting ψ can then be defined by

$$\psi(s) = \frac{1}{d} \sum_{\sigma \in G} \sigma(s),$$

where $G := \operatorname{Gal}(L/K)$ and $d := [L : K]$. We need to verify that $\psi(s) \in R$. By Galois theory, the sum $\sum_{\sigma \in G} \sigma(s)$ lies in K. Moreover, since each $\sigma(s)$ is integral over R, their sum is integral over R as well. Normality of R therefore implies that

$$\sum_{\sigma \in G} \sigma(s) \in R.$$

Since $\mathbb{Q} \subseteq R$, we get that $\psi(s) \in R$, as required. We omit the proof of the converse. □

Next we move to positive characteristic. The study of splinters in this setting originated in the work of Hochster and Huneke [HH92] and Smith's thesis [Smi94], was further shaped by contributions of Huneke-Lyubeznik [HL07] and Singh [Sin99c], and later developed by Blickle, Schwede, and Tucker [BST15], among others.

To prove the direct summand conjecture in characteristic $p > 0$, we will need the following lemma.

Lemma 4.5. *Let X be an affine variety over a field k of characteristic $p > 0$. Suppose that X is strongly F-regular. Then X is a splinter. Moreover, the converse is true when X is $\mathbb{Q}$-Gorenstein.*

An important open problem in the theory of singularities in positive characteristic asks for the converse to be true without the $\mathbb{Q}$-Gorenstein assumption.

Proof. The statement that $\mathbb{Q}$-Gorenstein splinters are strongly F-regular is a difficult theorem of Singh [Sin99c]. Below we will only explain the proof that strongly F-regular affine varieties are splinters.

Let $\phi\colon R \hookrightarrow S$ be a finite inclusion of domains, and set $K := \operatorname{Frac}(R)$ and $L := \operatorname{Frac}(S)$. By basic linear algebra, there exists a surjective K-linear map $L \to K$. Clearing denominators in the images of a finite set of R-module generators of S, we may assume that this map is induced by an R-linear map

$$\psi\colon S \to R.$$

Set $h := \psi(1)$. By strong F-regularity, there exists $e > 0$ such that the map in (3.17.1) splits. Explicitly, (3.17.1) is equal to the composition

$$\begin{array}{ccccccc} R & \xrightarrow{F^e} & F^e_* R & \xrightarrow{F^e_*\phi} & F^e_* S & \xrightarrow{F^e_*\psi} & F^e_* R \\ & \searrow^{\phi} & & & \nearrow_{F^e} & & \end{array} \qquad r \longmapsto F^e_*(hr^{p^e})$$

Since this map factors through $\phi\colon R \hookrightarrow S$, we get that ϕ splits. Thus R is a splinter. □

We are now ready to show the Direct Summand Conjecture in positive characteristic. It was originally proven by Hochster in 1973. For simplicity, we work under the assumptions of Setting 4.6.

Proof of Theorem 4.2 in positive characteristic. By Example 3.18, our regular ring R is strongly F-regular. Hence it is a splinter by the above Lemma 4.5. □

The Direct Summand Conjecture remained open from 1973 until around 2016, when André came up with his proof. It had been apparent for a long time that classical methods of commutative algebra were insufficient to resolve the conjecture and, indeed, André's proof draws on the then newly developed theory of perfectoid spaces of Scholze. We will not elaborate on André's argument here, as doing so would take us too far afield. Let us note, however, that there are now several proofs of the Direct Summand Conjecture, including one aided by prismatic cohomology and another obtained as a consequence of Bhatt's Theorem 4.8, to which we return later. We will also briefly treat perfectoid rings in Subsection 4.2.

4.1. **Splinter and +-rational.** From now until the end of this section, we specialise Setting 4.6 as follows.

Setting 4.6. All rings are assumed to be finitely generated over an excellent discrete valuation ring (V, ϖ) such that $\mathrm{char}(V/\varpi) > 0$. For such a ring R, we refer to $X := \operatorname{Spec} R$ as an affine variety over V.

When $\mathrm{char}(V) = 0$ but $\mathrm{char}(V/\varpi) > 0$, we say that V is of *mixed characteristic*. The key examples for us are the p-adic integers $\mathbb{Z}_p$ and the localisation $\mathbb{Z}_{(p)}$. When working over more general Dedekind domains, most commutative-algebraic questions can be easily reduced to this setting.

The study of singularities in mixed characteristic is subtle. On the one hand, birational methods[3] – which are central to the study of singularities in characteristic zero – are not sufficiently sensitive to capture the finer phenomena arising in positive and mixed characteristic. On the other hand, the Frobenius map, fundamental to the study of singularities in characteristic $p > 0$, is unavailable in mixed characteristic.

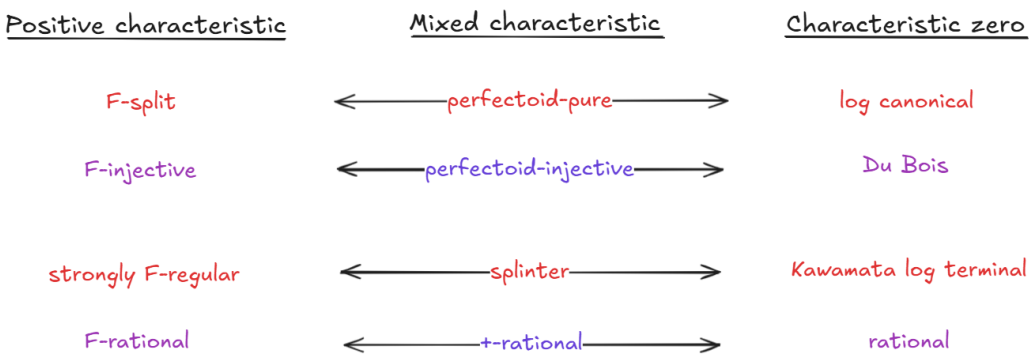
For this reason, splinters provide an effective notion of mild singularities in mixed characteristic. The Direct Summand Conjecture serves as the starting point, asserting that regular rings satisfy this condition – in other words, that they are splinters.

