Effective bounds in positive characteristic and other applications of Frobenius techniques

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Contents

CHAPTER 1

Introduction

The goal of this article is to present various applications of Frobenius-splitting techniques in birational geometry. We review some fundamental theorems and show new results obtained by the author.

The modern approach to positive characteristic birational geometry would not have been developed, if not for the breakthrough results of Hochster and Huneke. Their "tight closure theory" revolutionised geometric aspects of commutative algebra, enabling to prove various difficult facts in a surprisingly easy way. Although the theory captivated the mathematical community at that time, the climax of its importance came only many years later.

One of the most striking discoveries in algebraic geometry in the last twenty years, was that the notions of singularities arising from the tight closure theory posses unbelievable similarities to those coming from birational geometry and the minimal model program. The work aiming to comprehend this interplay has been undertaken by Smith, Hara, Watanabe, Schwede, and many others, and it led to profound breakthroughs in both of those areas of mathematics. One of the most fundamental off-springs of the theory is the development of the minimal model program for Kawamata log terminal threefolds in characteristic $p > 5$ (see [22]).

The main difficulty in dealing with birational geometry in positive characteristic is the lack of Kawamata-Viehweg vanishing theorem. One often applies this theorem to show that a certain restriction map on global sections of a line bundle is surjective. In many cases, one can show the surjectivity by proving that the corresponding first cohomology group in the long exact sequence vanish. Frobenius techniques allow us to show the surjectivity of restriction maps without showing the vanishing of cohomologies.

There are two approaches to this problem. One, called the evaporation technique, is based on the idea, that although the cohomology in question might not vanish, it may be torsion under the action of the Frobenius map. The second, more prolific technique, is based on the concept of a trace map. Mixing those two approaches, we are able to prove the following proposition as a corollary to [3].

Proposition 1.0.1. *Let X be a smooth projective threefold defined over $\overline{\mathbb{F}}_p$. Let A be an ample line bundle on X such that $K_X + A$ is nef and big. Further, suppose that there exists a smooth irreducible divisor $S \in |K_X + A|$. Then $K_X + A$ is semiample.*

This proposition is a weak version of the base point free theorem. Although our assumptions are rarely achievable, we believe that blends of similar approaches might shed a new light on positive characteristic birational geometry.

One of the questions that motivated the study of Frobenius techniques in birational geometry was the following conjecture.

Conjecture 1.0.2 (Fujita conjecture)**.** *Let X be a smooth projective variety of dimension n, and let A be an ample Cartier divisor on X. Then $K_X + (n+1)A$ is very ample.*

Fujita-type results play a vital role in understanding geometry of algebraic varieties, an example being the classification of Fano varieties. Even in characteristic zero, the conjecture is known only for $n \leqslant 4$.

If we assume that A is also base point free, the conjecture becomes easy in characteristic zero, but, in positive characteristic, it was widely open for many years. The development of the Frobenius trace map led to the solution of this problem (see [13, Theorem 1.1], c.f. Smith ([14] and [15]) and Hara ([17], [16])).

We obtain a new result in this direction, allowing singularities, while on the other hand demanding more positivity than usual.

Theorem 1.0.3. *Let X be an F-pure normal $\mathbb{Q}$-Gorenstein projective variety of dimension n and let H be an ample base point free Cartier divisor. If $H - K_X$ is ample, then $(n+2)H + N$ is very ample for any nef Cartier divisor N.*

Despite the significant progress in the research about Frobenius singularities and Frobenius techniques, there are very few important applications of them in the study of birational geometry of algebraic varieties in arbitrary dimension. The main results of this article concern algebraic surfaces.

The motivation for many results of this article centers around Fujita conjecture and the following question: given an ample Cartier divisor A, find $n \in \mathbb{N}$ for which nA is very ample. A famous theorem of Matsusaka states that one can find n which depends only on the Hilbert polynomial of A, when the variety is smooth and the characteristic of the field if equal to zero ([9]). This theorem plays a fundamental role in constructing moduli spaces. In positive characteristic, Kollar proved the same statement for normal surfaces, but without providing explicit bounds ([1, Theorem 2.1.2]).

In positive characteristic, Fujita conjecture is known only for curves, and those surfaces which are neither of general type, nor quasi-elliptic. This follows from a result of Shepherd-Barron which says that on such surfaces, rank two vector bundles which do not satisfy Bogomolov inequality are unstable ([10, Theorem 7]). Indeed, the celebrated proof by Reider of Fujita conjecture for characteristic zero surfaces, can be, in such a case, applied without any modifications (see [12], [11]).

