The Geometry of Smooth Quartics

Jakub Witaszek

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Advisor: Prof. Dr. Daniel Huybrechts

Mathematical Institute

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

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Introduction

Projective hypersurfaces played a vital role in the history of algebraic geometry and they are still a source of surprising phenomena. Mathematicians were for decades captivated by the idea of describing their geometry. It all started with the result of Cayley and Salmon, who proved that there are exactly 27 lines on any cubic surface. This marked a significant milestone in the development of algebraic geometry and led to future profound discoveries.

In this thesis we present a description of the geometry of very general smooth quartics in $\mathbb{P}^3$ in relation to the projective Gauss map. As far as we know, there is no reference containing a modern comprehensive study of this subject. Some of the proofs in this thesis were developed by the author, other are rewritten or simplified versions from other papers. The majority of the results have been known to the 19$^{\text{th}}$ century mathematicians, but we hope that the reader will appreciate our modern approach to this subject. We combine methods of the enumerative theory of singularities, degenerations of plane curves, stable maps and projective differential geometry.

Smooth quartics in $\mathbb{P}^3$ are the simplest examples of K3 surfaces. There are still many conjectures about K3 surfaces, like Bloch-Beilinson conjecture, which haven't been checked even for a single example. We hope that this thesis may help other mathematicians in verifying their hypothesis.

Our main objects of interest are three curves: *the parabolic curve* C_{par} which is the ramification locus of the Gauss map, *the double-cover curve* which is the non-injective locus of the Gauss map, and *the flecnodal curve* C_{hf}, which is the locus of points p of the property that there exists a line intersecting the surface at p with multiplicity four.

A curve on a surface is called *a constant cycle curve*, if all of its points are represented by the same element in the Chow group CH_0. They were introduced in [19] in order to better understand CH_0 of a K3 surface. In the same paper it is proved that the curve C_{hf} is a constant cycle curve. There is the following conjecture raised by Prof. Huybrechts:

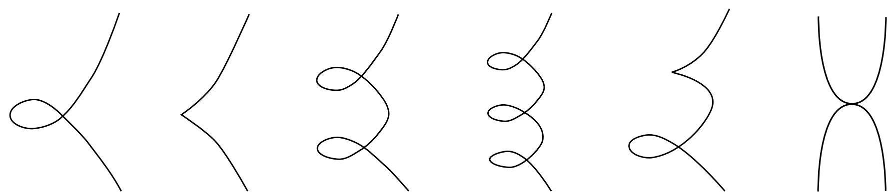


Figure 1: Singularities of hyperplane sections of very general smooth quartics

Conjecture. *The parabolic curve C_{par} and the double-cover curve C_{d} are constant cycle curves.*

We show that the method of proving that C_{hf} is a constant cycle curve, does not work in the case of C_{d}. Further, we show that the curve C_{par} is a constant cycle curve for the Fermat quartic.

The main reference for the theory of hypersurfaces in $\mathbb{P}^3$ is an old book [27]. A local description of the Gauss map and properties of the curve C_{par} and its interesection points with C_{hf} may be found in [24]. The properties of the curve C_{hf} are shown in [34]. The description of the singularities of dual surfaces, which uses a method of multitransversality, is contained in [6]. A modern proof of Cayley-Zeuthen's formulas, describing numerical data of hypersurfaces in $\mathbb{P}^3$, may be found in [25].

The paper is organised as follows. In the first chapter we review some properties of plane curves, stable maps, the enumerative theory and the Gauss map. In the second chapter we describe the geometry of very general smooth quartics in $\mathbb{P}^3$. We present the Gauss map in local coordinates, classify hyperplane sections, analyse enumerative properties and singularities of special curve, and describe the singularities of dual surfaces. In the last chapter, we discuss some properties of general smooth quartics based on the analysis of the space of smooth quartics.

Acknowledgements

The author would like to thank Professor Daniel Huybrechts for his indispensable help in writing this thesis.

Notations and conventions

In this thesis all the varieties are defined over $\mathbb{C}$.

General

$\hat{R}$	The completion of a ring R.
$\overline{pq}$	The line through points p and q.
$\check{\mathbb{P}}^3$	The space of hyperplanes in $\mathbb{P}^3$.
$\mathrm{Res}(\omega)$	The residue of a form ω.
$\mathbb{C}[[x_1, \ldots, x_k]]$	The ring of formal power series in variables $x_1, \ldots, x_k$.

Curves

$C_1 \pitchfork_p C_2$	Curves C_1 and C_2 interesect transversally at a point p.
$C_1 \curlyvee_p C_2$	Curves C_1 and C_2 are tangent at a point p.
TC_p	The reduction of the tangent cone of a curve C at a point p.
$\mu_p(C)$	The Milnor number of a point p on a curve C.
$r_p(C)$	The number of analytic branches of a curve C at a point p.
$\delta_p(C)$	The δ-invariant of a point p on a curve C.
$g(C)$	The genus of a curve C.
$p_a(C)$	The arithmetic genus of a curve C.
$\mathrm{Hess}_p(C)$	The Hessian matrix of a function f describing C locally around p.

Varieties

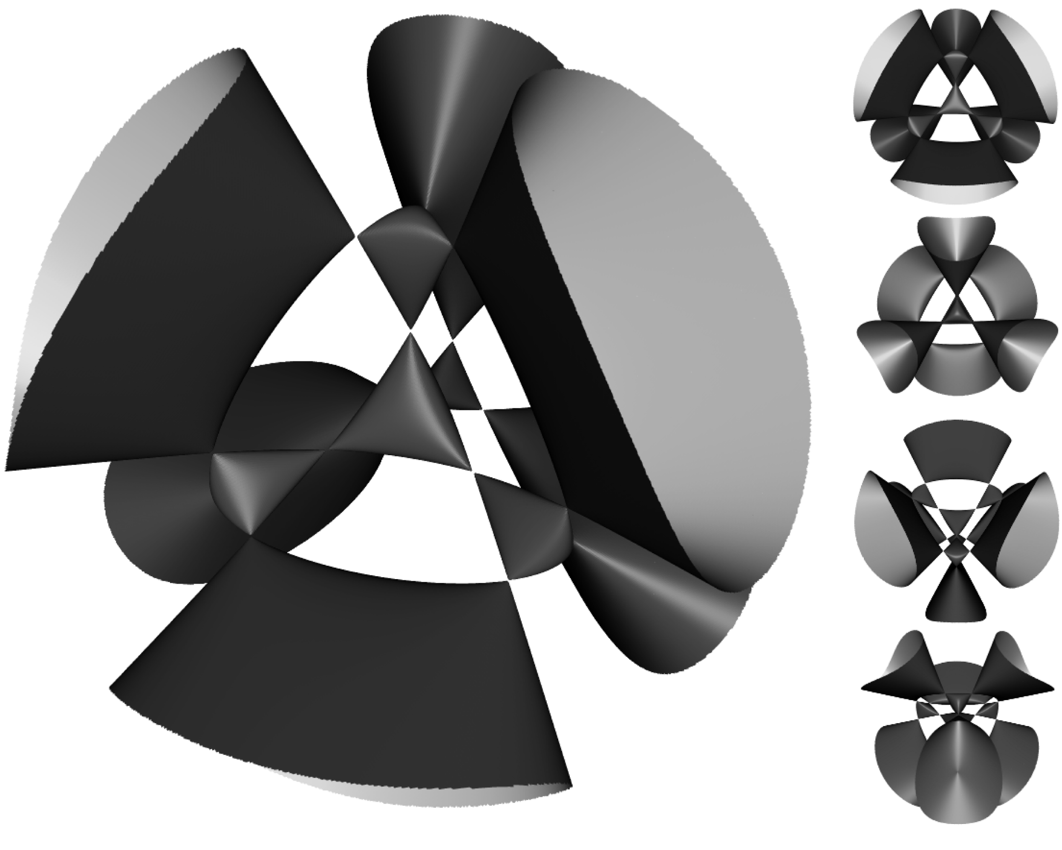
$\deg(X)$	The degree of an embedded variety $X \subseteq \mathbb{P}^n$.
$\mathrm{Sing}(X)$	The set of singular points of a variety X.
$\mathrm{Pic}(X)$	The Picard group of a scheme X.
$\mathbf{Pic}_{X/T}$	The Picard scheme of a T-scheme X.
$\mathrm{CH}_k(X)$	The Chow group of k-cycles on a variety X.
$[V]$	The class of a k-cycle V in $\mathrm{CH}_k(X)$.
T_pX	The tangent space of a variety X at a point p.
$\Sigma(\pi)$	The locus of points where a morphism π is not smooth.

Surfaces in $\mathbb{P}^3$

$\mathrm{I\!I}_p$	The projective second fundamental form at a point p.
E_pS	The tangent curve of a surface S at a point p.
C_{par}	The parabolic curve.
C_{d}	The double cover curve.
C_{hf}	The flecnodal curve.
$\mathrm{Swallowtail}(S)$	The set of Gauss swallowtail points on a surface S.
Γ_q	The polar locus of a point q.
$\lvert\mathcal{O}_{\mathbb{P}^3}(4)\rvert^{\mathrm{sm}}$	The space of smooth quartics in $\mathbb{P}^3$.

We say that *a general point* on a variety X satisfies a property P, if there exists an open dense subset of points of X satisfying the property P.

We say that *a very general point* on a variety X satisfies a property P, if there exists a countable union of closed subsets of X such that all the points outside of it satisfy the property P.



CHAPTER 1

Preliminaries

We recall the definition of the conductor. Let X be a variety and $p\colon \overline{X} \to X$ its normalization. We define *the conductor scheme* $\mathcal{C} \subseteq \overline{X}$ to be the inverse image of the non-normal locus of X. Formally, if $\mathcal{I} \subseteq \mathcal{O}_X$ is the maximal ideal sheaf for which $\mathcal{I}\mathcal{O}_{\overline{X}}$ is a subset of $\mathcal{O}_X$, then $\mathcal{C}$ is defined by the ideal $\mathcal{I}\mathcal{O}_{\overline{X}}$ in $\mathcal{O}_{\overline{X}}$.

1.1 Curves

In this section, we recall theorems about singularities of plane curves. We are particularly interested in their deformations.

1.1.1 Singularities of plane curves

We review properties of singularities of plane curves and classify singularities of plane curves of degree four.

Let $C \subseteq \mathbb{P}^2$ be a plane curve. Denote by $\operatorname{mult}_p(C)$ the multiplicity of a point $p \in C$.

Remark 1.1.1. A singular point $p \in C$ satisfies $\operatorname{mult}_p(C) = 2$ if and only if $\operatorname{Hess}_p(f) \neq 0$ at p, where $f \in \mathbb{C}[x, y]$ is an equation defining C locally around p.

Let $p \in \operatorname{Sing}(C)$ be a singular point with $\operatorname{mult}_p(C) = 2$. Then p is analytically isomorphic to the singularity defined by the equation $xy = 0$ or $x^2 - y^k$ for $k \geq 3$. We call the singularity $xy = 0$ *a node*, the singularity $x^2 - y^3 = 0$ *a cusp* and the singularity $x^2 - y^4 = 0$ *a tacnode*. We say that a singularity is *ordinary* if it is a node or a cusp. Note that this notation differs slightly from the usual one.

The tangent cone of C at p is defined by

$$v^T \operatorname{Hess}_p(f) v = 0,$$

where $v \in \mathbb{C}^2$ and $f \in \mathbb{C}[x,y]$ is an equation describing C locally around p. We define TC_p to be the reduction of the tangent cone of C at p. If p is a node, then TC_p is a union of two lines. Otherwise, TC_p is a line.

A crucial invariant of singularities is the Milnor number.

Definition 1.1.2. Take a power series $F \in \mathbb{C}[\![x,y]\!]$ which describes C locally around p. Define the *Milnor number* $\mu_p(C)$ by

$$\mu_p(C) := \dim \mathbb{C}[\![x,y]\!]/(F_x, F_y),$$

where F_x and F_y are derivatives of F. The Milnor number does not depend on the choice of F describing C.

Let $r_p(C)$ be the number of branches of C at a singular point p. We define *the δ-invariant* by the formula

$$\delta_p(C) := \frac{1}{2}(\mu_p(C) + r_p(C) - 1).$$

Lemma 1.1.3 ([33, Corollary 7.1.3])**.** *Let C be an irreducible plane curve of degree d. Then*

$$g(C) = \frac{1}{2}(d-1)(d-2) - \sum_{p \in \mathrm{Sing}(C)} \delta_p(C).$$

Remark 1.1.4. For $p \in \mathrm{Sing}(C)$, we have

$$\mu_p(C) \geq \binom{\mathrm{mult}_p(C)}{2},$$
$$\sigma_p(C) \geq \frac{1}{2}\binom{\mathrm{mult}_p(C)}{2}.$$

The invariants of the singularities of multiplicity two have the following values:

	Ordinary singularities				
	Node	Cusp	Tacnode	General singularity, $n \geq 3$	
Equation	xy	$x^2 - y^3$	$x^2 - y^4$	$x^2 - y^{2n-1}$	$x^2 - y^{2n}$
μ_p	1	2	3	$2n-2$	$2n-1$
r_p	2	1	2	1	2
δ	1	1	2	$n-1$	n

Figure 1.1: The numerical invariants of the singularities of multiplicity two

Remark 1.1.5. Assume that p is the singularity $x^2 - y^k$ for $k \geq 3$. Then

$$\mathrm{mult}_p(TC_p \cap C) = k.$$

We will use the following lemma to understand the singularities of curves in $|\mathcal{O}_S(1)|$, where $S \subset \mathbb{P}^3$ is a very general smooth surface of degree four.

Lemma 1.1.6. *Let $C \subseteq \mathbb{P}^2$ be a curve of degree four. Then C has at most three singularities. If $g(C) = 0$ and C is nodal, then C has exactly three nodes. If $g(C) = 1$, then* $\operatorname{Sing}(C)$ *consists of*

1. *one point of multiplicity three, or*
2. *one tacnode, or*
3. *two nodes, or*
4. *node and a cusp, or*
5. *two cusps.*

If $g(C) = 2$, then C has exactly one node or one cusp.

Proof. Lemma 1.1.3 shows that

$$g(C) = 3 - \sum_{p \in \operatorname{Sing}(C)} \sigma_p(C).$$

Since $g(C) \geq 0$ and $\sigma_p(C) \geq 1$ for $p \in \operatorname{Sing}(C)$, the curve C has at most three singularities.

If $g(C) = 0$ and C is nodal, then $\sigma_p(C) = 1$ for every $p \in \operatorname{Sing}(C)$, and so C has exactly three nodes.

If $g(C) = 2$, then by the genus formula there can only be one singularity, with δ-invariant equal to one. Thus, it must be a cusp or a node.

Let now $g(C) = 1$. Take $p \in \operatorname{Sing}(C)$. Note that p cannot be the singularity $x^2 - y^k$ for $k \geq 5$, because in such case $\operatorname{mult}_p(TC_p \cap C) \geq 5$, and this contradicts the assumption that $\deg(C) = 4$. We have

$$\begin{aligned} \sigma_p(C) &= 1, \text{ if } p \text{ is a node or a cusp,} \\ &\geq 2, \text{ if } p \text{ is a tacnode or } \operatorname{mult}_p(C) = 3, \\ &\geq 3, \text{ if } \operatorname{mult}_p(C) \geq 4. \end{aligned}$$

Thus, this part of the lemma also follows from the genus formula. □

Proposition 1.1.7. *Let E be an algebraic, not necessarily plane, curve. Assume that* $\operatorname{Sing}(E)$ *consists of nodes or cusps. Then,*

$$g(E) = p_a(E) - |\operatorname{Sing}(E)|,$$

where $p_a(E)$ is the arithmetic genus of E.

Proof. See [17, Exercise IV.1.8]. □

Remark 1.1.8. Since any smooth surface is locally analytically isomorphic to open subsets of $\mathbb{C}^2$, the properties of the singularities of plane curves extend to curves on smooth surfaces.

1.1.2 Simultaneous resolutions of singularities

The following section is based on [16, Section 2]. We show that a certian strong simultaneous resolution of singularities holds. Later, we will use it in Section 3.3.

Let

$$\begin{tikzcd} \mathcal{X} \subseteq T \times \mathbb{P}^n \arrow[d, "\pi"] \\ T \end{tikzcd}$$

be a flat family of projective reduced curves, where $\mathcal{X}$ and T are reduced varieties. Let $\phi(t)$ be the geometric genus of $\mathcal{X}_t$ for $t \in T$.

The following holds.

Lemma 1.1.9 ([16, Theorem 2.4])**.** *The function ϕ is lower semicontinous in the Zariski topology.*

Lemma 1.1.10 ([8, p. 80] or [16, Theorem 2.5])**.** *Assume that T is normal and ϕ is constant on T. Let $p\colon \mathcal{X}' \to \mathcal{X}$ be the normalization.*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "p"] \arrow[dr, "\pi \circ p"'] & \mathcal{X} \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

Then $\pi \circ p\colon \mathcal{X}' \to T$ is a smooth family of curves and each fiber of $\pi \circ p$ is the normalization of the corresponding fiber of π.

We need the following lemma in Section 3.3.

Lemma 1.1.11. *Assume that ϕ is constant on T. Then there exists a strong simultaneous resolution of singularities, that is a diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "\phi"] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ T' \arrow[r, "\psi"] & T \end{tikzcd}$$

together with divisors $D_1, \ldots, D_m$ on $\mathcal{X}'$, where

- *T' and $\mathcal{X}'$ are smooth,*
- *ψ is finite,*
- *$\mathcal{X}'_{t'}$ is the normalization of the curve $\mathcal{X}_{\psi(t')}$ for $t' \in T'$,*

- *$D_1|_{\mathcal{X}'_{t'}}, \dots, D_m|_{\mathcal{X}'_{t'}}$ are exactly the points of $\phi^*\left(\mathrm{Sing}\left(\mathcal{X}_{\psi(t')}\right)\right)$ for a general $t' \in T'$.*

Proof. By taking the base change with the normalization $\overline{T} \to T$, we may assume witout loss of generality that T is normal.

Let $p\colon \overline{\mathcal{X}} \to \mathcal{X}$ be the normalization of $\mathcal{X}$ and let $\mathcal{C} \subseteq \overline{\mathcal{X}}$ be the reduction of the conductor of p. By Lemma 1.1.10, the morphism $\overline{\mathcal{X}} \to T$ is a simultaneous resolution of singularities. Now, we need to modify $\overline{\mathcal{X}}$, so that we would be able to construct divisors $D_1, \dots, D_m$.

Since smoothness is an open condition, the subset $\Sigma(f) = \bigcup_{t\in T} \mathrm{Sing}(\mathcal{X}_t)$ is closed. Further, $\mathcal{X} \setminus \Sigma(f) \to T$ is smooth, and so $\mathcal{X}$ is normal outside of $\Sigma(f)$ by Lemma 1.1.12. On the other hand, $\overline{\mathcal{X}} \to T$ is a simultaneous resolution of singularities, and thus $\mathcal{X}$ cannot be normal at points of $\Sigma(f)$. It shows that $\Sigma(f) = p(\mathcal{C})^{\mathrm{red}}$. In particular, $\mathcal{C}_t := \mathcal{C}|_{\overline{\mathcal{X}}_t}$ for $t \in T$ is set theoretically equal to $p^*\left(\mathrm{Sing}\left(\mathcal{X}_t\right)\right)$.

Let μ_T be the generic point of T and let $k(\mu_T) = K(T)$ be its residue field. Consider a finite field extension $L \supseteq k(\mu_T)$ such that the zero-dimensional generic fiber $\mathcal{C}_{\mu_t}$ of $f \circ p|_{\mathcal{C}}\colon \mathcal{C} \to T$ splits over L into geometrically connected points.