Remark 4.7. Despite the proof of the direct summand conjecture, relatively little progress was initially made toward a broader theory of splinters. Instead, Schwede and Ma introduced and developed a different class of singularities, called *BCM-regular*,

[3]A common misconception the author of this survey often encounters is that the theories of F-singularities and splinters are needed only because resolution of singularities is unavailable in positive or mixed characteristic. In fact, the importance of these theories has little to do with the existence of resolutions. Even if resolutions were known, these arithmetic theories of singularities would remain essential: for example, they serve as replacements for vanishing theorems – such as Grauert-Riemenschneider vanishing – that play a central role in characteristic 0 geometry.

inspired by ideas underlying André's proof and conjecturally equivalent to splinters. This theory is rich and has led to many applications, playing a fundamental role in the development of mixed characteristic commutative algebra and birational geometry; however, due to space limitations, we will not discuss it further.

We enhance the picture from Section 3 to introduce mixed characteristic singularities.



A systematic study of splinter singularities became possible only after Bhatt's recent breakthrough theorem. In positive characteristic, the corresponding result was previously known by [HH92]. Recall that in this section the DVR V is assumed to be of positive or mixed characteristic. This result is false when $\operatorname{char}(V/\varpi) = 0$.

Theorem 4.8 ([Bha20])**.** *Let X be an affine variety over V of positive or mixed characteristic. Then $H^i_x(\mathcal{O}_X^+) = 0$ for all closed points $x \in X$ of residue characteristic $p > 0$ and $i < \dim \mathcal{O}_{X,x}$.*

Here, $\mathcal{O}_X^+$ is the sheaf associated to the *absolute integral closure* $R^+ := \bigcup_{R \subseteq S} S$, where the union ranges over all finite extensions $R \subseteq S$ of domains inside a fixed algebraic closure of the fraction field of R. Such vanishing of local cohomology is often referred to as *cohomological Cohen-Macaulayness*. For Noetherian rings, this notion is equivalent to the usual Cohen-Macaulay property; however, R^+ is not Noetherian.

Remark 4.9. A fundamental consequence of Bhatt's work is the proof of Kodaira vanishing up to finite covers. This result was subsequently used in [BMP+23] to establish the Minimal Model Program for threefolds in mixed characteristic when the residue characteristic is greater than 5.

We have not discussed this yet in the survey, but it is well known that Kawamata log terminal singularities, together with their cohomological and positive characteristic counterparts – rational, strongly F-regular, and F-rational – are all Cohen–Macaulay. As an easy corollary, we obtain the same result for splinters.

Corollary 4.10. *Let $X = \operatorname{Spec} R$ be an affine variety over V of positive or mixed characteristic. If X is a splinter, then it is Cohen-Macaulay at all closed points of residue characteristic $p > 0$.*

Proof. As every finite extension $R \hookrightarrow S$ splits in the category of R-modules, the map

$$H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R^+) = \varinjlim_{R \subseteq S} H^i_{\mathfrak{m}}(S)$$

is injective. Thus, $H^i_{\mathfrak{m}}(R) = 0$ for all $i < \dim R$ by Bhatt's result, and so R is Cohen-Macaulay as explained in Section 7. □

From the perspective of this survey, the key application of Bhatt's result is the deformation property for splinters. In fact, the result can be strengthened to yield a genuine inversion of adjunction, though we will not state it here.

Corollary 4.11. *Let X be a $\mathbb{Q}$-Gorenstein affine variety over V of positive or mixed characteristic and let D be a normal Cartier divisor. If D is a splinter, then X is a splinter along D as well.*

Note that this statement immediately implies the Direct Summand Conjecture. Indeed, on a regular variety X, through every point $x \in X$ there exists a hyperplane H that is also regular. By the above corollary, it therefore suffices to show that H is a splinter. Repeating this procedure, we reduce to the zero-dimensional case, where the statement is clear: a field is a splinter.

Remark 4.12. The key ingredients in the proof of Bhatt's result are the p-adic Riemann-Hilbert correspondence of Bhatt-Lurie and the prismatic cohomology of Bhatt-Scholze. These topics lie beyond the scope of this survey. Let us note, however, that the p-adic Riemann-Hilbert correspondence yields another important application: *evaporation* in mixed characteristic. More precisely, given a projective morphism $f\colon Y \to X$ to an affine variety over V and an integer $n > 0$, there exists a finite cover $g\colon Z \to Y$ such that the induced map $g^*\colon H^i(Y, \mathcal{O}_Y) \to H^i(Z, \mathcal{O}_Z)$ satisfies:

$$\operatorname{im}(g^*) \subseteq p^n H^i(Z, \mathcal{O}_Z) \quad \text{and} \quad g^*\colon H^i(Y, \mathcal{O}_Y)[p^n] \to H^i(Z, \mathcal{O}_Z)[p^n] \text{ is a zero map}$$

for every $i > 0$, where $H^i(Y, \mathcal{O}_Y)[p^n]$ denotes the p^n-torsion. In positive characteristic, evaporation asserts the existence of g such that $g^*\colon H^i(Y, \mathcal{O}_Y) \to H^i(Z, \mathcal{O}_Z)$ is zero; this was established earlier by Hochster-Huneke and Bhatt ([Bha12, Corollary 3.5]).