Given lack of any progress for positive characteristic surfaces of general type, Di Cerbo and Fanelli, undertook a different approach to the problem. They proved, among other things, that $2K_X + 4A$ is very ample, where A is ample, and X is a smooth surface of general type in characteristic $p \geqslant 3$. Then, they used it to obtain Mastusaka-type bounds.

One of the aims of this article is to generalize results of Fanelli and Di Cerbo ([8, Theorem 1.2] and [8, Theorem 1.4]) to singular surfaces. As far as we now, no effective bounds for singular surfaces in positive characteristic have been obtained before.

Theorem 1.0.4. *Let X be a projective surface with log terminal singularities defined over an algebraically closed field of characteristic $p > 5$. Assume that mK_X is Cartier for some $m \in \mathbb{N}$. Then, for an ample Cartier divisor A, we have that*

- $4mK_X + 14mA$ *is base point free, and*
- $16mK_X + 56mA$ *is very ample.*

The bounds are not sharp. See Theorem 3.3.1 for a slightly more general statement.

The proof consists of three main ingredients. First, we apply the result of Di Cerbo and Fanelli on a desingularization of X. This shows that the base locus of $2mK_X + 4mA$ is zero dimensional. Then, we apply a stupendous technique of Cascini, Tanaka and Xu (see [4, Theorem 3.8]), to construct certain "nearly-F-pure" centers, and use them to show that the base locus is empty. The last part follows by Theorem 1.0.3.

As far as we know, after the paper of Cascini, Tanaka and Xu ([4]) has been anounced, no one has yet applied their technique. We believe that down-to-earth examples provided in our paper may work as a gentle introduction to their prolific paper [4].

As a corollary to the main theorem, we obtain the following Matsusaka-type bounds.

Corollary 1.0.5. *Let A and N be, respectively, an ample and a nef Cartier divisor on a log terminal projective surface defined over an algebraically closed field of characteristic $p > 5$. Let $m \in \mathbb{N}$ be such that mK_X is Cartier. Then $kA - N$ is very ample for any*

$$k > \frac{2A \cdot (H + N)}{A^2}((K_X + 2A) \cdot A + 1),$$

where $H := 16mK_X + 56mA$.

After constructing a very ample line bundle, a natural question to ask is whether its higher cohomologies vanish. In characteristic zero, it often follows by Kodaira vanishing. In this article, we present an unpublished result of Hiromu Tanaka which deals with this issue in positive characteristic. It is used in the part of our article which concerns bounds on log del Pezzo surfaces. We are grateful to Hiromu Tanaka for allowing us to use his theorem in our article.

Theorem 1.0.6. *Let H and A be, respectively, a very ample and an ample Cartier divisor on a projective surface X. Assume that mK_X is Cartier for $m \in \mathbb{N}$, and $A + (m-1)K_X$ is nef. Then*

$$H^i(X, \overline{H}) = 0$$

for $i > 0$, where

- $\overline{H} := mK_X + H + A$, *if* $m \neq 1$
- $\overline{H} := 2K_X + H + A$, *if* $m = 1$.

In particular, $H^1(X, 4K_X + 21A) = 0$ for an ample Cartier divisor A on a smooth projective surface X, since $2K_X + 20A$ is always very ample, by [8].

The same way as Frobenius singularities correspond to singularities from the minimal model program, varieties that admit a global splitting of Frobenius, correspond to log Calabi-Yau and log Fano varities. For example, Schwede and Smith ([31]) proved that any globally F-regular variety in positive characteristic is a klt log Fano, and big enough reduction modulo p of a klt log Fano variety (X, Δ) in characteristic zero is globally F-regular.

We give a new proof of the latter statement, when X is smooth and Δ is a simple normal crossing. In particular, it gives a new proof of the following proposition.

Proposition 1.0.7. *Let X be a singular del Pezzo surface defined over an algebraically closed field of characteristic 0. Then, a reduction of X modulo $p \gg 0$ is globally F-regular.*

One of the fundamental conjectures in birational geometry is Borisov-Alexeev Boundedness conjecture, which says that ϵ-klt log Fano varieties are bounded. It is believed to be a crucial component that could lead to a proof of the ACC conjecture for minimal log discrepancy, and, henceforth, the termination of flips.