Let $\hat{T}$ be the closure of T in L. Over an affine $\mathrm{Spec}(A) \subseteq T$, the morphism $\hat{T} \to T$ restricts to the morphism $\mathrm{Spec}(A^L) \to \mathrm{Spec}(A)$, where A^L is the normal closure of A in the field L.

$$\begin{tikzcd} k(\mu_T) \arrow[r, hook] & L \\ A \arrow[u, hook] \arrow[r, hook] & A^L \arrow[u, hook] \end{tikzcd}$$

Take T' to be the resolution of singularities of $\hat{T}$. Let $\mu_{T'}$ be the generic point of T'. Consider the diagram

$$\begin{tikzcd} \mathcal{X}' := \overline{\mathcal{X}} \times_T T' \arrow[r, "\phi"] \arrow[d] & \overline{\mathcal{X}} \arrow[r, "p"] \arrow[d] & \mathcal{X} \arrow[dl, "f"] \\ T' \arrow[r] & T & \end{tikzcd}$$

Take divisors D_i to be the closures in $\mathcal{C} \times_T T'$ of closed points of $\mathcal{C} \times_{k(\mu_T)} k(\mu_{T'})$. This gives us a strong resolution of singularities. Note that $\mathcal{X}'$ is smooth, because T' is smooth and the morphism $\mathcal{X}' \to T'$ is smooth (see [17, Proposition 10.1]). □

Now, we show the lemma used in the proof above.

Lemma 1.1.12. *Let $X \to Y$ be a smooth morphism between varieties. Suppose that Y is normal. Then X is normal.*

Proof. By [14, Exposé 2], morphism $X \to Y$ decomposes into $X \to \mathbb{A}^n_Y \to Y$, where $X \to \mathbb{A}^n_Y$ is étale and $\mathbb{A}^n_Y \to Y$ is a projection. Since $\mathbb{A}^n_Y$ is normal, the variety X must also be normal by [14, Théorème 1.9.5]. □

In order to use the result from this section, we need to be able to check when a family of curves is flat over a base. For this, we recall the following theorem.

Theorem 1.1.13 ([17, Theorem 9.9])**.** *Let T be an integral noetherian scheme. Let $X \subseteq \mathbb{P}^n_T$ be a closed subscheme. For each point $t \in T$, we consider the Hilbert polynomial $P_t \in \mathbb{Q}[z]$ of the fiber X_t considered as a closed subscheme of $\mathbb{P}^n_{k(t)}$. Then X is flat over T if and only if the Hilbert polynomial P_t is independent of t.*

Recall that the Hilbert polynomial is uniquely determined by the equation $P_t(m) = \chi(\mathcal{O}_{X_t}(m))$ for $m \in \mathbb{Z}$.

Corollary 1.1.14. *A family of plane curves of fixed degree is flat.*

Proof. Recall the Riemann-Roch formula for singular curves (see [17, Exercise IV.1.9])

$$\chi(\mathcal{O}_C(D)) = \deg(D) + \chi(\mathcal{O}_C),$$

where C is a curve and D is a divisor such that $\mathrm{Supp}(D) \in C^{\mathrm{sm}}$. It implies that the Euler characteristic of a line bundle on a plane curve depends only on the degree of the line bundle and the degree of the curve. Thus, fibers in a family of plane curves of fixed degree have the same Hilbert polynomial. □

1.1.3 Spaces of plane curves of fixed degree

The following section is based on [16, p. 455] and [28, Subsection 4.7.2]. We present formulas for the dimensions of the spaces of plane curves of fixed degree. We will use them later on to show that certain curves on smooth quartic hypersurfaces degenerate in families and also that certain curves do not occur on general smooth quartics.

We consider curves of degree d in $\mathbb{P}^2$. They are represented by sections of $\mathcal{O}_{\mathbb{P}^2}(d)$ (homogenous polynomials of degree d in three variables). We say that $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(d))$ is the *space of plane curves* of degree d.

We define the Severi variety $\mathcal{V}_d^{\delta,\kappa}$ to be the subset of curves in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(d))$ with exactly δ nodes and κ cusps. By [28, Theorem 4.7.3], it is a locally closed subvariety of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(d))$. The following holds.

Proposition 1.1.15. *If $\kappa < 3d$, then*

$$\dim\left(\mathcal{V}_d^{\delta,\kappa}\right) = 3d + g - 1 - \kappa.$$

Proof. See [28, Corollary 4.7.8 and p. 312]. □

Now, we present a formula for the dimensions of spaces of plane curves of fixed degree and genus.

Lemma 1.1.16 ([16, Lemma 4.14]). *Let $U^{d,g}$ be the closure in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(d))$ of the locus of points corresponding to reduced curves of degree d and geometric genus g. Then every component of $U^{d,g}$ has dimension $3d+g-1$.*

Let

$$\begin{array}{ccc} \mathscr{C} & \subseteq & \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(d)) \times \mathbb{P}^2 \\ \big\downarrow \pi & & \\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(d)) & & \end{array}$$

be the universal family of plane curves of degree d. The fiber $\mathscr{C}_f \subseteq \mathbb{P}^2$ for $f \in \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(d))$ is the plane curve described by the equation $f = 0$. The family is flat by Corollary 1.1.14.

By the semicontinuity of the geometric genus (see Lemma 1.1.9) applied to $\mathscr{C}$, we see that curves in $U^{d,g}$ have geometric genus at most g.

1.1.4 Deformations of germs of singularities

The following section is based on [7]. We need to understand when a set of singularities can collide to a certain singularity.

Let

$$\begin{array}{c} (\mathcal{X}, x_0) \\ \big\downarrow \pi \\ (D, 0) \end{array}$$

be the germ of a flat family of reduced plane curves, where $D \subset \mathbb{C}$ is a small disc with center 0.

For $t \in D$ we define

$$\mu_t := \mu(\mathcal{X}_t) = \sum_{p \in \mathrm{Sing}(\mathcal{X}_t)} \mu(\mathcal{X}_t, p),$$
$$\delta_t := \delta(\mathcal{X}_t) = \sum_{p \in \mathrm{Sing}(\mathcal{X}_t)} \delta(\mathcal{X}_t, p).$$

Theorem 1.1.17. *Functions μ_t and δ_t are upper semicontinous in the Euclidean topology.*

Proof. See [7, Theorem 6.1.7]. □

In particular, we get that two singularities cannot collide to an ordinary singularity.

Corollary 1.1.18. *Assume that $\mathcal{X}_t$ has at least two singularities for $t \in D \setminus \{0\}$. Then $\delta_0 \geq 2$, and so x_0 cannot be an ordinary singularity of $\mathcal{X}_0$.*

□

Remark 1.1.19. The above theorem is also valid for nonplanar singularities.

1.2 Locally stable maps and their singularities

Given a holomorphic map $f\colon X \to Y$ between complex manifolds, we would like to describe it and its image locally. Without any assumptions on the map f, this is a very difficult task, because its singularities may be very complicated.

In this section, we recall the definition of locally stable maps, whose behaviour is much easier to describe. Then we explain under which assumptions the analytic branches of the image of a stable map at some point are transversal to each other.

1.2.1 Locally stable maps

The following material is based on [32]. The definitions in the differentiable setting may be found in [2] and [22].

Let X_1, X_2, Y_1, Y_2 be complex manifolds. We say that two holomorphic maps f_1 and f_2 are *equivalent* if there exist biholomorphisms $\psi\colon X_1 \to Y_1$ and $\phi\colon X_2 \to Y_2$, together with a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\psi"] & Y_1 \arrow[d, "\phi"] \\ X_2 \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

We say that two map-germs are *equivalent*, if they are germs of equivalent holomorphic maps.

Definition 1.2.1. Let X and Y be complex manifolds, S be a subset of points of X and p be a point on Y. A holomorphic multigerm $f\colon (X, S) \to (Y, p)$ satisfying $f(S) = p$ is *simultaneously stable*, if any small deformation of f is trivial.

Definition 1.2.2. Let X and Y be complex manifolds. We say that a holomorphic map $f\colon X \to Y$ is *locally stable* if for every point $p \in Y$ and every finite subset $S \subseteq f^{-1}(p)$, the multigerm $f\colon (X, S) \to (Y, p)$ is simultaneously stable.

We state a proposition of Mather, which says that stability is a Zariski open condition. For a morphism of smooth varieties $f\colon X \to Y$, we denote by $\Sigma(f)$ the subvariety of X where f is not smooth

$$\Sigma(f) := \{x \in X \mid \operatorname{rank}(df_x) < \dim(Y)\}\,.$$

Proposition 1.2.3 ([22, Proposition 1])**.** *Consider the following diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[dr] & \mathcal{Y} \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

where $\mathcal{X}$, $\mathcal{Y}$ and T are smooth varieties. For $t \in T$, we denote by $f_t \colon \mathcal{X}_t \to \mathcal{Y}_t$ the restriction of f to the fibers over t. Suppose that f and $\pi \circ f$ are smooth. Additionally, assume that $f|_{\Sigma(f)}$ and π are projective morphisms. Then

$$\{t \in T \mid f_t \textit{ is locally stable}\}$$

is a Zariski open subset of T.

The following result is crucial in describing the Gauss map locally.

Proposition 1.2.4 ([24, (3.4)])**.** *The space of simulatenously stable map-germs $f \colon (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$ consists of three equivalence classes:*

- *fold,* $(x_1, x_2, x_3, x_4) \overset{F_1}{\longmapsto} (x_1^2 + x_2^2, x_3, x_4)$,
- *cusp,* $(x_1, x_2, x_3, x_4) \overset{F_2}{\longmapsto} (x_1^2 + x_2^3 + x_2 x_3, x_3, x_4)$,
- *swallowtail point,* $(x_1, x_2, x_3, x_4) \overset{F_3}{\longmapsto} (x_1^2 + x_2^4 + x_2^2 x_3 + x_2 x_4, x_3, x_4)$.

1.2.2 The normal-crossing condition and transversality

We follow the presentation of [13, VI.5]. Let $f \colon X \to Y$ be a map between complex manifolds. We define $S_i(f)$ to be the set of points of X, where the rank of f drops by i. Inductively, we define

$$S_{i_1,\ldots,i_k}(f) = \left\{ x \in S_{i_1,\ldots,i_{k-1}}(f) \;\middle|\; \operatorname{rank}(f|_{S_{i_1,\ldots,i_{k-1}}}) \text{ drops by } i_k \right\}.$$

Definition 1.2.5. Let $I_1, \ldots, I_k$ be multi-indices. Take distinct points $x_1, \ldots, x_k$ such that $x_j \in S_{I_j}$ for $1 \leq j \leq k$ and

$$f(x_1) = \ldots = f(x_k).$$

Let H_j be the tangent space to S_{I_j} at x_j. We say that f satisfies the *NC* condition if

$$(df)_{x_1} H_1, \ldots, (df)_{x_k} H_k$$

lie in a general position in the tangent space to Y.

Proposition 1.2.6. *Holomorphic locally stable maps satisfy the NC condition.*

Proof. See [32, Corollary 1.5] and in the differentiable setting [13, Theorem 5.2]. □

To understand the importance of the NC condition, we first recall the definition of transversality.

Definition 1.2.7. Let X_1, X_2 and Y be complex manifolds. We say that two maps $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ are *transverse* to each other, if for every $x_1 \in X_1$ and $x_2 \in X_2$ such that $f_1(x_1) = f_2(x_2) = y$, we have

$$(df_1)_{x_1}(T_{x_1}X_1) + (df_2)_{x_2}(T_{x_2}X_2) = T_yY.$$

We use the same notation as in the definition of the NC condition. Take $k = 2$ and let U_1 and U_2 be small open neighbourhoods of x_1 and x_2 respectively. If we assume that $\dim((df)_{x_1}H_1) + \dim((df)_{x_2}H_2) \geq \dim Y$, then the NC condition implies that $f|_{U_1}$ and $f|_{U_2}$ are transverse.

The notion of transversality is important, because of the following proposition.

Proposition 1.2.8. *Let X_1, X_2 and Y be complex manifolds and let $f_1\colon X_1 \to Y$, $f_2\colon X_2 \to Y$ be two transverse morphisms. Assume that $f_2(X_2) \subseteq Y$ is smooth. Then $f_1^{-1}(f_2(X_2)) \subseteq X_1$ is smooth.*

Proof. See [15, Section §5]. □

1.3 Picard schemes

The following section is based on [11]. We use Picard schemes in Section 3.3.

Let $X \to T$ be a morphism of schemes.

Definition 1.3.1. We define the relative Picard functor $\operatorname{Pic}_{(X/T)}$ from the category of T-schemes to abelian groups by

$$\operatorname{Pic}_{(X/T)}(S) := \operatorname{Pic}(X \times_T S)/\operatorname{Pic}(S),$$

where S is a T-scheme.

We say that a morphism $f\colon Y' \to Y$ between schemes is an *étale covering* if it is étale and surjective. Note that we don't assume anything about the connectedness of Y or Y'.

Let $\operatorname{Pic}_{(X/T)(\text{ét})}$ be the sheafification in the étale topology of the functor $\operatorname{Pic}_{(X/T)}$. It is defined by the following conditions:

- Let S be a T-scheme. Elements in $\operatorname{Pic}_{(X/T)(\text{ét})}(S)$ are represented by those line bundles $\mathcal{L}' \in \operatorname{Pic}_{X/T}(S')$, where $S' \to S$ is an étale coverings, for which there exists an étale covering f

$$S'' \xrightarrow{f} S' \times_S S' \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} S'$$

such that $f^*(\pi_1^*\mathcal{L}') = f^*(\pi_2^*\mathcal{L}')$, where π_1 and π_2 are natural projections.

- Take two elements $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}_{(X/T)(\text{ét})}(S)$ represented by $\mathcal{L}'_1 \in \mathrm{Pic}_{(X/T)}(S'_1)$ and $\mathcal{L}'_2 \in \mathrm{Pic}_{(X/T)}(S'_2)$ for étale coverings $S'_1 \to S$ and $S'_2 \to S$. Then they are identified in $\mathrm{Pic}_{(X/T)(\text{ét})}(S)$ if and only if there exists an étale covering f

$$\begin{tikzcd} & & S'_1 \arrow[dr] & \\ S'' \arrow[r, "f"] & S'_1 \times_S S'_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & S \\ & & S'_2 \arrow[ur] & \end{tikzcd}$$

such that $f^*(\pi_1^*\mathcal{L}'_1) = f^*(\pi_2^*\mathcal{L}'_2)$, where π_1 and π_2 are natural projections.

The following theorem deals with the representability of the Picard functor.

Theorem 1.3.2 ([11, Theorem 9.4.8])**.** *Assume that $f\colon X \to T$ is projective Zariski locally over T, and is flat with integral geometric fibers. Then $\mathrm{Pic}_{(X/T)(\acute{e}t)}$ is representable by a separated scheme locally of finite type over T.*

Definition 1.3.3. *The Picard scheme* $\mathbf{Pic}_{X/T}$ is the scheme representing the functor $\mathrm{Pic}_{(X/T)(\text{ét})}$.

The Picard schemes behave well under base changes. Let $T' \to T$ be any morphism of schemes and assume that $\mathbf{Pic}_{X/T}$ exists. Then

$$\mathbf{Pic}_{X/T} \times_T T' = \mathbf{Pic}_{X \times_T T'/T'}\,.$$

Additionally, if $T = \mathrm{Spec}(k)$ for a field k, then

$$\mathbf{Pic}_{X/T}(k) = \mathrm{Pic}_{(X/T)(\text{ét})}(k) = \mathrm{Pic}(X).$$

Remark 1.3.4. Take a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. Its image in $\mathrm{Pic}_{(X/T)(\text{ét})}(T)$ defines us a T-point, that is a section

$$s_{\mathcal{L}}\colon T \longrightarrow \mathbf{Pic}_{X/T}\,.$$

Note that for $t \in T$ and a fiber X_t over it, we have

$$s_{\mathcal{L}}(t) = 0 \text{ if and only if } \mathcal{L}|_{X_t} = \mathcal{O}_{X_t}.$$

This follows from the fact that

$$\left(\mathbf{Pic}_{X/T}\right)_t (k(t)) = \mathbf{Pic}_{X_t/t}(k(t)) = \mathrm{Pic}(X_t),$$

where $\left(\mathbf{Pic}_{X/T}\right)_t$ is the fiber of $\mathbf{Pic}_{X/T} \to T$ over t and $k(t)$ is the residue field of t.

1.4 The enumerative theory

In this section, we recall three enumerative formulas: for the conductor of the normalization of a surface in $\mathbb{P}^3$, for the locus of points on a surface tangents at which contain a fixed point and for the number of nodal rational curves on $K3$ surfaces.

Note, that the following is true:

Fact 1.4.1. *Let $f\colon X \to Y$ be a morphism of smooth varieties. Assume that $\dim(X) = \dim(Y)$. Then the ramification locus of f is a divisor.*

Proof. The ramification locus of f is the zero locus of the section

$$\det(df) \in H^0\left(X, (\det T_X)^* \otimes f^*\left(\det T_Y\right)\right). \qquad \square$$

A stronger result is known.

Theorem 1.4.2 ([29, p. 247])**.** *Suppose $f\colon X \to Y$ is a finite dominant morphism, where X is a smooth and Y is a normal variety. Then, the ramification locus of f is a divisor.*

1.4.1 The double-point formula

Now, we follow the presentation of [9, p. 628]. Let $f\colon X \to Y$ be a morphism of smooth proper varieties of dimension m and n respectively. Let Z be the blow-up of the diagonal Δ in $X \times X$. Define $\tilde{D}(f)$ to be the strict transform of $X \times_Y X$ in Z. It is a subset of points $(x_1, x_2) \in X \times X$ for $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$, together with those tangents in $\mathbb{P}(N_{\Delta/X\times X}) = \mathbb{P}(T_X)$ which vanish under f.

Let $D(f)$ be the image of $\tilde{D}(f)$ in X under any of the two projections $Z \to X \times X \to X$. We call $D(f)$ the *double-point set*. The scheme $D(f)$ should be expected to usually have dimension $2m - n$.

The following double-point formula holds.

Theorem 1.4.3 ([12, Theorem 9.3])**.** *If $\dim(D(f)) = 2m - n$, then*

$$[D(f)] = f^*f_*[X] - (c(f^*TY)c(TX)^{-1})_{m-n} \in CH_{2n-m}(X),$$

where $\mathrm{CH}_k(X)$ is the Chow group of k-cycles on X, $\mathrm{CH}^k(X)$ is the Chow group of codimension k cycles on X and $c(E) = \sum_{i=0}^m c_i(E) \in \bigoplus_{i=0}^m \mathrm{CH}^i(X)$ is the Chern class of a vector bundle E on an m-dimensional variety X.

Later, we will use the following form of the double-point formula.

Proposition 1.4.4 ([9, p. 628])**.** *Let $Y = \mathbb{P}^3$ and let f be a finite map, which is birational onto its image. Then $D(f)$ is of pure dimension one and*

$$D(f) \equiv f^*S + f^*K_Y - K_X,$$

where $S := f(X)$.

Note that a map is finite and birational if and only if it is a normalization. In this case the scheme $D(f)$ is the conductor of f.

Proof. Assume by contradiction that $D(f)$ is not of pure dimension one. Then there exists an isolated point $p \in D(f)$.

First, assume that p lies in the non-injective locus of f, in other words, that there exists a point $q \in D(f)$ such that $f(p) = f(q)$ and $p \neq q$.