In turn, the evaporation leads to the following diagram (to clarify the notation, all colimits are p-adically completed):

$$\widehat{R^+} = \varinjlim_{\substack{f\colon Y \to X \\ \text{finite surj.}}} \widehat{R\Gamma(Y, \mathcal{O}_Y)} \xrightarrow{\cong} \varinjlim_{\substack{f\colon Y \to X \\ \text{alteration}}} \widehat{R\Gamma(Y, \mathcal{O}_Y)} \longleftarrow \varinjlim_{\substack{f\colon Y \to X \\ \text{proj. birational}}} \widehat{R\Gamma(Y, \mathcal{O}_Y)},$$

where the isomorphism is a direct consequence of evaporation together with Stein factorisation. Here, a morphism $f\colon Y \to X$ is called an *alteration* if it is a composition of a projective birational morphism and a finite surjective morphism.

Informally, the above remark can be summarised by the following guiding principle permeating the theory of arithmetic singularities:

Slogan. *From a cohomological perspective, projective birational morphisms in positive and mixed characteristic can be approximated by finite maps.*

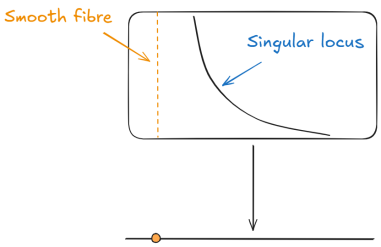
In turn, this allows for comparison of singularities in mixed characteristic defined via finite maps – like splinters and +-rationality – with those defined via birational maps.

Proposition 4.13. *Let X be a normal $\mathbb{Q}$-Gorenstein affine variety over V of positive or mixed characteristic. Suppose that X is a splinter. Then X is klt at every closed point of residue characteristic $p > 0$.*

A consequence of Proposition 3.19 in Section 3 was that if a reduction of a $\mathbb{Q}$-Gorenstein normal variety X over $\mathbb{Q}$ modulo a prime $p \gg 0$ is strongly F-regular, then X is klt. Using the proposition above, we obtain a substantial strengthening of this result: it now suffices to verify the assertion for a single prime p.

Corollary 4.14 ([MS21])**.** *Let X be a normal variety over $\mathbb{Q}$ with a $\mathbb{Q}$-Gorenstein model, and let $x \in X$ be a closed point. Suppose that $X \bmod p$ is strongly F-regular at a closed point y which is a specialisation of x. Then X is klt at x.*

We take care to specify the point x, since it can happen that X is singular while a special fibre of a chosen model is smooth. In the picture on the right, for instance, the generic point x of the singular locus does not specialise to any point y on the smooth special fibre. This phenomenon did not arise in Section 3, where we worked over fibres with $p \gg 0$. When X is projective, the issue disappears altogether, since every closed point of X specializes to a closed point on the fibre $X \bmod p$.



The above result has been obtained first by Ma and Schwede using their BCM-regular singularities. Below, we present a streamlined proof based on the theory of splinters.

Proof. Consider a model $\mathcal{X}$ over $\mathbb{Z}[1/N]$ such that $p \nmid N$. Let $\overline{x} \subseteq \mathcal{X}$ be the closure of $x \in X$. By Corollary 4.11 applied to $D = X \bmod p$ we get that $\mathcal{X}$ is a splinter along $\overline{x}$ up to multiplying N by a bigger number coprime to p. In turn, by Proposition 4.13, $\mathcal{X}$ is klt along $\overline{x}$. Hence X is klt at x, which concludes the proof.



□

Sato-Takagi managed to generalise the above statement to the log canonical case by ad-hoc methods without discussing mixed characteristic analogues of F-splittings.

Theorem 4.15 ([ST25])**.** *Let X be a normal variety over $\mathbb{Q}$ with a $\mathbb{Q}$-Gorenstein model and let $x \in X$ be a closed point. Suppose that $X \bmod p$ is normal and F-split at a closed point y which is a specialisation of x. Then X is log canonical at x.*

Both of the above results extend to general models as in Remark 3.10. Also note that both of the statements were known before for complete intersections thanks to the work of Zhu [Zhu17], using completely different methods, namely arc spaces.

Remark 4.16. The author of this survey finds the above results to be among the more *amusing* applications of mixed characteristic singularity theory, and often uses them to motivate mixed characteristic methods to complex algebraic geometers.

Let us imagine that the reader is given a complex singularity X. If the singularity is constructed geometrically (for instance, as a contraction inside a smooth variety), then checking that it is klt is often easy, since one can usually write down a resolution. On the other hand, if X is given by explicit equations, finding a resolution can be a nightmare. One can try to use Bernstein–Sato polynomials instead, but this approach only works for complete intersections and, even then, can be still rather slow.

Corollary 4.14 provides a very different strategy. One reduces modulo a suitably chosen small prime p and checks strong F-regularity using Glassbrenner's criterion (Theorem 3.21). If this holds, then X is klt. In practice, this test is often easy to carry out, and computers handle it extremely well. If you are a complex geometer wrestling with a singularity cut out by some truly fearsome system of polynomial equations and asking yourself "is this klt?", then this is absolutely the way to go!



Note, however, that the above method is poorly suited for formally proving that X is *not* klt, since that would require checking the condition for infinitely many primes.

By a result of Sato–Takagi, the same philosophy applies to log canonical singularities using Fedder's criterion (Theorem 3.13). In particular, for $f \in \mathbb{Z}[x_1, \dots, x_n]$:

$$f^{p-1} \bmod p \notin (x_1^p, \dots, x_n^p) \text{ for one prime } p \implies \int_U \frac{1}{|f|^{2t}} < \infty \quad \text{for all } t < 1,$$

where $U \subseteq \mathbb{C}^n$ is a small ball at 0. However, that this was already known by [Zhu17].