For surfaces, the theorem has been proved by Alexeev ([2, Theorem 6.9]) using Nikulin's diagram method, the "sandwich" argument and the beforementioned result about boundedness of polarized surfaces by Kollar ([1, Theorem 2.1.2]). The obtained bounds were highly inexplicit.

A different approach has been undertaken by Lai, who showed explicit bounds for the volume of an ϵ-klt log del Pezzo pair ([7, Theorem 4.3]). Instead of using the "sandwich argument" and Nikulin's diagram method, he applied a covering family of tigers.

For characteristic $p > 5$, in the proof of the boundedness of ϵ-klt log del Pezzo pairs, we can replace Kollar's result by Theorem 1.0.4, and hence obtain rough, but explicit bounds on the size of the bounded family.

Theorem 1.0.8. *Let (X, Δ) be a klt log del Pezzo surface of a $\mathbb{Q}$-factorial index m. Then, there exists a very ample divisor H on X such that $H^i(X, H) = 0$ for $i > 0$ and*

$$\begin{aligned} H^2 &\leqslant 128m^5(2m-1)a^2 + \max(64, 8m+4)^2 b^2, \\ H \cdot K_X &\leqslant 128m^5(2m-1)a, \\ H \cdot \Delta &\leqslant 3m \max(64, 8m+4)(a+b) + 128m^5(2m-1)a, \end{aligned}$$

where $a = 17m$ and $b = 59m$. In particular, X embeds into $\mathbb{P}^k$, where

$$k \leqslant 128m^5(2m-1)a^2 + \max(64, 8m+4)^2 b^2.$$

Note that ϵ-klt log del Pezzo surfaces have $\mathbb{Q}$-factorial index at each point bounded by $2(2/\epsilon)^{128/\epsilon^5}$ (see (d) in Proposition 4.2.1).

We apply this theorem to prove the existence of a bounded family of ϵ-klt log del Pezzo surfaces over $\operatorname{Spec} \mathbb{Z}$ (taken from a soon-to-be-published paper of Cascini, Tanaka, Witaszek). In particular,

Theorem 1.0.9 (from the soon-to-be-published paper of Cascini, Tanaka, Witaszek)**.** *Let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Take $\epsilon > 0$. Then there exists $p(I, \epsilon)$ which satisfies the following property:*

If (X, B) is an ϵ-klt log del Pezo surface defined over an algebraically closed field of characteristic $p > p(I, \epsilon)$ and such that the coefficients of B are contained in I, then (X, B) is globally F-regular.

In the soon-to-be-published paper of Cascini, Tanaka, Witaszek, a constant p is found which depends only on I.

The existence of the $\mathbb{Z}$-bounded family, shows that for big enough characteristic singular del Pezzo surfaces X of Gorenstein index m lift to characteristic zero. In particular, $H^1(X, L) = 0$ for all nef line bundles L. The following proposition gives an effective bound on characteristic, starting from which this property holds.

Theorem 1.0.10. *Let X be a singular del Pezzo surface of Gorenstein index m defined over an algebraically closed field of characteristic $p > 0$. If $p > 2m^2$, then $H^1(X, L) = 0$ for any nef line bundle L on X.*

The paper is organized as follows. In the first chapter, we introduce main notions in the theory of Frobenius splittings, and discuss results which concerns varieties of higher dimension. In the second chapter, we derive effective bounds for base point freeness, very ampleness and vanishing of cohomologies, in the case of surfaces. In the third chapter, we discuss global F-regularity, also focusing on the case of surfaces.

As far as we are concerned, the best source of knowledge about Frobenius singularities are unpublished notes of Karl Schwede [5]. We also recommend [6].

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Notation

If not mentioned otherwise, all varieties are defined over an algebraically closed field k of positive characteristic $p > 0$.

We refer to [21] for basic definitions in birational geometry like log discrepancy or log terminal singularities. We say that a pair (X, Δ) is a *log Fano* pair if $-(K_X + \Delta)$ is ample. In the case when $\dim(X) = 2$, we say that (X, Δ) is a *log del Pezzo* pair. If $-(K_X + \Delta)$ is big and nef, then we call (X, Δ) a *weak log Fano*, or, when $\dim(X) = 2$, a *weak log del Pezzo*.

A pair (X, B) is *ϵ-klt*, if the log discrepancy along any divisor is greater than ϵ. Note that the notion of being 0-klt is equivalent to klt.

We denote the base locus of a line bundle $\mathcal{L}$ by $\mathbb{B}(\mathcal{L})$. Note, that by abuse of notation, we use the notation for line bundles and the notation for divisors interchangeably.