Let U_1 and U_2 be disjoint small open neighbourhoods of p and q in X. Then $f(U_1)$ and $f(U_2)$ are two codimension one analytic branches intersecting in codimension three. This contradicts the Principal Ideal Theorem of dimension theory applied to $\widehat{\mathcal{O}}_{\mathbb{P}^3, f(p)}$ (cf. [10, Theorem 10.2]).

Let us now assume that p does not lie in the non-injective locus. Then, f ramifies at p and $f(p)$ is an isolated non-normal singularity of the hypersurface S. This contradicts Proposition 1.6.3.

The second part of the statement follows from the double-point formula. □

1.4.2 Polar loci

Now, we follow the presentation of [20, IV. B]. Let $X \subseteq \mathbb{P}^m$ be a smooth variety of dimension n and let A be a linear subspace in $\mathbb{P}^m$ of codimension k. In particular, for $k = m$, the set A is a point. We define the *k-th Polar locus* of A

$$\Gamma_A^k := \{x \in X \mid \dim (T_x X \cap A) \geq n - k + 1\},$$

where $1 \leq k \leq n + 1$. Note that a general linear subspace of dimension n in $\mathbb{P}^m$ intersects A along a linear subspace of dimension $n - k$. Hence, Γ_A^k is the subset of those points on X whose tangent space intersects A in one more dimension than expected.

Suppose that A is a general linear subspace of codimension k. Then Γ_A^k is empty or has pure codimension $n - k + 2$ (see [20, IV.B, p. 346]). Its class $[\Gamma^k] := [\Gamma_A^k]$ in CH^{n-k+2} is independent of A. The following recursive formula for $[\Gamma^k]$ holds:

Theorem 1.4.5 ([20, (IV,29)])**.**

$$c_i(T_X) = (-1)^{i+1}\binom{n+1}{i} c_1(\mathcal{L})^i + \sum_{j=0}^{i-1} (-1)^{j+1} \binom{n+1-i+j}{j} [\Gamma^{n-i+j+2}] c_1(\mathcal{L})^j\,,$$

where $\mathcal{L} = \mathcal{O}_X(1)$ and $1 \leq i \leq n$.

In particular for $i = 1$ we get

$$[\Gamma^{n+1}] = (n+1)c_1(O_X(1)) + c_1(K_X)\,.$$

This implies the following corollary:

Corollary 1.4.6. *Let $S \subseteq \mathbb{P}^3$ be a smooth surface. Take a point $q \in \mathbb{P}^3$ and define*

$$\Gamma_q := \{p \in S \mid q \in T_p S\}.$$

Then

$$\deg(\Gamma_q) = \deg(S) \cdot (\deg(S) - 1).$$

□

1.4.3 Nodal rational curves on $K3$ surfaces

This section is based on [18, Chapter 13, Section 4]. Let (X, H) be a very general polarized projective K3 surface of degree d. By a polarization of degree d we mean that the surface X is equipped with an ample line bundle H, which is indivisible in $\mathrm{Pic}(X)$ and $H^2 = d$. By the adjunction formula, any smooth curve in $|H|$ has genus g, where $2g - 2 = d$.

For the reader's convenience, we show the proof of the following fact:

Fact 1.4.7. *There are only finitely many rational curves in $|H|$.*

Proof. We apply the semicontinuity of the geometric genus (see Proposition 1.1.9) to the family

$$\begin{tikzcd} X & \mathcal{E} := \left\{ (p,s) \in X \times |H| \;\middle|\; s(p) = 0 \right\} \arrow[l, "\pi_X"'] \arrow[d, "\pi_{|H|}"] \\ & {|H|} \end{tikzcd}$$

and get that the locus $Z = \left\{ s \in |H| \;\middle|\; g(\mathcal{E}_s) = 0 \right\}$ is closed. Note that $\mathcal{Z} := \pi_{|H|}^{-1}(Z)$ is uniruled. If $\dim(\mathcal{Z}) \geq 2$, then $\pi_X|_{\mathcal{Z}} \colon \mathcal{Z} \to X$ would be surjective, and so X would also be uniruled, which is impossible, because X is a K3 surface. Thus, $\dim(\mathcal{Z}) \leq 1$ and $\dim(Z) \leq 0$. □

We say that a curve is *nodal* if all of its singularities are nodes.

Fact 1.4.8 ([18, Theorem 13.1.6])**.** *Every rational curve in $|H|$ is nodal.*

A natural question to ask is: how many rational curves are in $|H|$? One can show that this number depends only on $d = H^2$. We denote it by n_g for $g \geq 2$, where $2g - 2 = d$.

Theorem 1.4.9 ([18, 13.4.1])**.** *The following equality of formal power series in a variable q holds*

$$\sum_{g \geq 0} n_g q^g = \prod_{n \geq 1} (1 - q^n)^{-24},$$

where we set $n_0 = 1$ and $n_1 = 24$.

It is called the Yau-Zaslow formula. The first values in this series are the following (see [18, 13.4.2]):

$$\sum_{g\geq 0} n_g q^g = 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + \dots .$$

1.5 Constant cycle curves

The following section is based on [18]. Let X be a projective K3 surface. Take $x \in X$ such that x lies on some rational, not necessarily smooth, curve. The following holds.

Fact 1.5.1 ([5, Theorem 1a])**.** *The class $[x]$ in the Chow group of points $\mathrm{CH}_0(X)$ on X does not depend on the choice of x.*

We call this class the Beauville-Voisin class and denote it by c_X. For the reader's convenience, we prove this fact, following [5].

Proof. Let x lie on an irreducible rational curve R. First, we show that for any other point $x' \in R$, we have $[x] = [x']$. Let $p\colon \mathbb{P}^1 \to R$ be the normalization of R. Then $x = p_*\tilde{x}$ and $x' = p_*\tilde{x}'$ for some $\tilde{x}, \tilde{x}' \in \mathbb{P}^1$. Since points in $\mathbb{P}^1$ are rationally equivalent and taking pushforward preserves rational equivalence, we get

$$[x] = p_*[\tilde{x}] = p_*[\tilde{x}'] = [x'].$$

Take a point y on some other rational curve T. We need to show that $[x] = [y]$. Let H be any ample divisor on X. By [19, Corollary 13.13], H is equivalent to a union of rational curves $\bigcup_{i=1}^k C_i$, which is connected, because all ample divisors on a surface are connected (cf. [17, Exercise 11.3]). Clearly, $R \cdot H > 0$ and $T \cdot H > 0$, and so $R \cup T \cup \bigcup_{i=1}^k C_i$ is also connected. Hence, by the same argument as above, the classes of points on those curves are the same. In particular, $[x] = [y]$. $\square$

On every projective $K3$ surface there exists a rational curve, and so the class c_X is always well defined.

Let C be an integral curve. Take a point $x_0 \in X$ such that $[x_0] = c_X$ in $\mathrm{CH}_0(X)$. We define

$$\kappa_C := p^*(\Delta_C - \{x_0\} \times C) \in \mathrm{CH}_0\left(X \times k\left(\mu_C\right)\right),$$

where $k(\mu_C)$ is the residue field of the generic point μ_C of C, a curve $\Delta_C \subseteq X \times C$ is the graph of the inclusion $C \hookrightarrow X$, and $p\colon X \times k(\mu_C) \to X \times C$ is a natural inclusion. This class does not depend on the choice of x_0.

Definition 1.5.2. We call an integral curve C *a constant cycle curve* if the class $\kappa_C \in \mathrm{CH}_0\left(X \times k\left(\mu_C\right)\right)$ is torsion. An arbitrary curve is *a constant cycle curve*, if each irreducible component of its reduction is a constant cycle curve.

The following holds:

Proposition 1.5.3. *A curve $C \subseteq X$ is a constant cycle curve if and only if for any two points $p_1, p_2 \in C$ we have an equality $[p_1] = [p_2]$ in $\mathrm{CH}_0(X)$.*

Proof. See [19, Proposition 3.7]. □

Note that any ample divisor is equivalent to a union of rational curves (see [19, Corollary 13.13]), and so there must exist a rational curve intersecting C. In particular, C contains a point of class c_X. Hence, a curve C is a constant cycle curve if and only if $[p] = c_X$ for every $p \in C$.

It is usually nontrivial to show that a certain curve is a constant cycle curve. Some possible approaches are via the following propositions.

Proposition 1.5.4 ([19, Proposition 7.1])**.** *Let $f\colon X \xrightarrow{\sim} X$ be an automorphism of X of finite order such that $f^* \neq \mathrm{id}$ on $H^{2,0}(X)$. If each point of a curve $C \subseteq X$ is fixed by f, then C is a constant cycle curve.*

Proposition 1.5.5 ([5, Theorem 1b])**.** *The image of the intersection product*

$$\mathrm{Pic}(X) \otimes \mathrm{Pic}(X) \longrightarrow \mathrm{CH}_0(X)$$

is contained in $\mathbb{Z}c_X$.

Proof. This follows from the fact that every effective divisor is equivalent to a union of rational curves (see [18, Corollary 13.13]). □

Additionally, note that:

Proposition 1.5.6 ([18, Proposition 12.1.3])**.** *The Chow group of points $\mathrm{CH}_0(X)$ of a complex projective $K3$ surface is torsion free.*

1.6 Hypersurfaces in $\mathbb{P}^3$

In this section, we recall properties of hypersurfaces in $\mathbb{P}^3$ and their Gauss maps. Additionally, we define the second fundamental form and asymptotic directions.

Let V be a four-dimensional complex linear space and let $S \subset \mathbb{P}(V)$ be a smooth hypersurface of degree d. The embedding into the projective space gives us an ample divisor $\mathcal{O}_S(1)$ on S.

In this thesis, we frequently use the following result called Noether-Lefschetz theorem.

Theorem 1.6.1. *Let $S \subseteq \mathbb{P}^3$ be a very general smooth hypersurface of degree $d \geq 4$. Then $\mathrm{Pic}(S) = \mathbb{Z}\mathcal{O}_S(1)$.*

In particular, all curves in $|\mathcal{O}_S(1)|$ are irreducible and there are no lines on S. Additionally, every curve on S is connected.

Proof. See [35, Section 3.3.2]. □

Remark 1.6.2. By the above theorem, we have

$$C_1 \cdot C_2 = \frac{deg(C_1)\deg(C_2)}{\deg(S)},$$

where C_1 and C_2 are curves on a very general hypersurface $S \subseteq \mathbb{P}^3$. Hence, the intersection number of two curves does not depend on the position of those curves on the surface.

We also note the following.

Proposition 1.6.3 ([26, Appendix to §3])**.** *Let X be a locally complete intersection variety. Then X cannot have isolated non-normal singularities.*

Proof. Since X is a locally complete intersection, it is Cohen-Macaulay, and so it satisfies Serre's S_2 condition. Thus, the statement follows from Serre's criterion for normality. □

1.6.1 The Gauss map

In order to analyse the geometry of hypersurfaces, we consider the Gauss map.

Definition 1.6.4. We define the *Gauss map* $\phi\colon S \to \mathbb{P}(V^*)$ as

$$p \longmapsto \hat{T}_p S,$$

where $\mathbb{P}(V^*)$ is the space of hyperplanes in V and $\hat{T}_p S \subset V$ is the deprojectivization of the tangent space $T_p S$.

In what follows we write T_p and $\hat{T}_p$ instead of $T_p S$ and $\hat{T}_p S$.

Definition 1.6.5. We define the *dual variety* $S^* \subseteq \mathbb{P}(V^*)$ as

$$S^* := \phi(S).$$

Proposition 1.6.6. *The Gauss map ϕ is the normalization of S^*.*

Proof. See [30, Theorem 4.2]. □

In other words, ϕ is finite and birational.

Proposition 1.6.7. *The dual variety S^* has degree $d(d-1)^2$.*

Proof. See [30, Example 6.3, p. 112]. □

The following theorem relates the multiplicity of points on the dual variety with Milnor numbers of singularities of the corresponding curves.

Theorem 1.6.8. *Let $H \in S^*$ and suppose that the set $\operatorname{Sing}(S \cap H)$ is finite. Then*

$$\operatorname{mult}_H S^* = \sum_{p \in \operatorname{Sing}(S \cap H)} \mu_p(S \cap H).$$

Proof. See [30, Theorem 10.8]. □

1.6.2 The second fundamental form

Let

$$d\phi_p \colon T_p \longrightarrow T_{\hat{T}_p}\mathbb{P}(V^*)$$

be the derivative of the Gauss map ϕ. We would like to understand $d\phi$ in terms of the tangent and the normal bundle of S in $\mathbb{P}^3$. First, we use a natural identification

$$T_{\hat{T}_p}\mathbb{P}(V^*) \cong \mathrm{Hom}(\hat{T}_p, V/\hat{T}_p) \cong \mathrm{Hom}\left(\hat{T}_p, N\left(-1\right)_p\right),$$

where N is the normal bundle of $S \subset \mathbb{P}(V)$. The last isomorphism follows from a natural identification

$$N_p = \mathrm{Hom}\left(\hat{p}, V/\hat{T}_p\right),$$

where $\hat{p} \subset V$ denotes the deprojectivization of a point $p \in \mathbb{P}(V)$.

One can check that

$$\hat{p} \subset \ker d\phi_p(v)$$

for any $v \in T_p$, and so $d\phi_p$ factors to

$$\mathrm{I\!I}\colon T_p \longrightarrow \mathrm{Hom}\left(\hat{T}_p/\hat{p}, N\left(-1\right)_p\right) \cong \mathrm{Hom}\left(T_p, N_p\right),$$

where the last isomorphism follows from a natural identification

$$T_p = \mathrm{Hom}\left(\hat{p}, \hat{T}_p/\hat{p}\right).$$

Definition 1.6.9. We call $\mathrm{I\!I}$ defined above the *projective second fundamental form.*

The projective second fundamental form $\mathrm{I\!I}$ is symmetric. It is a section of $S^2T^* \otimes N$.

Remark 1.6.10. The projective second fundamental form can be also defined in the following way

$$\mathrm{I\!I}(v, w) = (\nabla_v w)^{\perp}$$

for $v, w \in TS$ and the Levi-Civita connection ∇ of $S \subset \mathbb{P}^3$.

Definition 1.6.11. We call $v \in T_pS$ an *asymptotic direction* if $\mathrm{I\!I}(v, v) = 0$.

Definition 1.6.12. Let T_pS be the tangent space to S at a point $p \in S$. We call $E_p := T_pS \cap S$ the *tangent curve* at p.

The tangent curve at $p \in S$ encapsulates many properties of the Gauss map around p (see for example Theorem 1.6.8 or Remark 1.6.14 below).

Remark 1.6.13. Note that p is a singularity of the tangent curve E_p. Tangent curves are exactly the divisors in $|\mathcal{O}_S(1)|$ which are singular.

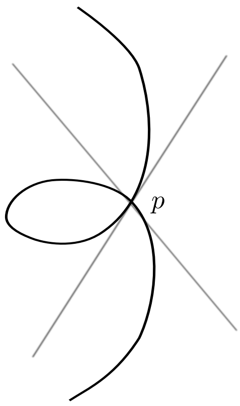


Figure 1.2: A tangent curve and asymptotic directions

Remark 1.6.14. We present the second fundamental form in local coordinates (see also [30, Example 3.9]). Let $p \in S$. Choose coordinates locally around p such that S is defined by $z + F(x, y) = 0$ for $F \in \mathbb{C}[\![x, y]\!]$ and $z = 0$ is tangent to S at 0. Then,

$$\mathrm{I\!I} = \begin{bmatrix} F_{x,x} & F_{x,y} \\ F_{y,x} & F_{y,y} \end{bmatrix} = \mathrm{Hess}(F).$$

Observe that the tangent curve E_p is defined by $F = 0$. Hence, asymptotic directions of S at p are exactly the tangents to the branches of E_p at p.

If there is only one asymptotic direction, say v, then it is a kernel of $\mathrm{I\!I}$, that is $\mathrm{I\!I}(v, \cdot) = 0$. In particular, the asymptotic direction at p is exactly the direction of ramification of ϕ, that is $d\phi_p(v) = 0$.

CHAPTER 2

The geometry of smooth quartics in $\mathbb{P}^3$

2.1 Properties of Gauss maps

Let $S \subset \mathbb{P}^3$ be a smooth quartic hypersurface described by an equation $f(x, y, z, w) = 0$ of degree four. Note that S is a $K3$ surface. Let $\phi\colon S \to \mathbb{P}^3$ be the Gauss map. We have

$$\phi\,(x : y : z : w) = (f_x : f_y : f_z : f_w)\,.$$

Unless stated otherwise, we assume that S is very general.

First, using the fact that for a general surface S, the Gauss map is stable, we describe it in local coordinates following [24, Section 3]. An easy corollary of this presentation is that the derivative of the Gauss map is always nonzero, which implies that hyperplane sections of a general quartic have only singularities of multiplicity two. Further, we classify all possible hyperplane sections and we use this classification to calculate multiplicities of singularities of dual surfaces.

2.1.1 A local description

All statements and proofs in this subsection work for any general hypersurface in $\mathbb{P}^3$ of degree $d \geq 3$. For the reader's convenience, we present the proof of the following proposition following [24, Section 3].

Proposition 2.1.1 ([24, (3.4)])**.** *Take $p \in S$. The Gauss map $\phi\colon S \to \mathbb{P}^3$ is locally equivalent at p to one of the following maps:*

(nonparabolic germ) • $(x, y) \xmapsto{\phi_1} (x, y, 0)$,

(general parabolic germ) • $(x, y) \xmapsto{\phi_2} (x^3, x^2, y)$,

(swallowtail germ) • $(x, y) \xmapsto{\phi_3} (3x^4 + x^2y, 2x^3 + xy, y)$.

Proof. Define a variety
$$\Gamma := \{(x, H) \mid x \in H\} \subseteq S \times \check{\mathbb{P}}^3,$$
where $\check{\mathbb{P}}^3$ is the space of hyperplanes in $\mathbb{P}^3$. The variety Γ is smooth, because we have a flat projection $\Gamma \to S$ with smooth fibers
$$\Gamma_x = \left\{H \in \check{\mathbb{P}}^3 \;\middle|\; x \in H\right\} \cong \mathbb{P}^2.$$
Let $\pi \colon \Gamma \to \check{\mathbb{P}}^3$ be another projection. Recall that $\Sigma(\pi)$ is the locus of points where π is not smooth. We have
$$\Sigma(\pi) = \{(x, T_x S) \mid x \in S\} \cong S.$$
This equality follows from the fact that $\Sigma(\pi) = \bigcup_{H \in \check{\mathbb{P}}^3} \operatorname{Sing}(\Gamma_H)$, where $\Gamma_H = H \cap S$ is the fiber of π over $H \in \check{\mathbb{P}}^3$. Clearly, Γ_H is singular at $x \in \Gamma_H$ if and only if $H = T_x S$.

Hence, $\pi|_{\Sigma(\pi)} \colon \Sigma(\pi) \to \check{\mathbb{P}}^3$ is exactly the Gauss map ϕ.