Remark 4.17. Similarly to F-rationality in characteristic $p > 0$, one can define $+$-rationality at a closed point $x \in X$ by requiring that the map

$$H^i_x(\mathcal{O}_X) \to H^i_x(\mathcal{O}_X^+)$$

is injective for every integer i. As in the case of F-rationality, this class of singularities is Cohen–Macaulay, satisfies deformation, and localises to rational singularities.

4.2. **Perfectoid-pure and perfectoid-injective.** We turn to the study of the mixed characteristic analogue of F-splitting. We first recall perfectoid rings (see the survey [MT25] for more details).

Definition 4.18. Fix a prime $p > 0$. A ring R is called *perfectoid* if:

(1) it is p-complete,
(2) the ring R/p is *semiperfect*: Frobenius $F\colon R/p \to R/p$ is surjective,
(3) there exists a p-th root of p in R up to a unit, meaning there exists $\varpi \in R$ and $u \in R^\times$ such that $\varpi^p = up$,

(4) the kernel of Fontaine's map $\theta\colon W(R^\flat) \to R$ is principal.

Let us, however, point out that when R is p-torsion-free, the fourth condition can be replaced by the requirement that Frobenius induce an isomorphism $R/\varpi R \cong R/pR$.

In characteristic $p > 0$, a ring is perfectoid if and only if it is perfect.

Example 4.19. A key example of a perfectoid ring is given by $A = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$, the p-adic completion of

$$\mathbb{Z}_p[p^{1/p^\infty}] := \bigcup_n \mathbb{Z}_p\left[p, p^{1/p}, p^{1/p^2}, \dots, p^{1/p^n}\right] \subseteq \overline{\mathbb{Q}}_p.$$

Over this ring, one can consider further perfectoid rings, for instance $R = \widehat{A[x^{1/p^\infty}]}$, where the completion is again taken with respect to the p-adic topology.

Mixed characteristic Kunz's theorem has been proven by Bhatt-Iyengar-Ma.

Theorem 4.20 ([BIM19])**.** *Let R be a Noetherian local ring of residue characteristic $p > 0$. Then R is regular if and only if there exists a faithfully flat map $R \to S$ where S is a perfectoid ring.*

We say that a map of R-modules $M \to N$ is *pure* if it is a filtered colimit of split maps $M \to N_i$. When k is a perfect field, one can check that a finitely generated k-algebra R is F-split if and only if the natural map $R \to R_{\mathrm{perf}}$ is pure, where R_{perf} denotes the perfection of R.

We continue working in Setting 4.6.

Definition 4.21 ([BMP+24])**.** We say that an affine variety X over V of positive or mixed characteristic is *perfectoid pure* at a closed point $x \in X$ of positive residue characteristic $p > 0$, if for $R := \mathcal{O}_{X,x}$, there exists a perfectoid R-algebra S such that the natural map $R \to S$ is pure. We say that X is *perfectoid pure* if it is perfectoid pure at every closed point of positive residue characteristic.

Let us state the following two key results about perfectoid purity.

Theorem 4.22 ([BMP+24])**.** *Let X be a normal $\mathbb{Q}$-Gorenstein affine variety over V of mixed characteristic. Assume that X is perfectoid pure. Then $X[1/p]$ is log canonical at every point which specialises to a characteristic p point.*

Theorem 4.23 ([BMP+24])**.** *Let X be an affine variety over V of positive or mixed characteristic and let D be a normal Cartier divisor. Assume that X is a locally complete intersection. If D is perfectoid-pure, then X is perfectoid pure along D.*

Remark 4.24. It would be very interesting to prove this theorem in the $\mathbb{Q}$-Gorenstein case without assuming that X is locally a complete intersection. Not only would this yield a more conceptual proof of Theorem 4.15 in full generality, but it would also resolve certain conjectures concerning the singularities of integral models of Shimura varieties.

Finally, one easily defines *perfectoid-injectivity* by requiring that the map to some perfectoid R-algebra is injective on local cohomology.

Remark 4.25. We briefly mention some other work related to singularities in mixed characteristic. One important direction is the generalization of the F-signature, leading to the notion of *perfectoid signature* [CLM+26] and applications to fundamental groups of mixed characteristic singularities. Another active area concerns explicit computations and the verification of the splinter property and the associated "splinter threshold" for specific singularities; see [CRMP+21, CPQG+25, BJP+26].

5. New directions

In this section, we discuss several new directions in the study of singularities.

5.1. **Quasi-F-splittings.** As mentioned earlier, F-splittings are closely related to the ordinarity of projective varieties, but smooth projective varieties carry finer – crystalline-type – invariants. Quasi-F-splittings, which are the focus of this subsection, are designed to detect such invariants. In particular, Yobuko introduced this notion to show that a broad class of Calabi–Yau varieties – those of finite Artin–Mazur height – admit liftings modulo p^2 [Yob19].

Let X be an affine variety over a perfect field k of characteristic $p > 0$. We say that X is *quasi-F-split* if there exists an integer $n > 0$ and a dashed arrow making the following diagram commute:

$$\begin{array}{ccc} W_n\mathcal{O}_X & \xrightarrow{F} & F_*W_n\mathcal{O}_X \\ {\scriptstyle \mathrm{res}}\downarrow & \swarrow & \\ \mathcal{O}_X. & & \end{array} \tag{5.0.1}$$

The condition becomes weaker as n increases. For $n = 1$, it coincides with F-splitting.

There are several related notions of quasi-singularities, such as quasi-F-regularity and quasi-splinters. These and many other properties are studied in [KTT+25a, KTT+24, KTT+25b]. A Fedder-type criterion for quasi-F-splittings was established in [KTY25]. In contrast to their classical counterparts, quasi-singularities do not in general satisfy inversion of adjunction or deformation properties, except in special cases, which makes their theory substantially more subtle. Further developments in the theory of quasi-singularities can be found in [Jag26, GJ25].

Remark 5.1. From a more concrete perspective, the advantage of quasi-F-splittings is that, despite being less restrictive, they retain many of the key properties of F-splittings. This makes them particularly useful for the purpose of applications in birational geometry. As mentioned earlier, all two-dimensional klt surface singularities are F-split in characteristic $p > 5$. However, in [CTW18] we showed that the analogous statement fails in dimension three: there is no bound p_0 such that every three-dimensional klt singularity in characteristic $p > p_0$ is F-split.

The situation changes dramatically when F-splitting is replaced by quasi-F-splitting. In [KTT+25a], we proved that every $\mathbb{Q}$-factorial three-dimensional klt singularity in characteristic $p \geq 42$ is quasi-F-split; this result was further strengthened to quasi-F-regularity in [KTT+25b]. In turn, this yielded a range of applications, including

liftability modulo p^2 for such klt singularities, extension theorems [KS25], and Steenbrink vanishing results [Kaw25].

Recently, Yoshikawa extended some of the framework of quasi-F-splittings to mixed characteristic and applied it to the study of perfectoid purity [Yos25a, Yos25b]. This extension opens the door to many further developments in the theory.

5.2. **Higher singularities in characteristic zero.** Before introducing the main objects of this subsection – higher Du Bois and rational singularities – we first give a brief overview of the Bernstein-Sato polynomial, as announced in the introduction.

5.2.1. *Berstein-Sato polynomial and the work of Mustaţă-Popa.* Let f be a polynomial in $\mathbb{C}[x_1, \dots, x_n]$. The motivating idea behind the Bernstein–Sato polynomial is to find a "minimal" differential operator

$$P \in \mathbb{C}\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}\rangle$$

such that Pf is a constant. In fact, one can do much more: there exists a family of differential operators $P(s)$, parameterised by s, such that $P(s)f^{s+1} = b(s)\, f^s$ for all integers $s > 0$, where both $P(s)$ and the scalar $b(s)$ depend polynomially on s.

Theorem 5.2 ([Ber71])**.** *With notation as above, there exists*

$$P(s) \in \mathbb{C}\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}, s\rangle$$

and a polynomial $b(s) \in \mathbb{C}[s]$ such that

$$P(s)f^{s+1} = b(s)f^s \tag{5.2.1}$$

for every integer s.

The set of all such polynomials $b(s)$ forms an ideal in $\mathbb{C}[s]$. Since $\mathbb{C}[s]$ is a PID, there is a unique monic polynomial $b(s)$ of minimal degree for which a differential operator $P(s)$ as above exists. This polynomial is called the *Bernstein-Sato polynomial*.

By plugging $s = -1$ in (5.2.1), we get that $P(-1)1 = b(-1)\frac{1}{f}$, and so

$$b(-1) = 0.$$

We define $\tilde{b}(s) := \frac{b(s)}{s+1}$ and call it the *normalised Bernstein-Sato polynomial*.

Example 5.3. Take $f = x^2 + y^2 + z^2 + w^2$. Then

$$(\partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2)(x^2 + y^2 + z^2 + w^2)^{s+1} = 4(s+1)(s+2)(x^2 + y^2 + z^2 + w^2)^s.$$

It is easy to read off from here that the Bernstein-Sato polynomial is $b(s) = (s + 1)(s + 2)$.

Example 5.4. Consider a cuspidal singularity $f = x^2 + y^3$. Then using the operator

$$P(s) = \frac{1}{12}\, y\, \partial_x^2 \partial_y + \frac{1}{27}\, \partial_y^3 + \frac{1}{4}\, s\partial_x^2 + \frac{3}{8}\, \partial_x^2,$$

one can calculate that $b(s) = (s + 1)(s + \frac{5}{6})(s + \frac{7}{6})$.

An important theorem of Kashiwara [Kas76] states that all the roots of the Bernstein-Sato polynomial $b(s)$ are negative rational numbers, that is, we can write:

$$\tilde{b}(s) = (s+\alpha_1)\cdots(s+\alpha_k) \quad \text{for } \alpha_i \in \mathbb{Q}_{>0}.$$

Remark 5.5. It turns out that $e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_k}$ are exactly the eigenvalues of the action of the monodromy operator on the nearby cycle complex [Mal83, Kas83]. We will not explain this in details, but let us point out that as a consequence, the set of fractional parts $\{\{\alpha_1\}, \ldots, \{\alpha_k\}\}$ is a topological invariant.

Definition 5.6. We set $\alpha(f) := \alpha_1$ and call it the *minimal exponent* of f.

Another fundamental theorem [Kol97] relates the minimal exponent and the log canonical threshold:

$$\mathrm{lct}(f) = \min\big(\alpha(f), 1\big).$$

In particular, if $\alpha(f) \leq 1$, then the minimal exponent and the lct coincide.

Remark 5.7. Returning to the motivation from the introduction, finding the log canonical threshold via Bernstein-Sato polynomials is often much faster than constructing log resolutions. Algorithms for finding Bernstein-Sato polynomials are coded in Macaulay2. Alas, this approach is still relatively slow, and is also limited to complete intersections. The arithmetic methods discussed above are more flexible and effective.

The minimal exponent provides a refinement of singularities that are already log canonical: the guiding principle is that larger minimal exponents correspond to milder singularities. In fact, if $\alpha(f) > \frac{n}{2}$, then the singularity is necessarily smooth.

Mustaţă and Popa have studied in depth singularities satisfying $\alpha(f) \geq k+1$, which they call *k-log canonical*. They also introduced a sequence of ideals measuring these singularities, known as the *k-Hodge ideals*. This theory has found numerous applications, including to conjectures concerning the singularities of theta divisors and the independence conditions imposed by hypersurfaces. The study of these notions is rich and active; we refer to [MP19, MP22, CDM25] for further details. However, at present, the theory applies mostly to hypersurfaces and complete intersections.