Lemma 2.1.2 ([24, Lemma 3.3])**.** *For a general smooth quartic S, the morphism π is locally stable.*

Proof. We need to show that the subset of smooth quartics, for which the Gauss map is stable, is Zariski open and non-empty. For non-emptiness, we refer to [24, p. 274-275], where the proof is based on topological transversality argument of [6]. We show only the openness. Define
$$\Lambda := \{(x, H, f) \mid x \in H \text{ and } f(x) = 0\} \subseteq \mathbb{P}^3 \times \check{\mathbb{P}}^3 \times |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}},$$
where $|\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}$ is the space of smooth quartic hypersurfaces in $\mathbb{P}^3$. The statement follows by applying Proposition 1.2.3 to
$$\begin{tikzcd} \Lambda \arrow[r, "\Pi"] \arrow[dr] & \check{\mathbb{P}}^3 \times |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \arrow[d] \\ & |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \end{tikzcd}$$
where the maps are natural projections. □

Now, we assume that S is general and π is locally stable. Note that π restricted to a fiber $\Gamma_x = \left\{H \in \check{\mathbb{P}}^3 \;\middle|\; x \in H\right\}$ of the projection $\Gamma \to S$ is an inclusion. Hence, π is of rank at least two at every point, and thus, by Proposition 1.2.4, it is locally equivalent to one of the following maps:
$$\begin{aligned}
(x_1, x_2, x_3, x_4) &\overset{\pi_1}{\longmapsto} (x_2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\overset{\pi_2}{\longmapsto} (x_1^2 + x_2^2, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\overset{\pi_3}{\longmapsto} (x_1^2 + x_2^3 + x_2 x_3, x_3, x_4), \\
(x_1, x_2, x_3, x_4) &\overset{\pi_4}{\longmapsto} (x_1^2 + x_2^4 + x_2^2 x_3 + x_2 x_4, x_3, x_4).
\end{aligned}$$

Let f_i for $1 \leq i \leq 4$ be the projection of π_i to the first coordinate. Then $\Sigma(\pi_i) = \left\{\frac{\partial f_i}{\partial x_1} = 0, \frac{\partial f_i}{\partial x_2} = 0\right\}$. We calculate

$$\begin{aligned}
\Sigma(\pi_1) &= \emptyset, \\
\Sigma(\pi_2) &= \{x_1 = 0, x_2 = 0\}, \\
\Sigma(\pi_3) &= \left\{x_1 = 0, 3x_2^2 + x_3 = 0\right\}, \\
\Sigma(\pi_4) &= \left\{x_1 = 0, 4x_2^3 + 2x_2x_3 + x_4 = 0\right\}.
\end{aligned}$$

Hence, the Gauss map is locally equivalent to one of the maps:

$$\begin{aligned}
&(x_3, x_4) \mapsto (0, 0, x_3, x_4) \overset{\pi_2}{\longmapsto} (0, x_3, x_4), \\
&(x_2, x_4) \mapsto (0, x_2, -3x_2^2, x_4) \overset{\pi_3}{\longmapsto} (-2x_2^3, -3x_2^2, x_4), \\
&(x_2, x_3) \mapsto (0, x_2, x_3, -4x_2^3 - 2x_2x_3) \overset{\pi_4}{\longmapsto} (-3x_2^4 - x_2^2x_3, x_3, -4x_2^3 - 2x_2x_3).
\end{aligned}$$

Now, we obtain the forms we are looking for, by applying a suitable change of coordinates. □

Definition 2.1.3. A point $p \in S$ is called a *planar point* if $d\phi_p = 0$.

Lemma 2.1.4. *A point $p \in S$ is planar if and only if* $\mathrm{mult}_p(E_p) > 2$.

Proof. Locally, the derivative $d\phi_p$, the second fundamental form $\mathrm{I\!I}_p$ and $\mathrm{Hess}_p(E_p)$ coincide, by construction of $\mathrm{I\!I}$ and Remark 1.6.14. Hence, the lemma follows from Remark 1.1.1. □

If a surface contains planar points, then it is much more difficult to understand its geometry. Fortunately, the following proposition holds.

Proposition 2.1.5 ([24, Theorem 3.1])**.** *There are no planar points on a general smooth quartic S.*

Proof. By Proposition 2.1.1, we have $d\phi_p \neq 0$ for every $p \in S$. □

Hence, on a general smooth quartic hypersurface $S \subset \mathbb{P}^3$, tangent curves have only singularities of mutiplicity two and the second fundamental form $\mathrm{I\!I}_p$ is nonzero for all $p \in S$.

2.1.2 A classification of tangent curves

First, observe that by Theorem 1.6.1, all tangent curves are irreducible. Further, by Fact 1.4.8 all rational tangent curves are nodal.

We say that a curve E is *elliptic cuspidal* if $g(E) = 1$ and the singularities of E are cusps. The following holds.

Lemma 2.1.6. *There are no elliptic cuspidal tangent curves on a general smooth quartic in $\mathbb{P}^3$.*

Proof. We use the notation from Subsection 1.1.3. Let $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^3}(4)\right)$ be the space of quartic hypersurfaces in $\mathbb{P}^3$ and let $V \subseteq \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^3}(4)\right)$ be the subvariety of those quartics, which contain an elliptic cuspidal tangent curve. In order to prove the lemma, it is enough to show that

$$\dim(V) < \dim\left(\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^3}(4)\right)\right) = \binom{7}{3} - 1 = 34.$$

Choose coordinates x, y, z, w of $\mathbb{P}^3$. Let $\overline{V} \subseteq V$ be the subvariety of those quartics for which the curve cut out by $w = 0$ is an elliptic cuspidal curve. Quartics in $\overline{V}$ are the zero loci of polynomials

$$F(x, y, z, w) = f(x, y, z) + wg(x, y, z, w),$$

where $g \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$, $f \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$, and $\{f = 0\} \in \mathcal{V}_4^{0,2}$. Hence,

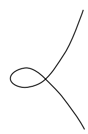
$$\dim\left(\overline{V}\right) = \dim\left(\mathcal{V}_4^{0,2}\right) + h^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)\right) = 10 + \binom{6}{3} = 30.$$

Let $\check{\mathbb{P}}^3$ be the space of hypersurfaces in $\mathbb{P}^3$. We have

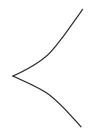
$$\dim(V) \leq \dim\left(\overline{V}\right) + \dim\left(\check{\mathbb{P}}^3\right) = 33 < \dim\left(\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^3}(4)\right)\right). \qquad \square$$

Proposition 2.1.7. *Let p be a point on a very general smooth quartic $S \subseteq \mathbb{P}^3$ and let $E_p \subseteq S$ be the tangent curve at p. The point p is of one of the following types:*

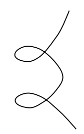
1. *(general point)*
   $g(E_p) = 2$ and E_p has one node,



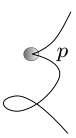
2. *(simple parabolic point)*
   $g(E_p) = 2$ and E_p has one cusp,



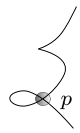
3. *(simple Gauss double point)*
   $g(E_p) = 1$ and E_p has two nodes,



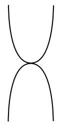
4. *(parabolic Gauss double point)*
   $g(E_p) = 1$, E_p has one node and one cusp, and p is its cusp,



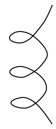
5. *(dual to parabolic Gauss double point)*
   $g(E_p) = 1$, E_p *has one node and one cusp, and* p *is its node,*



6. *(Gauss swallowtail)*
   $g(E_p) = 1$ *and* E_p *has one tacnode,*



7. *(Gauss triple point)*
   $g(E_p) = 0$ *and* E_p *has three nodes.*



Proof. The proof follows from Lemma 1.1.6 together with Fact 1.4.8 and Lemma 2.1.6. □

Remark 2.1.8. We will see later on that all the cases occur. We will prove that the subset of simple parabolic points, parabolic Gauss double points and Gauss swallowtails is closed of pure dimension one. We call it *the parabolic curve.* Similarily, *the double-cover curve* which is the subset of simple Gauss double points, parabolic Gauss double points, dual to parabolic Gauss double points, Gauss swallowtails and Gauss triple points is also closed and of pure dimension one. Note that a calculation similiar to the proof of Lemma 2.1.6, confirms that the double-cover curve and the parabolic curve should generically have dimension one.

Using Proposition 2.1.7 we can calculate the multiplicity of a point $\phi(p)$ in S^*, for $p \in S$.

Proposition 2.1.9. *Let* $p \in S$ *and set* $p^* = \phi(p)$. *Then* $\operatorname{mult}_{p^*} S = 2$ *if* p *is a simple parabolic point or a simple Gauss double point and* $\operatorname{mult}_{p^*} S = 3$ *if* p *is a parabolic Gauss double point, dual to parabolic Gauss double point, a Gauss swallowtail, or a Gauss triple point.*

In particular, since S^* is generically smooth, a general point p on S satisfies property (1), that is $g(E_p) = 2$ and E_p has exactly one node.

Proof. The proposition follows from Theorem 1.6.8 and the calculation of Milnor numbers of nodes, cusps and tacnodes in Subsection 1.1.1. □

Note, that in the following sections we will redefine the parabolic curve, and Gauss swallowtails, and then check that the new defintions agree with the ones from Proposition 2.1.7.

2.2 The parabolic curve

In this section, we analyze the ramification locus of the Gauss map, which we call *the parabolic curve.* We show that for a general smooth quartic, this curve is smooth of genus 129 and degree 32. It consists exactly of the points, at which the tangent curve has a cusp.

We call the points where the Gauss map restricted to the parabolic curve ramifies, *the Gauss swallowtails.* One can also define them as those points where the tangent direction to the curve and the asymptotic direction coincide. We will show in Subsection 2.4, that this definition agrees with the one from Proposition 2.1.7 and that there are 320 Gauss swallowtails.

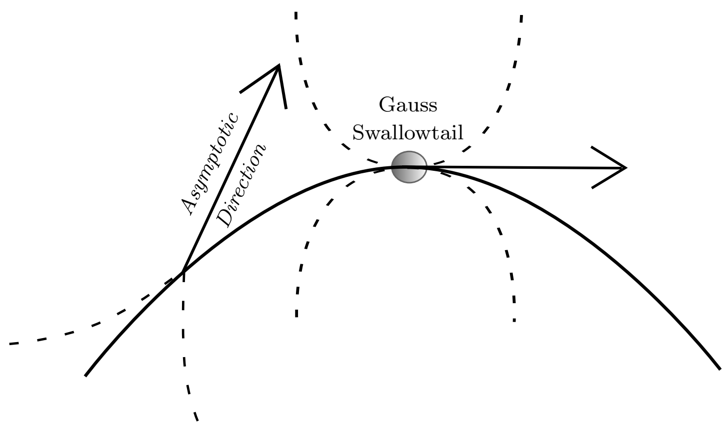


Figure 2.1: The parabolic curve

Definition 2.2.1. We define the *parabolic curve* C_{par} to be the ramification locus of the dual map ϕ. We say that a point p is *parabolic* if $p \in C_{\text{par}}$.

We need to check that this definition agrees with the one from Remark 2.1.8.

Proposition 2.2.2. *Let $p \in S$. Then p is parabolic if and only if p is a cusp or a tacnode of E_p.*

Proof. The point p is a cusp or a tacnode of E_p exactly when $\det(\mathrm{Hess}_p(E_p)) = 0$. Since $d\phi$ and $\mathrm{Hess}(E_p)$ coincide locally by Remark 1.6.14 and the construction of $\mathrm{I\!I}$, the proposition follows. $\square$

Hence, a point p is parabolic if and only if it has exactly one asymptotic direction.

We say that a point p is *simple parabolic*, if $g(E_p) = 2$ and p is a cusp of E_p.

Proposition 2.2.3. *The parabolic curve C_{par} is the zero locus of* $\det(\mathrm{Hess}(f))$ *on S. In particular, it is of pure dimension one and lies in $\mathcal{O}_S(8)$.*

Proof. Without loss of generality we can restrict ourselves to

$$U := \{(x : y : z : 1)\}.$$

Using that $xf_x + yf_y + zf_z + wf_w = 4f$ and analogous equations for second derivatives, for example $xf_{x,x} + yf_{x,y} + zf_{x,z} + wf_{x,w} = 3f_x$, we can conduct elementary transformations to get

$$\det(\mathrm{Hess}(f)) = c \det \begin{pmatrix} f_{x,x} & f_{y,x} & f_{z,x} & f_x \\ f_{x,y} & f_{y,y} & f_{z,y} & f_y \\ f_{x,z} & f_{y,z} & f_{z,z} & f_z \\ f_x & f_y & f_z & f \end{pmatrix}$$

for a nonzero constant c.

Take $p \in S$. By an analytic change of coordinates we can assume that locally around p we have $f = z - F(x, y)$, where $F \in \mathbb{C}[\![x, y]\!]$ and $z = 0$ is tangent to S. Note that terms of F have degree at least two. We get

$$\det(\mathrm{Hess}(f)) = c \det \begin{pmatrix} -F_{x,x} & -F_{y,x} & 0 & 0 \\ -F_{x,y} & -F_{y,y} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = c \det(\mathrm{I\!I}).$$

Since $\mathrm{I\!I}$ and $d\phi$ coincide by the construction of $\mathrm{I\!I}$, the proposition follows. $\square$

Proposition 2.2.4 ([24, Theorem 3.1])**.** *The parabolic curve C_{par} on a general smooth quartic S is smooth.*

Proof. Here, we use the notation of Proposition 2.1.1. Note that the parabolic curve C_{par} is equal to the locus $\Sigma(\phi)$ of the points where ϕ is not smooth. If the Gauss map ϕ is locally equivalent to a map $\phi_i(x, y) = (f_i(x, y), g_i(x, y), y)$, then we have

$$\Sigma(\phi_i) = \left\{ \frac{\partial f_i}{\partial x} = \frac{\partial g_i}{\partial x} = 0 \right\},$$

and so

$$\begin{aligned} \Sigma(\phi_1) &= \emptyset, \\ \Sigma(\phi_2) &= \{x = 0\}, \\ \Sigma(\phi_3) &= \left\{6x^2 + y = 0\right\}. \end{aligned}$$

In particular, C_{par} is smooth. $\square$

Corollary 2.2.5. *The curve C_{par} has genus* 129.

Proof. By the adjunction formula (see [4, Section II.11]) we get

$$g(C_{\text{par}}) = \frac{1}{2}C_{\text{par}}^2 + 1 = \frac{1}{2} \cdot 4 \cdot 8 \cdot 8 + 1 = 129.$$

□

Definition 2.2.6. We call $p \in S$ a *swallowtail of the Gauss map* (*Gauss swallowtail* in short) if p lies in the ramification divisor of $\phi|_{C_{\text{par}}}$.

Remark 2.2.7. In other words, by Remark 1.6.14, Gauss swallowtails are exactly those points on the parabolic curve C_{par} where the tangent direction of C_{par} at p and the asymptotic direction at p coincide.

Remark 2.2.8. Suppose that S is a general quartic. We use, here, the notation of Proposition 2.1.1. Take $p \in C_{\text{par}}$. The calculations in the proof of Proposition 2.2.4, show that $\phi|_{C_{\text{par}}}$ is locally at p equivalent to one of the following maps

$$\begin{aligned} y &\mapsto (0, y) \xmapsto{\phi_2|_{C_{\text{par}}}} (0, 0, y), \\ x &\mapsto (x, -6x^2) \xmapsto{\phi_3|_{C_{\text{par}}}} (-3x^4, -4x^3, -6x^2). \end{aligned}$$

In particular, ϕ_1 describes the Gauss map around nonparabolic points, ϕ_2 around parabolic points, which are not Gauss swallowtails, and ϕ_3 around Gauss swallowtails.

2.3 Bitangents, hyperflexes and the flecnodal curve

In this section, we analyse *the flecnodal curve*, which consists of those points, to which one of the asymptotic direction is tangent with multiplicity four. For a very general surface S, it is irreducible of genus 201. The points where both asymptotic directions are tangent with multiplicity four are contained in the singular locus of the flecnodal curve. In this section, we also find equations defining the flecnodal curve in local coordinates, which we will use later to calculate the number of Gauss swallowtails and show that the flecnodal curve has degree 80.

2.3.1 Bitangents

Definition 2.3.1. A line $l \subseteq \mathbb{P}^3$ is called a *bitangent* of S if it is tangent to S at each point $x \in S \cap l$. We call it a *hyperflex* if $|l \cap S| = 1$.

Since there are no lines on S, a bitangent l intersects S in at most two points. A line l is a hyperflex if and only if it intersects S at some point with multiplicity four.

Remark 2.3.2. Take $p, q \in S$ such that $p \neq q$. Then, the line $\overline{pq}$ is a bitangent if and only if $\overline{pq} \subseteq T_p \cap T_q = T_q E_p$, that is $\overline{pq}$ is tangent to E_p at q (or equivalently $\overline{pq}$ is tangent to E_q at p).

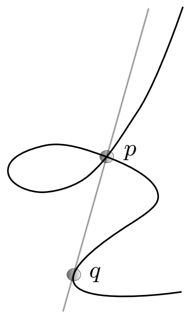


Figure 2.2: The tangent curve E_p and the bitangent $\overline{pq}$

Proposition 2.3.3. *A general point $p \in S$ has exactly six bitangents.*

Proof. Let $p \in S$ be a general point. Then, the curve E_p is singular only at p, where it has a node. Additionally, $g(E_p) = 2$ and asymptotic directions l_1, l_2 satisfy $\mathrm{mult}_p(E_p \cap l_i) = 3$. Take an arbitrary line $\mathbb{P}^1 \subseteq T_p$ and consider the projection $\pi\colon E_p \dashrightarrow \mathbb{P}^1$ from p. By Remark 2.3.2, bitangents correspond to ramification divisors of π.

Let $\tilde{E_p} \to E_p$ be the normalization of E_p and let $\tilde{\pi}\colon \tilde{E_p} \to \mathbb{P}^1$ be its composition with π. As $\deg(E_p) = 4$, it holds that $\deg(\tilde{\pi}) = \deg(\pi) = 2$. By Riemann–Hurwitz formula

$$\chi(\tilde{E_p}) = 2\chi(\mathbb{P}^1) + \deg(R),$$

where R is the ramification divisor, we get that $\deg(R) = 6$. Let $\{p_1, p_2\}$ be the inverse image of p under the normalization of E_p, where p_i corresponds to the asymptotic direction l_i. It is sufficient to show that $\tilde{\pi}$ does not ramify at p_i.

Since $\deg(S) = 4$ and $\mathrm{mult}_p(E_p \cap l_i) = 3$, the asymptotic direction l_i intersects E_p at exactly one another point, say q_i. Then,

$$\tilde{\pi}(p_i) = \tilde{\pi}(q_i),$$

and so $\tilde{\pi}$ does not ramify at p_i. □

Remark 2.3.4. By the above calculation, we see, that a point $q \in S$ has less than six bitangents if and only if E_q has more than one singularity or there is a hyperflex through q.

2.3.2 Global properties of the flecnodal curve

Definition 2.3.5. A point $p \in S$ is called a *flec point* if there exists a hyperflex through it. The *flecnodal curve* C_{hf} is the reduced subscheme of flec points.

Proposition 2.3.6. *The flecnodal curve C_{hf} is closed and of pure dimension one.*

Proof. Define the universal family of bitangents

$$\begin{array}{ccc} B_S & \xrightarrow{\ \mathrm{p}\ } & S \\ \Big\downarrow{\scriptstyle \mathrm{q}} & & \\ F_S & & \end{array}$$

where

$$B_S := \{(p, l) \mid l \text{ is bitangent to } S \text{ at } p\}$$
$$F_S := \{l \mid l \text{ is bitangent to } S\}\,.$$

The surfaces B_S, F_S are irreducible and smooth (see [31] or [34]). Since C_{hf} is the image under p of the ramification locus of q, the proposition follows from Fact 1.4.1. □

Proposition 2.3.7. *For a very general quartic S, the curve C_{hf} is irreducible. It has geometric genus* 201*.*

Proof. See [19, Proposition 8.8]. □

Proposition 2.3.8. *Suppose that S is a general quartic. Then a general point $p \in C_{\mathrm{hf}}$ has exactly one hyperflex.*

Proof. The proof is similiar to the argument of Section 3.3. □

Proposition 2.3.9. *The flecnodal curve C_{hf} on a very general smooth quartic S is a constant cycle curve.*

For the reader's convenience, we present the proof following [19, Proposition 8.7].