We note that although we have formulated the discussion in terms of Bernstein-Sato polynomials, in practice much of the theory is developed using Hodge modules.

Remark 5.8. This survey began with an example motivating log canonicity and the log canonical threshold through the study of integrals. It is therefore natural to ask whether there is an analytic interpretation of the *minimal exponent*, an invariant that refines the log canonical threshold. The answer is yes!

To explain this, let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial, and view our favourite integral from the introduction as giving a function on $H := \{z \in \mathbb{C} \mid \Re(z) \leq 0\}$ by

$$t \longmapsto \int_{\mathbb{C}^n} \frac{1}{|f(x_1, \ldots, x_n)|^{2t}}\, \mu,$$

where μ is a compactly supported volume form (in practice one must vary over various forms, but we suppress these details).

Using the Bernstein-Sato polynomial and the functional equation (5.2.1), this function admits a meromorphic continuation to $\mathbb{C}$ so that the poles lie on the real line: $\mathbb{R}_{>0} = \{z \in \mathbb{C} \mid \Re(z) > 0 \text{ and } \Im(z) = 0\}$. A moment's inspection shows that the first pole is nothing other than the log canonical threshold. Moreover, if f is smooth, then the poles occur precisely at the positive integers $1, 2, 3, \ldots$, each with multiplicity one.

And now comes the analytic meaning of the minimal exponent. It is the smallest number at which a *new phenomenon* appears: either the first non-integer pole, or the first integer pole that is no longer simple but has higher multiplicity. This interpretation was proved by Loeser for isolated singularities [Loe85], and most recently in full generality by Davis, Lőrincz, and Yang [DLY25].

5.2.2. *Higher Du Bois and rational singularities.* We now change gears briefly, after which we connect the following discussion with the minimal exponent studied above.

One of the key results in algebraic geometry is the Hodge decomposition:

$$H^k(X, \mathbb{C}) \cong \bigoplus_{i+j=k} H^i\big(X, \Omega_X^j\big) \tag{5.8.1}$$

for a smooth projective variety X. This extends to singular varieties via the theory of mixed Hodge structures. Namely, if X is a projective variety over a field of characteristic 0, there exist complexes $\underline{\Omega}_X^i \in D^b_{\mathrm{coh}}(X)$, called the *Deligne-Du Bois complexes*. As a consequence of mixed Hodge theory, one obtains a (non-canonical) isomorphism:

$$H^k(X, \mathbb{C}) \cong \bigoplus_{i+j=k} \mathbb{H}^i(X, \underline{\Omega}_X^j). \tag{5.8.2}$$

We have already encountered the complex $\underline{\Omega}_X^0$. The remaining complexes are constructed similarly. With notation as in (2.16.1), we construct an object in $D^b_{\mathrm{coh}}(X)$:

$$\underline{\Omega}_X^i := \mathrm{Tot}\left((Rf_1)_*\Omega^i_{Y_1} \xrightarrow{d_0^* - d_1^*} (Rf_2)_*\Omega^i_{Y_2} \xrightarrow{d_0^* - d_1^* + d_2^*} (Rf_3)_*\Omega^i_{Y_3} \to \ldots \right). \tag{5.8.3}$$

Remark 5.9. At first glance, this definition may appear somewhat unnatural. It is more fruitful to view $\underline{\Omega}_X^i$ as the shefification of the functor Ω^i in the h- or eh-topology. Rather than formalising this perspective, we explain the underlying idea.

A key property of Ω^i is the following. Let $f\colon Y \to X$ be the blow-up of a smooth variety X along a smooth closed subvariety Z, with exceptional divisor E. Then

$$\begin{array}{ccc} Rf_*\Omega_E^i & \longleftarrow & Rf_*\Omega_Y^i \\ \uparrow & & \uparrow \\ \Omega_Z^i & \longleftarrow & \Omega_X^i \end{array}$$

is a homotopy pullback square. Equivalently, in the language of triangulated categories, this means that there is a natural exact triangle

$$\Omega_X^i \longrightarrow \Omega_Z^i \oplus Rf_*\Omega_Y^i \longrightarrow Rf_*\Omega_E^i \xrightarrow{+1}.$$

The guiding idea behind $\underline{\Omega}^i_X$ is that it is a universal complex which agrees with Ω^i_X when X is smooth and which, for arbitrary – not necessarily smooth – X and Z, satisfies an analogous descent property. Namely, for any such blow-up $f\colon Y \to X$:

$$\begin{array}{ccc} Rf_*\underline{\Omega}^i_E & \longleftarrow & Rf_*\underline{\Omega}^i_Y \\ \uparrow & & \uparrow \\ \underline{\Omega}^i_Z & \longleftarrow & \underline{\Omega}^i_X \end{array}$$

is a homotopy pullback square. Since X and each stratum appearing in its resolution can be resolved recursively, this property uniquely determines $\underline{\Omega}^i_X$.

Following the original work and motivation of Steenbrink, the study of singularities via the complexes $\underline{\Omega}^i_X$ was initiated in [MOPW23], where the implication ($\Longleftarrow$) in the theorem below was established. Shortly thereafter, [JKSY22] proved the converse implication ($\Longrightarrow$) and introduced the term *higher Du Bois* singularities.

Theorem 5.10 ([MOPW23, JKSY22])**.** *Let Z be a hypersurface in $\mathbb{A}^n_{\mathbb{C}}$. Let $m \geq 0$ be an integer. Then $\underline{\Omega}^i_Z \cong \Omega^i_Z$ for every $i \leq m$ if and only if $\alpha(Z) \geq m+1$.*

This motivates the following more general definition of m-Du Bois singularities introduced by Shen, Venkatesh, Vo [SVV23].

Definition 5.11. We say that an affine variety X over $\mathbb{C}$ is m-Du-Bois if $\underline{\Omega}^i_X \cong \Omega^{[i]}_X$ for all $i \leq m$ and $\operatorname{codim}_X(\operatorname{Sing}(X)) \geq 2m+1$.

In particular, for varieties with k-Du Bois singularities, Hodge decomposition (5.8.2) admits a particularly easy form. As a beautiful application, Friedman and Laza generalised Theorem 2.19 and consequently showed that the unobstructedness of deformations of Calabi-Yau varieties extends to singular ones with 1-Du Bois singularities.

Theorem 5.12 ([FL24b])**.** *Let $f\colon Y \to S$ be a flat proper family of complex algebraic varieties. Assume that all fibres of f have m-Du Bois locally complete intersection singularities. Then for all $i \geq 0$ and $j \leq m$, the sheaves $R^i f_*\Omega^j_{Y/S}$ are locally free and compatible with base change.*

Theorem 5.13 ([FL24b])**.** *Let Y be a Calabi-Yau variety over $\mathbb{C}$ with 1-Du Bois lci singularities. Then the deformation theory of Y is unobstructed.*

It is unknown at this point if there is a way to modify the statements so that the local complete intersection assumption can be dropped.

The study of higher Du Bois singularities was quickly followed by the introduction of higher rational singularities by Friedman and Laza in the aforementioned works [FL24b, FL24a] (see also [KL25]). Adopting the most general definition due to Shen-Venkatesh-Vo, we say that an affine variety X over $\mathbb{C}$ is *m-rational* if

$$\Omega^{[i]}_X \cong Rf_*\Omega^i_Y(\log E) \tag{5.13.1}$$

for all $0 \leq i \leq m$, where $f\colon Y \to X$ is a strong[4] resolution of singularities with exceptional divisor $E = \mathrm{Exc}(f)$, and $\mathrm{codim}_X(\mathrm{Sing}(X)) > 2m+1$.

Remark 5.14. There has been significant recent progress in the study of higher Du Bois and higher rational singularities such as [SVV23, SVV24, PSV26]. For instance, (5.13.1) can now be reformulated as

$$I\underline{\Omega}^i_X \cong \Omega^{[i]}_X,$$

where $I\underline{\Omega}^i_X$ denotes the aforementioned complex of intersection differential forms associated to one of the central objects in modern geometry, the intersection complex IC_X (see [DOR25, PP24, PSV26]). Moreover, the deformation property for higher rational singularities was established in [KW25]; see [MP19, Che25] for the case of complete intersections.

5.3. **Higher singularities in characteristic $p > 0$.** It is natural to ask whether arithmetic analogues of higher singularities exist. The answer is affirmative: *higher F-injectivity* was recently introduced by the author of this survey and Kawakami. All rings below are finitely generated over a perfect field k of characteristic $p > 0$.

Remark 5.15. To explain the definition, we first recall Cartier operators. Consider the de Rham complex and note that the maps are not R-linear:

$$0 \to R \xrightarrow{d} \Omega^1_R \xrightarrow{d} \Omega^2_R \xrightarrow{d} \cdots.$$

A key feature of characteristic $p > 0$ is that, after applying Frobenius, the differentials in $F_*\Omega^\bullet_R$ become R-linear. Indeed, locally for α in Ω^i_R and $r \in R$, since $p = 0$ in R:

$$d(r \cdot F_*\alpha) = F_*d(r^p\alpha) = F_*\big(pr^{p-1}dr \wedge \alpha + r^p d\alpha\big) = F_*(r^p d\alpha) = r \cdot d(F_*\alpha).$$

Let $Z\Omega^i_R$ and $B\Omega^i_R$ denote the cycles and boundaries in the complex $F_*\Omega^\bullet_R$, respectively. Now consider the map of sets

$$\gamma\colon R \to \Omega^1_R, \qquad r \mapsto r^{p-1}dr.$$

A direct computation shows that for $r, s \in R$:

(1) $\gamma(rs) = r^p\gamma(s) + \gamma(r)s^p$,
(2) $d\gamma(r) = 0$, and
(3) $\gamma(r+s) - \gamma(r) - \gamma(s) \in B\Omega^1_R$.

It follows that γ extends uniquely to an R-linear map $C^{-1}\colon \Omega^1_R \to F_*\Omega^1_R/B\Omega^1_R$. Taking exterior powers yields analogous maps for all i called *inverse Cartier operators*:

$$C^{-1}\colon \Omega^i_R \to F_*\Omega^i_R/B\Omega^i_R.$$

The standard proof of the Hodge decomposition over $\mathbb{C}$ (5.8.1) involves solving difficult elliptic partial differential equations. Cartier operators play a vital role in the purely algebraic proof, due to Deligne-Illusie, which proceeds via reduction modulo p.

We are now ready to define the higher F-injective condition. For simplicity, we restrict to the case of hypersurfaces; a general definition exists, but is more technical.

[4]That is, f is an isomorphism over the smooth locus of X.

Definition 5.16. We say that an affine variety X over k (assumed to be a hypersurface in $\mathbb{A}^n_k$) is *m-F-injective* if, for every closed point $x \in X$, every integer $0 \leq i \leq m$, and every integer $j \geq 0$ with $i + j \leq d$, the following map is injective

$$C^{-1} \colon H^j_x(\Omega^i_X) \longrightarrow H^j_x\big(F_*\Omega^i_X / B\Omega^i_X\big)\,.$$

In analogy with the classical Du Bois condition, we proved that if a reduction of a characteristic zero singularity X modulo $p \gg 0$ is m-F-injective, then X is m-Du Bois. Conversely, assuming the ordinarity conjecture – a strengthening of the weak ordinarity conjecture mentioned earlier – we showed that if X is m-Du Bois, then its reduction is m-F-injective for infinitely many primes p.