Proof. Take $p \in C_{\mathrm{hf}}$ and a hyperplane H distinct from T_pS, but containing a hyperflex at p. Since $\deg(E_p) = 4$, the hyperplane H intersects E_p only at the point p, and so, by Proposition 1.5.5, we have

$$4[p] = 4c_X.$$

Since $\mathrm{CH}_0(X)$ is torsion free (see Proposition 1.5.6), we get $[p] = c_x$. Thus, by Proposition 1.5.3, the flecnodal curve C_{hf} is a constant cycle curve. □

2.3.3 The flecnodal curve in local coordinates

We use the following lemma to find equations for C_{hf} in local coordinates.

Lemma 2.3.10. *Take $p \in S$ and a local chart $U \subseteq S$ with coordinates x, y. Choose a vector v in the asymptotic direction at p and extend it to a vector field in asymptotic directions. Let $f\colon U \to \mathbb{C}$ be a slope function $\frac{y}{x}$ of this vector field. Then $\mathbb{C}v$ is a hyperflex if and only if $df_p(v) = 0$.*

The condition $df_p(v) = 0$ does not depend on the choice of a chart and local coordinates. In the language of differential geometry, it says that flec points are inflection points of *asymptotic curves* (integral curves of asymptotic directions). Note that asymptotic curves do not need to be algebraic.

Proof. We can assume that locally around p the surface S is defined by $z - F(x, y)$, where $F \in \mathbb{C}[\![x, y]\!]$ and $z = 0$ is tangent to S. Consider the continuous function $f(q) = \frac{v_x}{v_y}$, where $v = (v_x, v_y, v_z)$ is an asymptotic direction at q.

By Remark 1.6.14, the slope $f(q)$ satisfies

$$f(q)^2 F_{x,x} + 2f(q)F_{x,y} + F_{y,y} = 0.$$

Taking the derivative and evaluating on v we get

$$df_p(v)(2fF_{x,x} + 2F_{x,y}) + v_y^{-2} M = 0, \tag{2.1}$$

where

$$M = F_{x,x,x}v_x^3 + 3F_{x,x,y}v_x^2 v_y + 3F_{x,y,y}v_x v_y^2 + F_{y,y,y}v_y^3.$$

Note that v is a hyperflex if and only if $M = 0$. Thus, if $df_p(v) = 0$, then $\mathbb{C}v$ is a hyperflex.

Now assume that $\mathbb{C}v$ is a hyperflex. We want to prove that $df_p(v) = 0$. Since C_{par} and C_{hf} does not have common components, by continuity we can assume without loss of generality that $p \notin C_{\text{par}}$, that is p has two asymptotic directions. Then,

$$2f(q)F_{x,x} + 2F_{x,y} \neq 0,$$

and so $df_p(v) = 0$ by (2.1). $\square$

Let $p \in S$. Choose a local chart $U \subseteq S$ with coordinates x, y such that p is sent to 0 and

$$\mathrm{I\!I} = \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix},$$

where $\mu, \lambda \in \mathbb{C}[\![x, y]\!]$. Since C_{par} is the zero locus of $\det(\mathrm{I\!I})$, we can assume that it is given by $\mu = 0$. Additionally, we can assume without loss of generality that λ is nowhere vanishing on U, as there are no planar points on S by Proposition 2.1.5.

Note that for a point $p \in C_{\text{par}}$, it holds that $\mu_x = 0$ if and only if p is a Gauss swallowtail (see Remark 2.2.7). The following lemma is a reformulation of the result of McCrory and Shifrin (see [24, Lemma 1.11]).

Lemma 2.3.11. *The flecnodal curve C_{hf} is described locally around p by*

$$(\mu_x\lambda - \mu\lambda_x)^2 \lambda + (\mu_y\lambda - \mu\lambda_y)^2 \mu = 0.$$

Proof. A vector $[a, b]$ is an asymptotic direction at $x \in U \setminus C_{\text{par}}$ if and only if

$$a^2\mu + b^2\lambda = 0.$$

Hence, the slope function p of asymptotic curves satisfies

$$p^2 = -\frac{\mu}{\lambda}.$$

Choose $x \in U \setminus C_{\mathrm{par}}$ and fix a branch of $\sqrt{\ }$ around x. Define

$$\begin{aligned}\omega &:= d\left(-\sqrt{\frac{\mu}{\lambda}}\right) \\ &= \frac{1}{2}\sqrt{\frac{\lambda}{\mu}}\left(\frac{\mu_x\lambda - \mu\lambda_x}{\lambda^2}dx + \frac{\mu_y\lambda - \mu\lambda_y}{\lambda^2}dy\right).\end{aligned}$$

By Lemma 2.3.10, the curve C_{hf} is described outside of C_{par} by

$$\omega(\sqrt{\lambda}, \sqrt{\mu}) \cdot \omega(\sqrt{\lambda}, -\sqrt{\mu}) = 0,$$

which is equivalent to

$$\frac{\lambda}{\mu}\left(\left(\frac{\mu_x\lambda - \mu\lambda_x}{\lambda^2}\right)^2 \lambda + \left(\frac{\mu_y\lambda - \mu\lambda_y}{\lambda^2}\right)^2 \mu\right) = 0.$$

Thus, the curve C_{hf} is contained in the curve $\tilde{C}_{\mathrm{hf}}$ described by

$$(\mu_x\lambda - \mu\lambda_x)^2\,\lambda + (\mu_y\lambda - \mu\lambda_y)^2\,\mu = 0.$$

Those two curves coincide on $U \setminus C_{\mathrm{par}}$. Note that $\tilde{C}_{\mathrm{hf}}$ intersects C_{par} exactly at points of C_{par} satisfying $\mu_x = 0$, that is at Gauss swallowtails. Since the set of Gauss swallowtails is finite, the curves C_{hf} and $\tilde{C}_{\mathrm{hf}}$ must coincide. □

Corollary 2.3.12. *Let $p \in C_{\mathrm{hf}}$ be a point with two distinct hyperflexes. Then p is a singularity of C_{hf}.*

Proof. We use, here, the notation of Lemma 2.3.11. Since p has two asymptotic directions, it holds that $p \notin C_{\mathrm{par}}$, and so we may assume that locally around p functions μ and λ are nowhere vanishing. In particular, we may choose a well defined branch of $\sqrt{\mu}$ and $\sqrt{\lambda}$.

Locally around p the curve C_{hf} is the union of two curves defined by

$$\begin{aligned}\omega(\sqrt{\lambda}, \sqrt{\mu}) &= 0, \\ \omega(\sqrt{\lambda}, -\sqrt{\mu}) &= 0.\end{aligned}$$

Those two branches are distinct by Proposition 2.3.8. Since p lies on both, the curve C_{hf} is singular at p. □

One could also prove this corollary by using the argument from the proof of Proposition 2.3.8.

2.4 Gauss swallowtails

In this section, following [24], we show that Gauss swallowtails are exactly the intersection points of the flecnodal curve and the parabolic curve. It implies that a point is a Gauss swallowtail if and only if the tangent curve at it has a tacnode. The parabolic curve and the flecnodal curve always intersect each other with multiplicity two. Further, the flecnodal curve is smooth at Gauss swallowtails. Using these facts, we show that there are 320 Gauss swallowtails and the flecnodal curve has degree 80.

Let Swallowtail(S) be the set of Gauss swallowtails.

Remark 2.4.1. Recall that on the parabolic curve C_{par} we have unique asymptotic directions. Let $\mathcal{L} \subseteq TS|_{C_{\text{par}}}$ be the subbundle of asymptotic directions. Take a projection $\pi \colon TS|_{C_{\text{par}}} \to N_{C_{\text{par}}/S}$. Then, by Remark 2.2.7, Gauss swallowtails are exactly zeroes of $\pi|_{\mathcal{L}}$.

We say that a Gauss swallowtail $p \in \text{Swallowtail}(S)$ is *nondegenerate* if it is a simple zero of $\pi|_{\mathcal{L}}$.

Proposition 2.4.2. *All Gauss swallowtails of a general smooth quartic are nondegenerate.*

Proof. See [24, Theorem 3.1]. □

Proposition 2.4.3. *A point $p \in C_{\text{par}}$ is a Gauss swallowtail if and only if it is a flec point. In other words*

$$\text{Swallowtail}(S) = C_{\text{par}} \cap C_{\text{hf}}.$$

Proof. Here, we use the notation of Lemma 2.3.11. Recall that the equation $\mu = 0$ defines C_{par} and for every q in the local chart $U \subseteq S$ we have $\lambda(q) \neq 0$.

Take $p \in S$. If p is a Gauss swallowtail, that is $\mu(p) = 0$ and $\mu_x(p) = 0$, then p satisfies the equation of Lemma 2.3.11, and so $p \in C_{\text{hf}}$. Other way round, if $\mu(p) = 0$ and p satisfies the equation of Lemma 2.3.11, then $\mu_x(p)^2\lambda(p)^3 = 0$, and so $\mu_x(p) = 0$, which is equivalent to p being a Gauss swallowtail. □

Note that a point $p \in C_{\text{par}}$ is a flec point if and only if E_p has a tacnode at p (see Remark 1.1.5). Therefore, we have finally verified that our definition of a Gauss swallowtail agrees with the one from Proposition 2.1.7.

Proposition 2.4.4. *For every Gauss swallowtail point $p \in \text{Swallowtail}(S)$, the curve C_{hf} is smooth at p and $C_{\text{hf}} \curlyvee_p C_{\text{par}}$ with $\text{mult}_p(C_{\text{hf}} \cap C_{\text{par}}) = 2$.*

We simplify the proof from [24, Theorem 1.8].

Proof. Take $p \in \text{Swallowtail}(S)$. Here, we use the notation of the proof of Lemma 2.3.11. The asymptotic direction at p is given by $y = 0$. By Remark 2.2.7, it holds that $\mu_x(p) = 0$ and, since C_{par} is smooth, $\mu_y(p) \neq 0$. We have $\mu_{x,x}(p) \neq 0$ by Proposition 2.4.2.

Take

$$M := \frac{\mu_x \lambda - \mu \lambda_x}{\mu_y \lambda - \mu \lambda_y}.$$

Since $\mu_y \lambda - \mu \lambda_y \neq 0$, using Lemma 2.3.11 we get that C_{hf} is described by

$$\mu + \lambda M^2 = 0.$$

As $M(p) = 0$, we have $(\lambda M^2)_x(p) = (\lambda M^2)_y(p) = 0$, and so

$$d\mu(p) = d(\mu + \lambda M^2)(p).$$

Hence, C_{hf} is smooth at p and $C_{\text{hf}} \curlyvee_p C_{\text{par}}$. Easy calculation shows that at the point p

$$(\lambda M^2)_{x,x}(p) = 2\lambda(p) \left(\frac{\mu_{x,x}(p)}{\mu_y(p)} \right)^2 \neq 0.$$

Hence, $\text{mult}_p(C_{\text{hf}} \cap C_{\text{par}}) = 2$. □

Proposition 2.4.5 ([24, Theorem 2.5])**.** *There are* 320 *Gauss swallowtails.*

We rewrite the proof from [24].

Proof. By Remark 2.4.1 and Proposition 2.4.2, we have that

$$\begin{aligned} |\,\text{Swallowtail}(S)| &= \deg \mathcal{H}om\left(\mathcal{L}, N_{C_{\text{par}}/S}\right) \\ &= -\deg\left(\mathcal{L}\right) + \deg\left(N_{C_{\text{par}}/S}\right) \\ &= -\deg\left(\mathcal{L}\right) + C_{\text{par}} \cdot C_{\text{par}} \\ &= -\deg\left(\mathcal{L}\right) + 256. \end{aligned}$$

We need to calculate $\deg(\mathcal{L})$. Consider the map $\mathrm{I\!I}\colon TS \otimes TS \to N_{S/\mathbb{P}^3}$. On C_{par} it gives us an isomorphism

$$\mathcal{M} \otimes \mathcal{M} \xrightarrow{\cong} N_{S/\mathbb{P}^3}|_{C_{\text{par}}},$$

where $\mathcal{M} := TS|_{C_{\text{par}}}/\mathcal{L}$. Thus,

$$\begin{aligned} \deg(\mathcal{L}) &= \deg(TS|_{C_{\text{par}}}) \; - \frac{1}{2} \deg\left(N_{S/\mathbb{P}^3}|_{C_{\text{par}}}\right) \\ &= -\deg(K_S|_{C_{\text{par}}}) - \frac{1}{2} \deg\left(\mathcal{O}_{C_{\text{par}}}(4)\right) \\ &= -64. \end{aligned}$$

Therefore, we have $|\,\text{Swallowtail}(S)| = 320$. □

The following may be found in [19, Proposition 8.8] or [34]. We present a different proof based on the calculation of Gauss swallowtails.

Corollary 2.4.6. *It holds that* $\deg(C_{\mathrm{hf}}) = 80$.

Proof. Recall that C_{par} lies in $|\mathcal{O}_S(8)|$. By Proposition 2.4.4 and Proposition 2.4.5 we have

$$320 = |\operatorname{Swallowtail}(S)| = |C_{\mathrm{hf}} \cap C_{\mathrm{par}}| = \frac{8\deg(C_{\mathrm{hf}})}{2}. \qquad \square$$

In particular, C_{hf} lies in $|\mathcal{O}_S(20)|$, since a very general smooth quartic S has the Picard rank one (see Theorem 1.6.1).

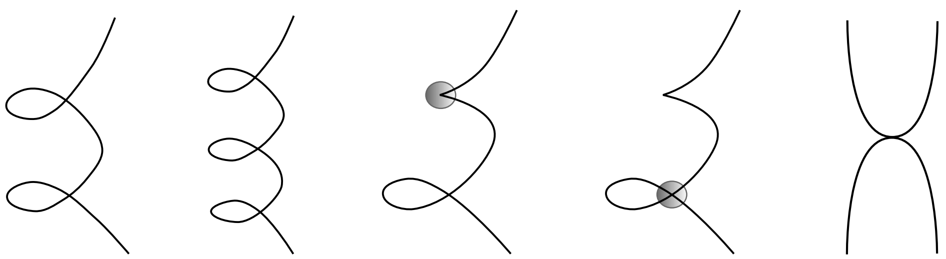
Note that by the adjunction formula (see [4, Section II.11])

$$p_a(C_{\mathrm{hf}}) = \frac{1}{2}C_{\mathrm{hf}}^2 + 1 = 801,$$

and so by Proposition 2.3.7, the curve C_{hf} is singular.

2.5 The double-cover curve

In this section, we analyse the double-cover curve, which is the subset of points of the following type:



Simple Gauss Double | Gauss Triple | Parabolic Gauss Double | Dual to Parabolic Gauss Double | Gauss Swallowtail

We show that the locus of those points is a curve, which means that there are no isolated points of the above type. One could also define the double-cover curve to be the closure of the locus of simple Gauss double points.

The double-cover curve has nodes at Gauss triple points, cusps at dual to parabolic Gauss double points and is smooth everywhere else. The parabolic curve intersects the double-cover curve at Gauss swallowtails with multiplicity two and at parabolic Gauss double points with multiplicity one.

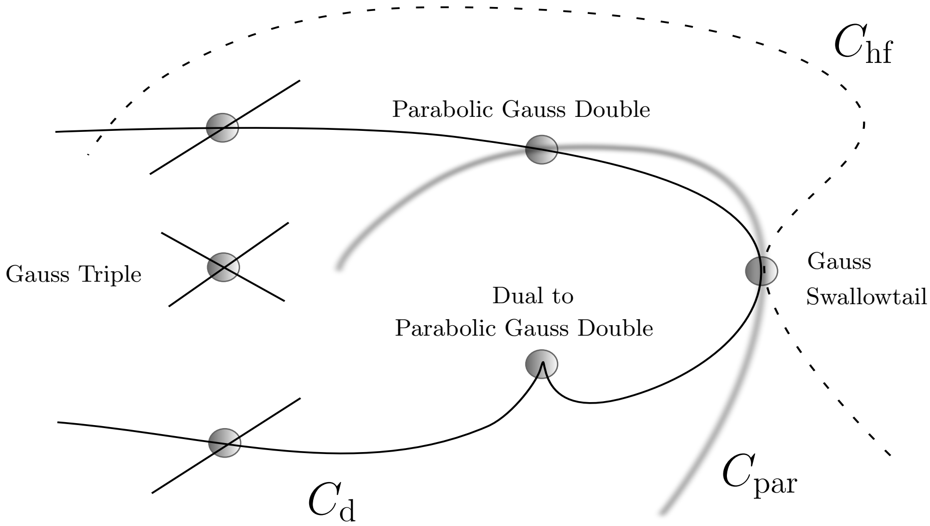


Figure 2.3: The geometry of a very general smooth quartic surface

We calculate that there are 3200 Gauss triple points and 1920 parabolic Gauss double points. The double-cover curve is irreducible. It has genus 1281 and degree 320.

2.5.1 Deformations of tangent curves

Definition 2.5.1. We define the *double-cover curve* $C_{\rm d}$ to be the subset of points $p \in S$ satisfying $g(E_p) \leq 1$.

Recall the definitions of points with $g(E_p) \leq 1$ from the classification of tangent curves (see Proposition 2.1.7). We call a point $p \in S$ *a Gauss double point*, if E_p has exactly two singular points and *a Gauss triple point*, if E_p has three singular points.

For a Gauss double point $p \in C_{\rm d}$, we define the *dual point* $\hat{p} \in C_{\rm d}$ to be the second singularity of E_p. We have a natural rational involution $i\colon C_{\rm d} \dashrightarrow C_{\rm d}$ which takes p to $\hat{p}$.

We say that a Gauss double point p is *parabolic*, if the tangent curve E_p has a cusp at p. We call a point p *a dual to a parabolic Gauss double point*, if $\hat{p}$ is a parabolic Gauss double point. We call those points *nonsimple Gauss double points.* We say that a Gauss double point is *simple*, if it is neither parabolic, nor dual to a parabolic point.

By the classification of tangent curves (see Proposition 2.1.7), $C_{\rm d}$ consists exactly of simple Gauss double points, parabolic Gauss double points, dual to parabolic Gauss double points, Gauss swallowtails and Gauss triple points. Recall that there are only finitely many nonsimple Gauss double points and Gauss swallowtails.

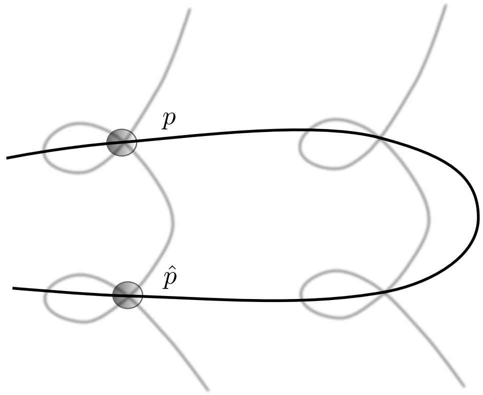


Figure 2.4: The double cover curve and the tangent curve at a point p

The aim of this subsection is to show that the double-cover curve is of pure dimension one. In other words, we need to show that Gauss triple points, nonsimple Gauss double points and Gauss swallowtails lie in the closure of the locus of simple Gauss double points.

First, we need to show the following.