Remark 5.17. There is more work to come, including the notion of m-F-rational singularities and other further developments. One of the projects of the Colorado Bootcamp, for which this article serves as a survey, has been to study m-F-injectivity for cone singularities.

6. Appendix 1: proof of Fedder's criterion

In this appendix, we explain the proof of Fedder's criterion.

Proof of Fedder's criterion, Theorem 3.13. Let $X := \operatorname{Spec} k[x_1, \ldots, x_n]$. Set $\mathfrak{m} := (x_1, \ldots, x_n)$ and $\mathfrak{m}^{[p]} := (x_1^p, \ldots, x_n^p)$. By Theorem 3.12, the divisor D is F-split at $\mathfrak{m}$ if and only if $(\mathbb{A}^n, D)$ is F-split at $\mathfrak{m}$ that is

$$k[x_1, \ldots, x_n]_{\mathfrak{m}} \xrightarrow{F} F_*k[x_1, \ldots, x_n]_{\mathfrak{m}} \xrightarrow{\cdot F_* f^{p-1}} F_*k[x_1, \ldots, x_n]_{\mathfrak{m}} \tag{6.0.1}$$

splits. Up to killing denominators, this is equivalent to the existence of a $k[x_1, \ldots, x_n]$-module homomorphism

$$\phi \colon F_*k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$$

such that $\phi(F_* f^{p-1}) \notin \mathfrak{m}$.

We argue by case-by-case analysis. First, suppose that $f^{p-1} \in \mathfrak{m}^{[p]}$. Then $\phi(F_* f^{p-1}) \in \mathfrak{m}$ for any such ϕ, and hence D is not F-split.

Now suppose that $f^{p-1} \notin \mathfrak{m}^{[p]}$. Writing f^{p-1} as a sum of distinct monomials, there is a nonzero coefficient c of some monomial $x_1^{b_1} \cdots x_n^{b_n}$ with $0 \leq b_1, \ldots, b_n < p$. Since $F_*k[x_1, \ldots, x_n]$ is a free $k[x_1, \ldots, x_n]$-module with basis

$$\left\{F_* x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_1, \ldots, a_n < p\right\},$$

we can define ϕ on these generators by

$$\phi(F_* x_1^{a_1} \cdots x_n^{a_n}) = \begin{cases} 1 & \text{if } (a_1, \ldots, a_n) = (b_1, \ldots, b_n), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h := \phi(F_* f^{p-1}) \in c + \mathfrak{m}$, so D is F-split at $\mathfrak{m}$, as required. □

Exercise 6.1. In the proof of the above theorem we invoked Theorem 3.12, which in full generality requires some work. However, in the special case $X = \operatorname{Spec} k[x_1, \dots, x_d]$ the argument is simpler. Show that for $f \in \mathfrak{m} := (x_1, \dots, x_d)$

$$(\operatorname{Spec} k[x_1, \dots, x_d], V(f)) \text{ is } F\text{-split at } \mathfrak{m} \iff k[x_1, \dots, x_d]/f \text{ is } F\text{-split at } \mathfrak{m}$$

Remark 6.2. In this remark we give a hint for the above exercise. Given a splitting ϕ of (6.0.1), define $\psi \colon F_*(k[x_1, \dots, x_n]_{\mathfrak{m}}/f) \to k[x_1, \dots, x_n]_{\mathfrak{m}}/f$ by setting

$$\psi(F_*[g]) := [\phi(gf^{p-1})]$$

where $g \in k[x_1, \dots, x_n]_{\mathfrak{m}}$. For the opposite direction, use that a map on a free module can be defined on the generators to lift ψ to a map

$$\psi_0 \colon F_* k[x_1, \dots, x_n]_{\mathfrak{m}} \to k[x_1, \dots, x_n]_{\mathfrak{m}}$$

so that $\phi(F_* h) := \frac{1}{f}\psi_0(F_* hf)$ splits (6.0.1).

7. Appendix 2: local cohomology

Let $X = \operatorname{Spec} R$ for a ring R, and let $Z := V(I) \subseteq X$ be a closed subset defined by an ideal $I \subseteq R$. Let $\Gamma_Z(-)$ denote the functor which assigns to a coherent sheaf on X the R-module of sections supported on Z. The i-th *local cohomology* of a coherent sheaf $\mathcal{M}$ on X along Z is defined by

$$H^i_Z(\mathcal{M}) := R^i\Gamma_Z(\mathcal{M}).$$

Remark 7.1. Local cohomology admits a more explicit description. For $i \geq 2$, one has

$$H^i_Z(\mathcal{M}) = H^{i-1}(U, \mathcal{M}|_U),$$

where $U := X \setminus Z$ is the complement of Z. For $i \leq 1$, there is a long exact sequence

$$0 \to H^0_Z(\mathcal{M}) \to H^0(X, \mathcal{M}) \to H^0(U, \mathcal{M}|_U) \to H^1_Z(\mathcal{M}) \to 0.$$

Example 7.2. Let $R = k[x]$ for a field k, and let $\mathfrak{m} = (x)$. Then

$$H^0_{\mathfrak{m}}(R) = 0,$$
$$H^1_{\mathfrak{m}}(R) = k[x, x^{-1}]/k[x].$$

A key property of local cohomology is that if R is a Noetherian ring and M is a finitely generated R-module, then M is Cohen–Macaulay if and only if

$$H^i_{\mathfrak{m}}(M) = 0 \quad \text{for all maximal ideals } \mathfrak{m} \subset R \text{ and all } i < \dim \operatorname{Supp}(M_{\mathfrak{m}}).$$

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