Lemma 2.5.2. *The subset C_{d} is closed.*

Proof. Define the family of tangent curves on S

$$\begin{array}{l} \mathcal{E} := \left\{(p,x) \in S \times \mathbb{P}^3 \,\middle|\, x \in E_p\right\} \\ \Big\downarrow \pi \\ S \end{array}$$

which is flat by Corollary 1.1.14. By the semicontinuity of the geometric genus (see Lemma 1.1.9), we get that the locus of points $p \in S$ satisfying $g(E_p) \leq 1$ is closed. □

Proposition 2.5.3. *The closed subset C_{d} has pure dimension one.*

In other words, C_{d} is a curve.

Proof. Take a point p in C_{d}. We show that p cannot be an isolated point of C_{d}.

Case 1. Suppose that E_p has at least two singularities.

Under this assumption, there exists a point q such that $p \neq q$ and $\phi(p) = \phi(q)$.

Let U_1 and U_2 be disjoint, sufficiently small open neighbourhoods of p and q in S. Images $\phi(U_1)$ and $\phi(U_2)$ are codimension one analytic branches of S^* at $\phi(q)$, and so, by dimension theory, their interesection is of codimension at most two (more precisely, apply the Principal Ideal Theorem to $\widehat{\mathcal{O}}_{\mathbb{P}^3,\phi(p)}$, cf. [10, Theorem

10.2]). Since the locus where the Gauss map ϕ is not injective is contained in C_{d}, we get that $\phi^{-1}(\phi(U_1) \cap \phi(U_2)) \subseteq C_{\mathrm{d}}$, and so the point p cannot be isolated.

Case 2. Suppose that E_p has exactly one singularity. We use global deformation theory of plane curves (see Subsection 1.1.3) to show that p cannot be an isolated point. Note, that the following argument works, also, in the case when p is a Gauss double point, but not when p is a Gauss triple point (see Remark 2.5.4).

Let V be the image of $|\mathcal{O}_{\mathbb{P}^3}(1)|$ in $|\mathcal{O}_S(1)|$. Elements of V are exactly the intersections of hyperplanes in $\mathbb{P}^3$ with S. We will show that V contains an irreducible subvariety $\overline{V}$ of dimension at least one such that $E_p \in \overline{V}$ and $g(C) \leq 1$ for every $C \in \overline{V}$. This would imply that p is not an isolated point of C_{d}, because the closed subset $\bigcup_{C \in \overline{V}} \operatorname{Sing}(C)$ is contained in C_{d} and p is the only singularity of E_p.

First, note that $\dim(V) = 3$, as $\dim |\mathcal{O}_{\mathbb{P}^3}(1)| = 3$ and any two distinct hyperplanes in $\mathbb{P}^3$ intersect S in two distinct curves.

Every curve in V is planar of degree four. By taking a projection of T_qS to T_pS for points $q \in S$ in some Zariski open neighbourhood of p, we get a natural morphism $\psi\colon U \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(4))$ parametrizing curves in V, where $U \subseteq V$ is a Zariski open neighbourhood of $E_p \in V$.

If $\dim(\psi(U)) < 3$, then $\psi^{-1}(\psi(E_p)) \subseteq V$ is at least one-dimensional family of curves with genus smaller or equal than one. Hence, we may assume that $\dim(\psi(U)) = 3$. Since, $U^{4,1}$ has codimension two in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(4))$ (see Lemma 1.1.16), the image $\psi(U)$ must intersect $U^{4,1}$ along a subvariety of dimension at least one (once again by dimension theory). The inverse image of the irreducible component of $\psi(E_p)$ in this intersection is the family we were looking for. □

We get that C_{d} is the closure of the locus of simple Gauss double points. Note that the definition of the double-cover curve and the propositions above work for all smooth quartics in $\mathbb{P}^3$.

Remark 2.5.4. It is important to consider two cases of the proof separatedly. By applying the argument of Case 2 to a Gauss triple point, we could get that nodal rational curves degenerate in families, but it is not clear how to get from this that nodes of rational curves degenerate in families.

Remark 2.5.5. In the case when S is a general quartic surface and the Gauss map π is stable, one can show that Gauss swallowtails are not isolated points of C_{d} by a local calculation (see the proof of Proposition 2.5.11).

Proposition 2.5.6. *The curve C_{d} is irreducible.*

Proof. See [25] or [27]. □

2.5.2 Singularities

In this subsection we analyse the local behaviour of the curve C_{d}.

Proposition 2.5.7. *Let p be a Gauss double point. Assume it is not dual to a parabolic Gauss double point. Then the double-cover curve C_{d} is smooth at p.*

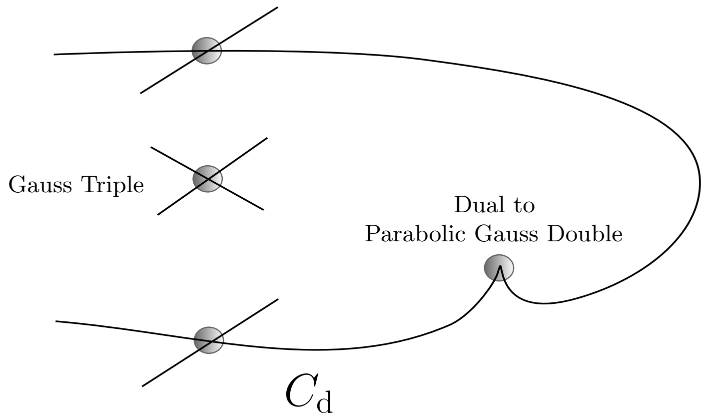


Figure 2.5: The singularities of the double cover curve

Proof. Since ϕ is stable, it satisfies the NC condition (see Proposition 1.2.6). Take two small disjoint neighbourhoods $U_p, U_{\hat{p}} \subseteq S$ containing p and $\hat{p}$ respectively. Then $\phi|_{U_p}$ and $\phi|_{U_{\hat{p}}}$ are transverse. Also, note that $\phi(U_{\hat{p}})$ is smooth, because ϕ does not ramify at $\hat{p}$. Hence, by Proposition 1.2.8, we get that

$$C_d \cap U_p = \phi|_{U_p}^{-1}\left(\phi\left(U_{\hat{p}}\right)\right)$$

is smooth. □

Definition 2.5.8. We say that two analytic branches B_1 and B_2 of C_{d} at points $p_1 \in C_{\mathrm{d}}$ and $p_2 \in C_{\mathrm{d}}$ respectively are dual to each other, if locally $i(B_1) = B_2$.

Consider a sequence of points p_i on the double-cover curve. Note that if this sequence converges, then the sequence $\hat{p}_i$ also converges.

Lemma 2.5.9. *A sequence of points $p_i \in C_{\mathrm{d}}$ converges to a Gauss swallowtail if and only if the sequences p_i and $\hat{p}_i$ converge to the same point.*

Proof. Assume that the sequence p_i converges to a point p. If $\hat{p}_i$ also converges to the point p, then by Corollary 1.1.18 the point p cannot be an ordinary singularity. By the classification of tangent curves (see Proposition 2.1.7), the point p must be a Gauss swallowtail.

Now, assume that p is a Gauss swallowtail. In particular, E_p is singular only at p by the classification of tangent curves (see Proposition 2.1.7). Since the sequence $\hat{p}_i$ converges to a singular point of E_p (see for example Theorem 1.1.17), it must converge to p. □

Proposition 2.5.10. *Let $p \in C_{\mathrm{d}}$ be a parabolic Gauss double point. Then C_{d} has a cusp at $\hat{p}$.*

Proof. See Proposition 2.7.3. □

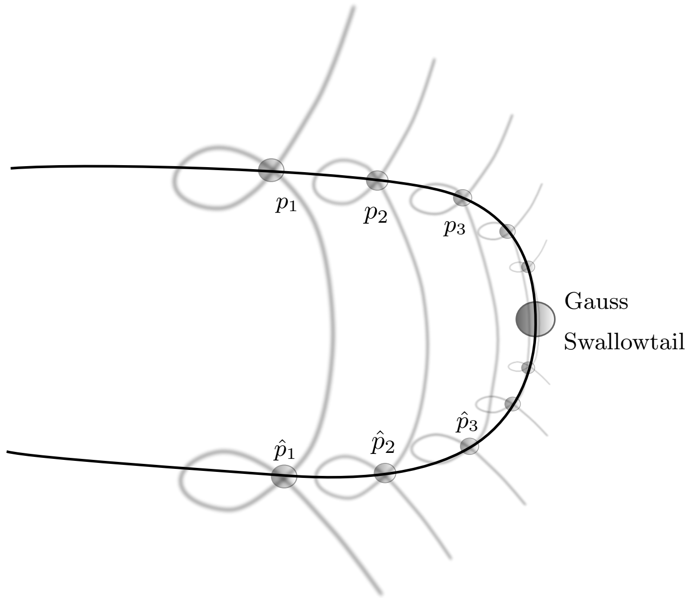


Figure 2.6: A sequence of points converging to a Gauss swallowtail

Proposition 2.5.11. *The double-cover curve C_{d} is smooth at Gauss swallowtails.*

Proof. By Remark 2.2.8, we need to find an equation for C_{d} at the chart ϕ_3

$$(x, y) \overset{\phi_3}{\longmapsto} (3x^4 + x^2y, 2x^3 + xy, y).$$

Assume that we have two points $p_1 := (x, y)$ and $p_2 := (\overline{x}, \overline{y})$ such that $\phi_3(x, y) = \phi_3(\overline{x}, \overline{y})$. Then clearly $y = \overline{y}$. We claim that $p_2 = (x, y)$ or $p_2 = (-x, y)$. Assume by contradiction that neither of this holds. We have

$$3x^4 + x^2y = 3\overline{x}^4 + \overline{x}^2y,$$
$$2x^3 + xy = 2\overline{x}^3 + \overline{x}y.$$

By dividing the first equation by $x^2 - \overline{x}^2$ and the second by $x - \overline{x}$, we get

$$3(x^2 + \overline{x}^2) + y = 0,$$
$$2(x^2 + x\overline{x} + \overline{x}^2) + y = 0.$$

By substracting both equations we obtain

$$x^2 - 2x\overline{x} + \overline{x}^2 = 0,$$

which is a contradiction. Hence, $\overline{x} = x$ or $\overline{x} = -x$.

The set of points (x, y) such that $\phi(x, y) = \phi(-x, y)$ is described by the equation $2x^3 + xy = 0$, and so the curve C_{d} is the zero locus of $2x^2 + y$. In particular, C_{d} is smooth at 0. □

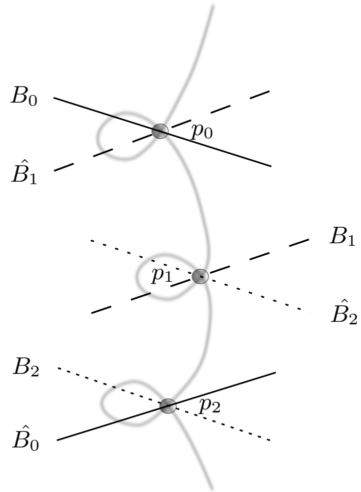


Figure 2.7: Branches of the double cover curve at Gauss triple points

Proposition 2.5.12. *Let p_0, p_1 and p_2 be Gauss triple points, which lie on the same tangent curve E. Then the curve C_d has at those points all together six analytic branches $B_0, \hat{B}_0, B_1, \hat{B}_1, B_2, \hat{B}_2$, where $\hat{B}_i$ is the dual of B_i. The branches B_i and $\hat{B}_{(i+1) \bmod 3}$ intersect each other transversally at p_i. Additionally, all the branches are smooth.*

Proof. This follows from the fact that ϕ satisfies the NC condition (see the proof of Proposition 2.5.7). □

Now, we describe the tangent T_pC_d at a point $p \in C_\mathrm{d}$ only in terms of properties of the tangent curve E_p.

Proposition 2.5.13. *Take $p \in C_\mathrm{d}$ and assume that C_d is smooth at this point. Then $T_pC_\mathrm{d} \subseteq T_pS$ is orthogonal to the line $\overline{p\hat{p}}$ with respect to the second fundamental form II_p, that is $\mathrm{II}_{p_i}\left(T_pC_\mathrm{d}, \overline{p\hat{p}}\right) = 0$.*

Proof. By continuity, it is enough to show the proposition in the case when C_d is also smooth at $\hat{p}$ and the dual map ϕ does not ramify at points p and $\hat{p}$. To easy the notation, we set $p_1 := p$ and $p_2 := \hat{p}$. Define $H := T_{p_i}S$ and $p^* = \phi(p_i)$. We write $T_{p_i}S$ when we want to specify that the origin is p_i and we write H otherwise. Since the Gauss map ϕ does not ramify at points p_i and C_d is smooth at those points, the curve $\phi(C_\mathrm{d})$ is smooth at p^* and we have

$$T_{p^*}\phi(C_\mathrm{d}) = d\phi_{p_1}(T_{p_1}C_\mathrm{d}) = d\phi_{p_2}(T_{p_2}C_\mathrm{d}).$$

Hence, $d\phi_{p_i}(T_{p_i}C_\mathrm{d}) \subseteq d\phi_{p_1}(H) \cap d\phi_{p_2}(H)$.

In order to identify the line $T_{p_i}C_\mathrm{d}$ we use the construction of the second fundamental form II (see Definition 1.6.9). First, recall that $d\phi_{p_i}(H)$ is contained in

$$T_{\hat{H}}\check{\mathbb{P}}^3 \cong \mathrm{Hom}(\hat{H}, \mathbb{C}^4/\hat{H}),$$

where $\hat{H} \subseteq \mathbb{C}^4$ is the deprojectivization of H and $\check{\mathbb{P}}^3$ is the space of hyperplanes in $\mathbb{P}^3$. The image $d\phi_{p_i}(H)$ is equal to

$$\mathrm{Hom}\left(\hat{H}/\hat{p}_i, \mathbb{C}^4/\hat{H}\right) \cong \mathrm{Hom}\left(T_{p_i}S, N\right),$$

where $\hat{p}_i$ is the deprojectivization of p_i and N is the orthogonal line to $T_{p_i}S$ in $\mathbb{P}^3$. Now, observe that

$$\begin{aligned} d\phi_{p_1}(H) \cap d\phi_{p_2}(H) &\cong \mathrm{Hom}\left(\hat{H}/\hat{p}_1, \mathbb{C}^4/\hat{H}\right) \cap \mathrm{Hom}\left(\hat{H}/\hat{p}_2, \mathbb{C}^4/\hat{H}\right) \\ &\cong \mathrm{Hom}\left(T_{p_i}S/(\overline{p_1p_2}), N\right), \end{aligned}$$

where the second isomorphism holds, because under the indentification of $\hat{H}/\hat{p}_1$ with $T_{p_1}S$, the line $\hat{p}_2$ is sent to $\overline{p_1p_2}$.

Since $d\phi_{p_i}(T_{p_i}C_{\mathrm{d}}) \subseteq d\phi_{p_1}(H) \cap d\phi_{p_2}(H)$, we have

$$\mathrm{I\!I}_{p_i}\left(T_{p_i}C_{\mathrm{d}}, \overline{p_1p_2}\right) = 0. \qquad \square$$

2.5.3 Intersection points with the parabolic curve

We want to understand how the double-cover curve and the parabolic curve intersect each other. By the classification of tangent curves, the intersection points are exactly parabolic Gauss double points and Gauss swallowtails.

Lemma 2.5.14. *Let $p \in C_{\mathrm{d}}$ be a parabolic Gauss double point. Then the asymptotic direction l at the point p is tangent to C_{d} at p.*

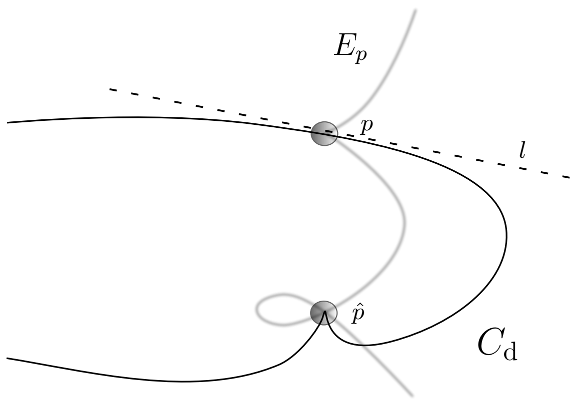


Figure 2.8: The double cover curve C_{d} and the tangent curve E_p at the parabolic Gauss double point p

We will prove at the end of this subsection that the lemma also holds for Gauss swallowtails.

Proof. First, note that $\mathrm{mult}_p(E_p \cap l) = 3$. It implies that $\hat{p} \notin l$, because otherwise $E_p \cdot l \geq 5$, which contradicts the fact that $\deg(E_p) = 4$.

Thus, we have that $\mathrm{II}(\overline{p\hat{p}}, \cdot)$ is nontrivial and its kernel is l. Hence, l is the only line orthogonal to $\overline{p\hat{p}}$ with respect to II_p, and so it must be the tangent to C_{d} at p by Proposition 2.5.13. □

Proposition 2.5.15. *Let $p \in C_{\mathrm{d}}$ be a parabolic Gauss double point. Then $C_{\mathrm{par}} \pitchfork_p C_{\mathrm{d}}$ (curves interesect each other transversally at p).*

Proof. Let l be the asymptotic direction at p. By the above lemma, $T_pC_{\mathrm{d}} = l$. On the other hand, since p is not a Gauss swallowtail, we have that $T_pC_{\mathrm{par}} \neq l$ by Remark 2.2.7. Hence, $T_pC_{\mathrm{par}} \neq T_pC_{\mathrm{d}}$, and so $C_{\mathrm{par}} \pitchfork_p C_{\mathrm{d}}$. □

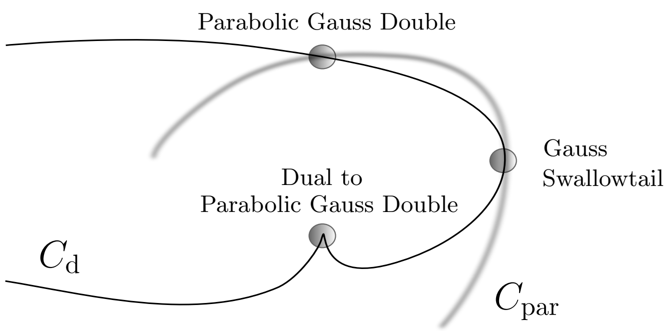


Figure 2.9: Intersection points of the double cover curve and the parabolic curve

Proposition 2.5.16. *Let $p \in C_{\mathrm{d}}$ be a Gauss swallowtail. Then $C_{\mathrm{par}} \curlyvee_p C_{\mathrm{d}}$ with $\mathrm{mult}_p(C_{\mathrm{par}} \cap C_{\mathrm{d}}) = 2$.*

Proof. Recall that at the local chart ϕ_3 (see the notation of Proposition 2.1.1), the curve C_{par} is defined by the equation $6x^2 + y = 0$ (see the proof of Proposition 2.2.4) and the curve C_{d} is defined by $2x^2 + y = 0$ (see the proof of Proposition 2.5.11). Thus, they are tangent with multiplicity two. □

Note, that since the tangents to C_{par} at Gauss swallowtails are the asymptotic directions, also the tangents to C_{d} at Gauss swallowtails are the asymptotic directions.

2.5.4 Enumerative properties

We want to calculate the degree of C_{d}. By Theorem 1.6.1, we know that $C_{\mathrm{d}} \in |\mathcal{O}_S(N)|$ for some $N \in \mathbb{N}$. Our strategy for calculating N is the following. First, we get a formula for the degree of $C_{\mathrm{d}}^* := \phi(C_{\mathrm{d}})$ depending on N, and then we plug it into a certain version of the double-point formula to obtain an equation for N.

A curve E is called *nodal elliptic* if $g(E) = 1$ and E has two nodes.

Lemma 2.5.17. *A general point $q \in S$ lies on exactly $6N$ nodal elliptic curves.*

Before proving the proposition, we make some general comments. Let $\check{\mathbb{P}}^3$ be the space of hyperplanes in $\mathbb{P}^3$. Since, we have chosen the basis, we have a natural isomorphism $\mathbb{P}^3 \to \check{\mathbb{P}}^3$. For a point $q \in \mathbb{P}^3$, we denote the corresponding hyperplane in $\check{\mathbb{P}}^3$ by H_q. It is the image in $\mathbb{P}^3$ of the hyperplane in $\mathbb{C}^4$ orthogonal to the deprojectivization of q. Similarly for a line $l \subseteq \mathbb{P}^3$ we denote the corresponding line in $\check{\mathbb{P}}^3$ by L_l.

Take a point $q \in \mathbb{P}^3$. Recall the definition of the polar locus curve of q

$$\Gamma_q := \{p \in S \mid q \in T_pS\}.$$

Since $\mathrm{Pic}(S) \cong \mathbb{Z}$, Corollary 1.4.6 implies that $\Gamma_q \in \mathcal{O}_S(3)$.

We have

$$q \in T_pS \text{ if and only if } H_q \ni \phi(p),$$

and so $\Gamma_q = \phi^{-1}(H_q \cap S^*)$.

We need the following lemma.

Lemma 2.5.18. *Let C be a curve on S. Then for a general point $q \in S$, the curve Γ_q intersects C transversally.*

Proof. Define $C^* := \phi(C) \subseteq \mathbb{P}^3$. It is enough to show that $\phi(\Gamma_q) \pitchfork C^*$, or equivalently that $H_q \pitchfork C^*$.

Let

$$\mathcal{H}_S := \{H_p \mid p \in S\} \subseteq \check{\mathbb{P}}^3$$

be the two-dimensional subspace of hyperplanes corresponding to points of S. We also define

$$\mathcal{H}_{\tan} := \{(x, H) \mid H \curlyvee_x C^*\} \subseteq C^* \times \check{\mathbb{P}}^3$$

to be the space of hyperplanes tangent to C^*. Its fiber $(\mathcal{H}_{\tan})_x$ over a general point $x \in C^*$ is equal to

$$\{H \mid H \curlyvee_x C^*\} = \{H \mid T_xC^* \subseteq H\} \cong \mathbb{P}^1,$$

and so $\dim(\mathcal{H}_{\tan}) = 2$.

Let $\pi_{\check{\mathbb{P}}^3} \colon \mathcal{H}_{\tan} \to \check{\mathbb{P}}^3$ be the projection onto $\check{\mathbb{P}}^3$. In order to show the claim, we need to prove that $\mathcal{H}_S \not\subseteq \pi_{\check{\mathbb{P}}^3}(\mathcal{H}_{\tan})$. Assume by contradiction that $\mathcal{H}_S \subseteq \pi_{\check{\mathbb{P}}^3}(\mathcal{H}_{\tan})$. Since $\mathcal{H}_S$ and $\mathcal{H}_{\tan}$ are of the same dimension, some fiber

$$(\mathcal{H}_{\tan})_x = \{H \mid T_xC^* \subseteq H\} \subseteq \check{\mathbb{P}}^3$$

is contained in $\mathcal{H}_S$ for a point $x \in C^*$. Hence,

$$L_{T_xC^*} \subseteq S,$$

which is impossible, because there are no lines on S by Theorem 1.6.1. Thus, the claim is proved. □

Proof of Lemma 2.5.17. Using the lemma above, we get that a general point $q \in S$ lies on exactly

$$\frac{1}{2}|\Gamma_q \cap C_{\mathrm{d}}| = \frac{1}{2} \cdot 4 \cdot 3 \cdot N = 6N$$

nodal elliptic curves. □

Define $C_{\mathrm{d}}^* := \phi(C_d)$.

Corollary 2.5.19. *The curve C_{d}^* has degree $6N$.*

Proof. From the proof of Lemma 2.5.18 we see that a general hyperplane H intersects C_{d}^* transversally, and so we have

$$\deg(C_{\mathrm{d}}^*) = |H \cap C_{\mathrm{d}}^*| = \frac{1}{2}|\phi^*(H \cap S^*) \cap C_{\mathrm{d}}| = \frac{1}{2}|\Gamma_q \cap C_{\mathrm{d}}| = 6N.$$

□

Proposition 2.5.20. *The curve C_{d} lies in $|\mathcal{O}_S(80)|$.*

Proof. By Proposition 1.4.4, we have

$$C_{\mathrm{d}} + 2C_{\mathrm{par}} \equiv f^*S + f^*K_{\mathbb{P}^3}.$$

Recal that $C_{\mathrm{par}} \in \mathcal{O}_S(8)$ and $S \in \mathcal{O}_{\mathbb{P}^3}(36)$. Since $f_*C_{\mathrm{d}} = 2C_{\mathrm{d}}^*$, by using the projection formula we get

$$\begin{aligned}(C_{\mathrm{d}} + 2C_{\mathrm{par}}) \cdot C_{\mathrm{d}} &= 2\,(S \cdot C_{\mathrm{d}}^*) + 2\,(K_{\mathbb{P}^3} \cdot C_{\mathrm{d}}^*) \\ &= 2 \cdot 36 \cdot \deg(C_{\mathrm{d}}^*) - 2 \cdot 4 \cdot \deg(C_{\mathrm{d}}^*) \\ &= 64 \deg C_{\mathrm{d}}^*.\end{aligned}$$

Since $\deg(C_{\mathrm{d}}^*) = 6N$ and

$$(C_{\mathrm{d}} + 2C_{\mathrm{par}}) \cdot C_{\mathrm{d}} = 4N^2 + 2 \cdot 4 \cdot 8 \cdot N,$$

we get

$$N^2 + 16N = 16 \cdot 6 \cdot N,$$

and so $N = 80$. □

Corollary 2.5.21. *A general point $q \in S$ lies on exactly* 480 *nodal elliptic curves. The curve C_{d}^* has degree 480.*

□

Proposition 2.5.22 ([25, Example 4, p. 233])**.** *There are* 9600 *Gauss triple points and* 1920 *parabolic Gauss double points.*

Proof. By Yau-Zaslow formula (see Theorem 1.4.9) we know that there are 3200 nodal rational curves in $|\mathcal{O}_S(1)|$. Every such nodal rational curve must be a tangent curve with three nodes. A node of a tangent curve determines the whole curve, and so those nodes are pairwise distinct. Those nodes are exactly Gauss triple points and there are 9600 of them.

Recall that C_{d} and C_{par} intersect exactly at Gauss swallowtails and parabolic Gauss double points with multiplicity two and one respectively. Also, recall that $C_{\mathrm{d}} \in \mathcal{O}_S(80)$, $C_{\mathrm{par}} \in \mathcal{O}_S(8)$ and there are 320 Gauss swallowtails. The number of parabolic Gauss double points is equal to

$$C_{\mathrm{d}} \cdot C_{\mathrm{par}} - 2 \cdot 320 = 4 \cdot 80 \cdot 8 - 2 \cdot 320 = 1920. \qquad \square$$

Proposition 2.5.23. *The curve C_{d} has genus* 1281.

Proof. Recall that C_{d} has ordinary singularities at dual to parabolic Gauss double points and at Gauss triple points. It is smooth everywhere else. Thus, by the adjunction formula (see [4, Section II.11]) and the above propositions, we get

$$\begin{aligned} g(C_{\mathrm{d}}) &= \frac{1}{2} C_{\mathrm{d}}^2 + 1 - (9600 + 1920) \\ &= 12801 - 11520 \\ &= 1281. \end{aligned} \qquad \square$$

2.6 The parabolic curve on the Fermat quartic

In this section, we show that the parabolic curve of the Fermat quartic is a constant cycle curve.

Assume that our smooth surface S is the Fermat quartic given by the equation $f = x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$.

Proposition 2.6.1. *The curve C_{par} on the Fermat quartic S is a constant cycle curve.*

Proof. The parabolic curve is described by $x_1^2 x_2^2 x_3^2 x_4^2 = 0$ (see Proposition 2.2.3). Without loss of generality it is enough to show that the curve C defined by $x_1 = 0$ is a constant cycle curve.

We have a natural automorphism ϕ of S

$$\phi\left((x_1 : x_2 : x_3 : x_4)\right) = (-x_1 : x_2 : x_3 : x_4).$$

Every point of C is fixed by ϕ. By Proposition 1.5.4, in order to show that C is a constant cycle curve, it is sufficient to prove that $\phi^* \neq \mathrm{id}$ on $H^{2,0}(X) = H^0(X, K_X)$.

We know that (cf. [18, (1.2)])

$$\omega := \mathrm{Res}\left(\frac{\sum_{i=1}^4 (-1)^i x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_4}{f}\right)$$

is a trivializing section of K_X. Let us take an affine chart $x_4 = 1$ with coordinates y_1, y_2, y_3. At this chart, the trivializing section is equal to

$$\begin{aligned}\operatorname{Res}\left(\frac{dy_1 \wedge dy_2 \wedge dy_3}{f(y_1, y_2, y_3, 1)}\right) &= \frac{1}{f_{y_1}} dy_2 \wedge dy_3 + \frac{1}{f_{y_2}} dy_1 \wedge dy_3 + \frac{1}{f_{y_3}} dy_1 \wedge dy_2 \\ &= \frac{1}{4y_1^3} dy_2 \wedge dy_3 + \frac{1}{4y_2^3} dy_1 \wedge dy_3 + \frac{1}{4y_2^3} dy_1 \wedge dy_2.\end{aligned}$$

We have

$$\phi^* \omega = -\omega,$$

and so $\phi^* \neq \mathrm{id}$ on $H^{2,0}$. $\square$

2.7 The geometry of dual surfaces

In this section, we describe the singularities of the dual surface. Further, we calculate the degree and the genus of the images of the double cover curve and the parabolic curve.

Define $C_{\mathrm{d}}^* := \phi(C_{\mathrm{d}})$ and $C_{\mathrm{par}}^* := \phi(C_{\mathrm{par}})$. Since ϕ is birational, it is an isomorphism outside of the union of the ramification locus and the locus where the map is not injective. Hence, $\mathrm{Sing}(S^*) = C_{\mathrm{d}}^* \cup C_{\mathrm{par}}^*$.

Definition 2.7.1. A point $p \in S^*$ is called

- a swallowtail point[1], if S^* is locally at p analytically isomorphic to the image of the map $(x, y) \mapsto (3x^4 + x^2y, 2x^3 + xy, y)$

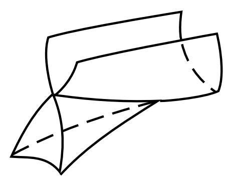


Figure 2.10: A swallowtail

- a triple point, if $\hat{\mathcal{O}}_{S^*,p} \cong \mathbb{C}[\![t_1, t_2, t_3]\!]/(t_1t_2t_3)$

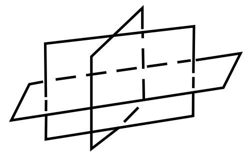


Figure 2.11: A triple point

[1]The picture is based on `http://www.encyclopediaofmath.org/index.php/Thom_catastrophes`

- a simple double point, if $\hat{\mathcal{O}}_{S^*,p} \cong \mathbb{C}[\![t_1, t_2, t_3]\!]/(t_1 t_2)$

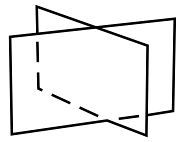


Figure 2.12: A simple double point

- a simple cuspidal point, if $\hat{\mathcal{O}}_{S^*,p} \cong \mathbb{C}[\![t_1, t_2, t_3]\!]/(t_1^3 - t_2^2)$

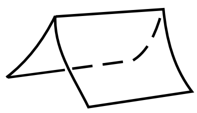


Figure 2.13: A simple cuspidal point

- a cuspidal double point, if $\hat{\mathcal{O}}_{S^*,p} \cong \mathbb{C}[\![t_1, t_2, t_3]\!]/((t_1^3 - t_2^2)t_3)$

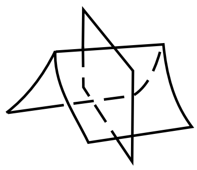


Figure 2.14: A cuspidal double point

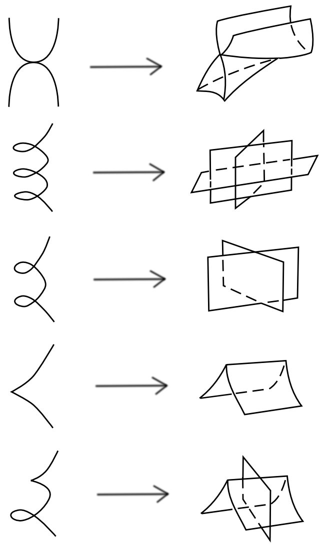
Remark 2.7.2. Let p be a Gauss swallowtail. In [25], it is stated that the local ring of S^* at $\phi(p)$ is isomorphic to the local ring of the Whitney umbrella

$$\hat{\mathcal{O}}_{S^*,p} \cong \mathbb{C}[\![t_1, t_2, t_3]\!]/(t_2^2 - t_1^2 t_3).$$

We believe that it is a mistake. From local calculations, we know that the singularity at $\phi(p)$ is the swallowtail. The swallowtail singularity and the Whitney umbrella are not isomorphic. Though, it must be noted that they are not very far from each other - they are topologically equivalent (see [1, Section 1.3]).

Proposition 2.7.3. *The Gauss map ϕ sends*

- *Gauss swallowtails to swallowtail points*
- *Gauss triple points to triple points*
- *simple Gauss double points to simple double points*
- *simple parabolic points to simple cuspidal points*
- *parabolic Gauss double points to cuspidal double points*



Proof. The description of the singularities for the images of Gauss swallowtails and simple parabolic points follows from the local description of the Gauss map (see Proposition 2.1.1).

Take $p \in S^*$. First, consider the case when p is the image of a parabolic Gauss double point. Then S^* consists of two branches at p and we can change coordinates so that $\hat{O}_{S^*,p} \cong \mathbb{C}[\![t_1, t_2, t_3]\!]/((t_1^3 - t_2^2)f)$, where f is a polynomial in t_1, t_2, t_3. Since the Gauss map ϕ satisfies the NC condition, we have $\frac{\partial f}{\partial t_3} \neq 0$. Hence, we can replace coordinates t_1, t_2, t_3 by coordinates t_1, t_2, f. Then $\hat{O}_{S^*,p} \cong \mathbb{C}[\![t_1, t_2, f]\!]/((t_1^3 - t_2^2)f)$.

Other cases are treated analogously. □

Proposition 2.7.4. *The curve C_{d}^* is smooth outside of triple points and cuspidal double points. At triple points it has three-branched nodes and at cuspidal double points it has cusps. The curve C_{par}^* is smooth outside of swallowtail points.*

Proof. Take $p \in C_{\mathrm{d}}^*$. Assume that p is a swallowtail. We saw in the proof of Proposition 2.5.11 that at the chart ϕ_3

$$(x, y) \overset{\phi_3}{\longmapsto} (3x^4 + x^2y, 2x^3 + xy, y),$$

the curve C_{d} is defined by the equation $2x^2 + y = 0$. In particular, the map $\phi_3|_{C_{\mathrm{d}}} : C_{\mathrm{d}} \to \mathbb{P}^3$ is defined locally at $\phi_3^{-1}(p)$ by the formula

$$x \mapsto (x, -2x^2) \overset{\phi_3}{\longmapsto} (x^4, 0, -2x^2).$$

In particular, the point p is smooth.

Now, consider the case when p is not a swallowtail. The description of the singularities of C_{d}^* follows from Proposition 2.7.3, if we note that locally around p the curve C_{d}^* is the union of the intersections of branches of S^* at p.

As for C^*_{par}, recall that $\phi|_{C_{\mathrm{par}}}\colon C_{\mathrm{par}} \to C^*_{\mathrm{par}}$ is injective and ramifies exactly at Gauss swallowtails. Thus, C^*_{par} is singular at swallowtail points and is smooth everywhere else. $\square$

Proposition 2.7.5 ([25, Example 4, p. 233])**.** *The curve* C^*_{d} *has degree* 480 *and genus* 561*. The curve* C^*_{par} *has degree* 96 *and genus* 129*.*

Proof. By Corollary 2.5.21, we know that $\deg(C^*_{\mathrm{d}}) = 480$. In order to calculate its genus we use the Riemann-Hurwitz formula. Consider the following diagram

$$\begin{tikzcd} \widetilde{C_{\mathrm{d}}} \arrow[r, "\psi"] \arrow[d, "p_1"] & \widetilde{C^*_{\mathrm{d}}} \arrow[d, "p_2"] \\ C_{\mathrm{d}} \arrow[r, "\phi|_{C_{\mathrm{d}}}"] & C^*_{\mathrm{d}} \end{tikzcd}$$

where p_1 and p_2 are the normalizations of C_{d} and C^*_{d} respectively. The morphism p_1 ramifies over cusps of C_{d}, which are exactly dual points to parabolic Gauss double points. The morphism $\phi|_{C_{\mathrm{d}}}$ ramifies at parabolic Gauss double points and Gauss swallowtails. The morphism p_2 ramifies over cuspidal double points. All the ramifications are of degree two, and so ψ ramifies exactly at the preimages under p_1 of Gauss swallowtails. There are 320 Gauss swallowtails, and so by the Riemann-Hurwitz formula

$$2g(C_{\mathrm{d}}) - 2 = 2(2g(C^*_{\mathrm{d}}) - 2) + 320.$$

Since $g(C_{\mathrm{d}}) = 1281$, we have $g(C^*_{\mathrm{d}}) = 561$.

Now we consider the case of C^*_{par}. Since $\phi|_{C_{\mathrm{par}}}\colon C_{\mathrm{par}} \to C^*_{\mathrm{par}}$ is birational, we have $g(C^*_{\mathrm{par}}) = g(C_{\mathrm{par}}) = 129$.

For calculating $\deg(C^*_{\mathrm{par}})$, we use the notation from Corollary 2.5.19. Recall that the polar locus curve Γ_q lies in $|\mathcal{O}_S(3)|$, and also that $C_{\mathrm{par}} \in |\mathcal{O}_S(8)|$. From the proof of Lemma 2.5.18 we see that a general hyperplane H intersects C^*_{par} transversally, and so we have

$$\deg(C^*_{\mathrm{par}}) = |H \cap C^*_{\mathrm{par}}| = |\phi^*(H) \cap C_{\mathrm{par}}| = |\Gamma_q \cap C_{\mathrm{par}}| = \Gamma_q \cdot C_{\mathrm{par}} = 4 \cdot 3 \cdot 8 = 96.$$

$\square$

Résumé

The double-cover curve C_{d}

Genus	1281
Degree	320
Singularities	Gauss Triple points, *Nodes*
	Dual to Parabolic Gauss Double points, *Cusps*

The dual curve C_{d}^*

Genus	561
Degree	480
Singularities	Gauss Triple points, *Three-branched nodes*
	Cuspidal Double points, *Cusps*

The parabolic curve C_{par}

Genus	129
Degree	32
Singularities	No

The dual curve C_{par}^*

Genus	129
Degree	96
Singularities	Swallowtail points

The flecnodal curve C_{hf}

Genus	201
Degree	80
Singularities	Yes

Special points

Gauss Swallowtails	320
Gauss Triple	9600
Parabolic Gauss Double	1920

CHAPTER 3

The space of smooth quartics in $\mathbb{P}^3$

Take a very general point $p \in C_{\mathrm{d}}$ on a very general smooth quartic in $\mathbb{P}^3$ and let $\tilde{E}_p$ be the normalization of the tangent curve E_p. Our main aim is to prove that there is no relation between p and the dual point $\hat{p}$ inside $\mathrm{Pic}(\tilde{E}_p)$. This shows that the method of proving that C_{hf} is a constant cycle curve, does not work in the case of C_{d}.

Additionally, we prove that a general point on C_{hf} has exactly one hyperflex.

3.1 Embeddings of elliptic curves in $\mathbb{P}^2$ with two nodes

We need the following lemma for the proof of Theorem 3.1.2.

Lemma 3.1.1. *Let $V \subset \mathbb{C}^4$ be a three-dimensional linear subspace. Then there exist two nonzero vectors $v_1, v_2 \in V$ of the form $v_1 = (p, q, 0, 0)$, $v_2 = (0, 0, r, s)$, where $p, q, r, s \in \mathbb{C}$.*

Proof. There exist nonzero $v_1 \in V \cap (\mathbb{C}^2 \times \{0\} \times \{0\})$ and $v_2 \in V \cap (\{0\} \times \{0\} \times \mathbb{C}^2)$, since the intersection of a two-dimensional space with a three-dimensional space in a four-dimensional space must be at least one dimensional. $\square$

Theorem 3.1.2. *Let E be a smooth elliptic curve and let $P_1, P_2, Q_1, Q_2 \in E$ be pairwise distinct points such that $P_1 + P_2 \not\sim Q_1 + Q_2$. Then, there exists a morphism $\phi\colon E \to \mathbb{P}^2$, which is an isomorphism onto its image outside of P_1, P_2, Q_1, Q_2 and such that $\phi(P_1) = \phi(P_2) \neq \phi(Q_1) = \phi(Q_2)$.*

Proof. Let $D := P_1 + P_2 + Q_1 + Q_2$ and $X := \mathrm{Supp}(D)$. Consider the following exact sequence:

$$0 \longrightarrow H^0(E, \mathcal{O}_E) \longrightarrow H^0(E, \mathcal{O}_E(D)) \longrightarrow H^0(X, \mathcal{O}_X).$$

As $h^0(E, \mathcal{O}_E) = 1$ and $h^0(E, \mathcal{O}_E(D)) = 4$, the image of $H^0(E, \mathcal{O}_E(D))$ in $H^0(X, \mathcal{O}_X)$ is three-dimensional. We have

$$H^0(X, \mathcal{O}_X) = H^0(P_1, \mathcal{O}_{P_1}) \times H^0(P_2, \mathcal{O}_{P_2}) \times H^0(Q_1, \mathcal{O}_{Q_1}) \times H^0(Q_2, \mathcal{O}_{Q_2}) \simeq \mathbb{C}^4.$$

From the lemma above we can find three linearly independent sections s_0, s_1, s_2 of $\mathcal{O}_E(D)$ such that the image of s_0 in $H^0(X, \mathcal{O}_X)$ is zero, $s_1(Q_1) = s_1(Q_2) = 0$ and $s_2(P_1) = s_2(P_2) = 0$.

Let V be a linear system in $\mathcal{O}_E(D)$ spanned by s_0, s_1, s_2. Note that $|D|$ defines a closed embedding of E into $\mathbb{P}^3$. By [3, Lemma 2.1] elliptic curves in $\mathbb{P}^3$ have no multisecants, so $s_1(P_1), s_1(P_2) \neq 0$ and $s_2(Q_1), s_2(Q_2) \neq 0$. As s_0 is vanishing only on X, it holds that V is base point free.

Let $\phi\colon E \to \mathbb{P}^2$ be the morphism associated to V. By definition of V, we get

$$\phi(P_1) = \phi(P_2) \neq \phi(Q_1) = \phi(Q_2). \tag{3.1}$$

Claim 3.1.3. *The morphism ϕ is birational and* $\deg(\phi(E)) = 4$.

Proof. We have $\deg(\phi) \in \{1, 2, 4\}$, because $\deg(\mathcal{O}_E(D)) = 4$. It is sufficient to prove that $\deg(\phi) = 1$.

If $\deg(\phi) = 4$, then $\deg(\phi(E)) = 1$. In this case

$$\phi(P_1) = \phi(P_2) = \phi(Q_1) = \phi(Q_2),$$

because

$$D = (s_0) = \phi^*(H \cap \phi(E)),$$

for some hyperplane $H \subseteq \mathbb{P}^2$ and a point $H \cap \phi(E)$. This contradicts equation (3.1).

If $\deg(\phi) = 2$, then $\deg(\phi(E)) = 2$, and so $\phi(E)$ is a smooth rational curve. In this case $\phi(P_1) \sim \phi(Q_1)$ in $\mathrm{Pic}(\phi(E))$, and thus

$$P_1 + P_2 = \phi^*(\phi(P_1)) \sim \phi^*(\phi(Q_1)) = Q_1 + Q_2,$$

contradicting the assumption of the theorem. □

The points $\phi(P_1)$ and $\phi(Q_1)$ are singularities of $\phi(E)$. As $g(\phi(E)) = 1$ and $\deg(\phi(E)) = 4$, the curve E cannot have more singularities by Lemma 1.1.3. Thus, the morphism ϕ must be an isomorphism outside P_1, P_2, Q_1, Q_2. □

Note that $\phi(E)$ has two branches at each $\phi(P_1)$ and $\phi(Q_1)$. Since $\phi(E)$ has geometric genus one and arithmetic genus three, its singularities must be nodes (locally analytically isomorphic to $xy = 0$) by Lemma 1.1.3.

3.2 Plane curves on smooth quartics

The following theorem is crucial in the proofs of Theorem 3.3.2 and Proposition 2.3.8.

Theorem 3.2.1. *Every plane curve C of degree 4 is the intersection of some smooth quartic in $\mathbb{P}^3$ with a hyperplane.*

Proof. Let $f \in \mathbb{C}[x, y, z]$ be a homogenous polynomial of degree 4 describing the curve C. We show that there exist $a, b, c \in \mathbb{C}$ such that

$$F^{a,b,c} := f(x, y, z) + w(ax^3 + by^3 + cz^3) + w^4 = 0 \tag{3.2}$$

describes a smooth surface $S^{a,b,c}$. Note that $C = S^{a,b,c} \cap \{w = 0\}$.

The partial derivatives of $F^{a,b,c}$ are:

$$\begin{aligned} F_w^{a,b,c} &= ax^3 + by^3 + cz^3 + 4w^3, \\ F_x^{a,b,c} &= f_x + 3awx^2, \\ F_y^{a,b,c} &= f_y + 3bwy^2, \\ F_z^{a,b,c} &= f_z + 3cwz^2. \end{aligned} \tag{3.3}$$

By equations (3.2) and (3.3), for $(x, y, z) \in \operatorname{Sing}(S^{a,b,c})$ it holds that

$$f(x, y, z) = 3w^4. \tag{3.4}$$

Let $T := \{(a, b, c) \mid S^{a,b,c} \text{ is singular}\} \subseteq \mathbb{C}^3$. It is sufficient to show that $\dim(T) \leq 2$. Define

$$\begin{aligned} T_1 &:= \Big\{ ((a, b, c), [x : y : z]) \;\Big|\; S^{a,b,c} \text{ is singular at } [x : y : z : 0] \Big\} \subseteq \mathbb{C}^3 \times \mathbb{P}^2, \\ T_2 &:= \Big\{ ((a, b, c), (x, y, z)) \;\Big|\; S^{a,b,c} \text{ is singular at } [x : y : z : 1] \Big\} \subseteq \mathbb{C}^3 \times \mathbb{C}^3. \end{aligned}$$

We need to show that $\dim(T_1) \leq 2$ and $\dim(T_2) \leq 2$.

By the equations above, we have that $T_1 \subseteq \mathbb{C}^3 \times \operatorname{Sing}(C)$ is defined by $ax^3 + by^3 + cz^3 = 0$, and so $\dim(T_1) \leq 2$.

The scheme T_2 is contained in the zero locus of

$$\begin{aligned} f - 3 &= 0, \\ f_x + 3ax^2 &= 0, \\ f_y + 3by^2 &= 0, \\ f_z + 3cz^2 &= 0. \end{aligned}$$

Those equations are independent, and so $\dim(T_2) \leq 2$. $\square$

3.3 Relations between nodes of tangent curves

Let $|\mathcal{O}_{\mathbb{P}^3}(4)|$ be the space of quartic hypersurfaces in $\mathbb{P}^3$ and let $|\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \subseteq |\mathcal{O}_{\mathbb{P}^3}(4)|$ be the open subset of the smooth ones. We define

$$\begin{tikzcd} \mathscr{S} \arrow[d] & \subseteq & |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \times \mathbb{P}^3 \\ |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \end{tikzcd}$$

to be the universal family of smooth quartics in $\mathbb{P}^3$. For $f \in |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}$, the fiber $\mathscr{S}_f$ is the surface in $\mathbb{P}^3$ defined by $f = 0$.

Let $\mathscr{C}_d \to |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}$ be the universal family of double-cover curves of smooth quartics. For $f \in |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}$, the fiber $(\mathscr{C}_d)_f \subseteq \mathscr{S}_f$ is defined to be the double-cover curve of $\mathscr{S}_f$.

$$\begin{tikzcd} \mathscr{C}_d \arrow[r, hook] \arrow[d] & \mathscr{S} \arrow[dl] \\ |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \end{tikzcd}$$

Further, let

$$\begin{tikzcd} \mathscr{E} \arrow[d] & \subseteq & \mathscr{S} \times \mathbb{P}^3 \\ \mathscr{S} \end{tikzcd}$$

be the universal family of tangent curves on smooth quartics. For

$$(f, p) \in \mathscr{S} \subseteq |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \times \mathbb{P}^3,$$

the fiber $\mathscr{E}_{(f,p)} \subseteq \mathbb{P}^3$ is the tangent curve of $\mathscr{S}_f$ at p.

We say that a plane curve E of degree four is a *general double curve*, if E has exactly two nodes and $g(E) = 1$. We define

$$\overline{\mathscr{C}}_d := \{x \in \mathscr{S} \mid \mathscr{E}_x \text{ is a general double curve}\} \subseteq \mathscr{C}_d.$$

to be the subset of those points on smooth quartics, whose tangent curve is a general double curve.

Finally, we define

$$\overline{\mathcal{E}}_d := \overline{\mathscr{C}}_d \times_{\mathscr{S}} \mathcal{E}$$

to be the universal family of tangent curves restricted to $\overline{\mathscr{C}}_d$. By definitiom of $\overline{\mathscr{C}}_d$, all the tangent curves $(\overline{\mathscr{E}}_d)_x$ for $x \in \overline{\mathscr{C}}_d$ are general double curves.

The above construction is presented in the following diagram.

$$\begin{tikzcd}
\mathbb{P}^3 \times \overline{\mathscr{C}_d} \supseteq \overline{\mathscr{E}}_d \arrow[rr, hook] \arrow[d] & & \mathscr{E} \arrow[d] \\
\overline{\mathscr{C}}_d \arrow[r, hook] & \mathscr{C}_d \arrow[r, hook] & \mathscr{S} \arrow[d] \\
& & |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}
\end{tikzcd}$$

Note that the morphism $\overline{\mathscr{E}}_d \to \overline{\mathscr{C}}_d$ is flat by Corollary 1.1.14 and by definition all the fibers have the same geometric genus.

Definition 3.3.1. Let E be a singular curve and $p\colon \tilde{E} \to E$ its normalization. Let

$$\left(p^{-1}\operatorname{Sing}(E)\right)^{\mathrm{red}} = \bigcup_{i=1}^{m} p_i,$$

where p_i are closed points of $\tilde{E}$. We say that E satisfies *singular divisorial relation* (*SDR*) of degree m if there exists $a_1, \ldots, a_m \in \mathbb{Z}$ such that

- $|a_i| \leq m$,
- $\sum_{i=1}^m a_i = 0$,
- $\mathcal{O}_{\tilde{E}}(a_1 p_1 + \ldots + a_m p_m)$ is trivial in $\operatorname{Pic}(\tilde{E})$.

Theorem 3.3.2. *Fix $m \in \mathbb{N}$. Then, for a general $x \in \overline{\mathscr{C}}_d$, the curve $(\overline{\mathscr{E}}_d)_x$ does not satisfy SDR of degree m.*

In other words, the tangent curve at a general point of a double-cover curve of a general smooth quartic does not satisfy SDR of degree m.

Proof. Define

$$V := \left\{x \in \overline{\mathscr{C}}_d \;\middle|\; (\overline{\mathscr{E}}_d)_x \text{ does not satisfy SDR of degree } m\right\}.$$

Claim 3.3.3. *V is nonempty.*

Proof. Take an arbitrary smooth elliptic curve E' and points P_1, P_2, Q_1, Q_2 such that $P_1 + P_2 \not\sim Q_1 + Q_2$ in $\operatorname{Pic}(E')$ and there is no divisorial relation of degree m between P_1, P_2, Q_1, Q_2.

By Theorem 3.1.2, there exists a plane curve E of degree four and a morphism $p\colon E' \to E$ which is the normalization of E with $P_1 \cup P_2 \cup Q_1 \cup Q_2$ being the inverse image under p of the singular points.

By Theorem 3.2.1, the curve E is an intersection of some smooth quartic $\mathscr{S}_f$ with a hyperplane. Since E is singular, this hyperplane must be tangent to $\mathscr{S}_f$, and so E is a tangent curve. The curve E has exactly two nodes, and thus E is a general double curve. Hence V is nonempty. $\square$

We need to prove that V is open and dense. Since $\overline{\mathscr{C}}_d$ is irreducible (see Lemma 3.3.4) and V is nonempty, it is sufficient to show that V is open.

We use a strong simultaneous resolution of singularities (see Lemma 1.1.11)

$$\begin{tikzcd} \widetilde{\mathscr{E}_d} \arrow[r, "\phi"] \arrow[d] & \overline{\mathscr{E}}_d \arrow[d] \\ \widetilde{\mathscr{C}_d} \arrow[r, "\psi"] & \overline{\mathscr{C}}_d \end{tikzcd}$$

where ψ is finite and a fiber $(\widetilde{\mathscr{E}_d})_x$ for $x \in \widetilde{\mathscr{C}_d}$ is the resolution of singularities of the curve $(\overline{\mathscr{E}}_d)_{\psi(x)}$. Let $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ be divisors on $\widetilde{\mathscr{E}_d}$ such that $(\mathcal{D}_i)|_{(\widetilde{\mathscr{E}_d})_x}$ for $i \in \{1,2,3,4\}$ are all the points in $\phi^* \operatorname{Sing}\left(\left(\overline{\mathscr{E}}_d\right)_{\psi(x)}\right)$.

Let $a = (a_1, a_2, a_3, a_4)$ be a quadruple of integers. Take a divisor $\mathcal{D}_{(a)} := a_1\mathcal{D}_1 + a_2\mathcal{D}_2 + a_3\mathcal{D}_3 + a_4\mathcal{D}_4 \in \operatorname{Div}(\widetilde{\mathscr{E}_d})$. Since $\widetilde{\mathscr{E}_d}$ is smooth, $\mathcal{D}_{(a)}$ is Cartier. As in Remark 1.3.4, the divisor $\mathcal{D}_{(a)}$ gives us a $\widetilde{\mathscr{C}_d}$-point of $\mathbf{Pic}_{\widetilde{\mathscr{E}_d}/\widetilde{\mathscr{C}_d}}$, that is a section

$$s_{\mathcal{D}_{(a)}} : \widetilde{\mathscr{C}_d} \longrightarrow \mathbf{Pic}_{\widetilde{\mathscr{E}_d}/\widetilde{\mathscr{C}_d}}\,.$$

Take $x \in \widetilde{\mathscr{C}_d}$ and a curve $E := (\widetilde{\mathscr{E}_d})_x$. Recall from Remark 1.3.4 that $s_{\mathcal{D}_{(a)}}(x) = 0$ if and only if $\mathcal{O}_E(\mathcal{D}_{(a)}|_E)$ is trivial in $\operatorname{Pic}(E)$. Thus, we have

$$V = \bigcap_{a \in A} \psi(U_{\mathcal{D}_{(a)}}),$$

where

$$U_{\mathcal{D}_{(a)}} := \left\{ x \in \widetilde{\mathscr{C}_d} \;\middle|\; s_{\mathcal{D}_{(a)}}(x) \neq 0 \right\} \subseteq \widetilde{\mathscr{C}_d}$$

and

$$A := \left\{ (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 \;\middle|\; \sum_{i=1}^{4} a_i = 0 \text{ and } |a_i| \leq m \text{ for } i \in \{1,2,3,4\} \right\}.$$

Since sets $U_{\mathcal{D}_{(a)}}$ are open, we get that V is open as well. □

Lemma 3.3.4. *The variety $\overline{\mathscr{C}}_d$ is irreducible.*

Proof. Since

$$\begin{tikzcd} \overline{\mathscr{C}}_d \arrow[d, "p"] \\ |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \end{tikzcd}$$

is surjective and $|\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}$ is irreducible, there exists an irreducible component $W \subseteq \overline{\mathscr{C}}_d$, which is mapped by p surjectively onto $|\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}$.

From Proposition 2.5.6, we know that there exists an open subset $U \subseteq |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}$, such that double-cover curves $\left(\overline{\mathscr{C}}_d\right)_u$ are irreducible for all $u \in U$. In particular $p^{-1}(U) \subseteq W$, and so any other potential irreducible component $W' \subseteq \overline{\mathscr{C}}_d$ cannot be mapped by p surjectively onto $|\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}}$. Hence, in order to prove the irreducibility of $\overline{\mathscr{C}}_d$, it is enough to show that for every $x \in \overline{\mathscr{C}}_d$ there exists an irreducible closed subset of $\overline{\mathscr{C}}_d$ of dimension $\dim |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} + 1$ containing x.

We use, here, the notation of Subsection 1.1.3. Take $x \in \overline{\mathscr{C}}_d \subseteq \mathscr{S}$ and consider an open subset $U_{\mathscr{S}} \subseteq \mathscr{S}$ containing x. For sufficiently small $U_{\mathscr{S}}$, we have a morphism

$$\begin{aligned} \phi \colon U_{\mathscr{S}} &\longrightarrow U^{4,2} \\ u &\longmapsto \mathscr{E}_u, \end{aligned}$$

which sends a point $u \in U_{\mathscr{S}}$ to its tangent curve. We treat all the curves $\mathscr{E}_u$ as plane curves in $T_p(\mathscr{S}_f)$ by taking projections, where

$$x = (f, p) \in \mathscr{S} \subseteq |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} \times \mathbb{P}^3.$$

Since $\dim(U_{\mathscr{S}}) = \dim |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} + 2$ and $U^{4,1}$ has codimension one in $U^{4,2}$ (see Lemma 1.1.16), there exists an irreducible closed subset $T \subseteq U_{\mathscr{S}}$ satisfying

- $x \in T$,
- $\dim(T) = \dim |\mathcal{O}_{\mathbb{P}^3}(4)|^{\mathrm{sm}} + 1$,
- $\phi(t) \in U^{4,1}$ for all $t \in T$, in other words $g(\mathscr{E}_t) \leq 1$.

In particular, $T \subseteq \mathscr{C}_d$ and the closure of T in $\overline{\mathscr{C}}_d$ is the subset we were looking for. $\square$

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The first picture comes from `http://en.wikipedia.org/wiki/File:Kummer_surface.png`