The following facts are used in the course of the proofs in this paper.

Theorem 1.0.11 ([19, Theorem 1.8.5])**.** *Let X be a projective variety, and let M be a globally generated ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf on X such that $H^i(X, \mathcal{F} \otimes M^{-i}) = 0$ for $i > 0$. Then $\mathcal{F}$ is globally generated.*

Theorem 1.0.12 ([19, Theorem 1.4.35] and [19, Remark 1.4.36])**.** *Let X be a projective variety, and let H be an ample divisor on X. Given any coherent sheaf $\mathcal{F}$ on X, there exists an integer $m(\mathcal{F}, H)$ such that*

$$H^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)\big) = 0,$$

for all $i > 0$, $m \geqslant m(\mathcal{F}, H)$ and any nef divisor D on X.

Theorem 1.0.13 (Log-concavity of volume)**.** *For two big Cartier divisors D_1 and D_2 on a normal variety X of dimension n we have that*

$$\mathrm{vol}(D_1 + D_2)^{\frac{1}{n}} \geqslant \mathrm{vol}(D_1)^{\frac{1}{n}} + \mathrm{vol}(D_2)^{\frac{1}{n}}.$$

Recall that

$$\mathrm{vol}(D) := \limsup_{m \to \infty} \frac{H^0(X, mD)}{m^n/n!}.$$

Proof. See [8, Theorem 2.2] (cf. [20, Theorem 11.4.9] and [24]). □

CHAPTER 2

Frobenius singularities and the trace map

All the rings in this sections are assumed to be geometric and of positivie characteristic, that is finitely generated over an algebraically closed field of characteristic $p > 0$.

2.1 Local and global Frobenius splitting

This subsection is partially taken from author's minor first-year-of-PhD project "Different viewpoints on multiplier ideal sheaves and singularities of theta divisors". It is based on [5] and [6].

One of the most amazing discoveries of singularity theory, is that properties of the Frobenius map may reflect how singular a variety is. This observation is based on the fact that for a smooth local ring R, the e-times iterated Frobenius map $R \to F^e_* R$ splits. Further, the splitting does not need to hold when R is singular.

This lead to a definition of *F-split rings*, rings R such that for divisible enough $e \gg 0$ the e-times iterated Frobenius map $F^e \colon R \to F^e_* R$ splits. For log pairs, we have the following definition.

Definition 2.1.1. We say that a log pair (X, Δ) is *globally F-split* if for enough divisible $e \gg 0$ the e-times iterated Frobenius map $\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$ splis.

The problem with this property is that, in general, it does not behave well. For example, it is neither an open nor a close condition on fibers of a family.

Note, that $F^e \colon R \to F^e_* R$ splits if and only if there exists a map $\phi \colon F^e_* R \to R$ such that $1 \in \phi(F^e_* R)$. This suggests to consider rings which are F-split "under all small perturbations". More formally we say that R is *F-regular* if for every $c \in R \setminus 0$, there exists $e > 0$ a map $\phi\colon F^e_* R \to R$ such that $1 \in \phi(F^e_* cR)$.

For log pairs, we have the following definition.

Definition 2.1.2. We say that a log pair (X, Δ) is *globally F-regular* if for every principal divisor $D \subseteq X$ there exists $e > 0$ and a map

$$\phi \in \mathrm{Hom}_X(F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta + D \rceil), \mathcal{O}_X)$$

such that $1 \in \phi(F^e_* \mathcal{O}_X)$.

This definition may seem a bit mysterious in the first glance. Let us untangle it. Firstly, Grothendieck duality gives

$$\mathrm{Hom}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \simeq H^0(X, \omega_X^{1-p^e}).$$

This explains the following crucial proposition.

Proposition 2.1.3 ([33, Theorem 3.11, 3.13])**.** *There is a natural bijecion.*

$$\left\{ \begin{array}{c} \textit{Non-zero } \mathcal{O}_X\textit{-linear maps} \\ \phi\colon F^e_* \mathcal{O}_X \to \mathcal{O}_X \textit{ up to} \\ \textit{pre-multiplication by units.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Effective } \mathbb{Q}\textit{-divisors } \Delta \textit{ such that} \\ (1 - p^e)\Delta \sim -(1 - p^e)K_X \end{array} \right\}$$

The $\mathbb{Q}$-divisor corresponding to $\phi\colon F^e_* \mathcal{O}_X \to \mathcal{O}_X$ will be denoted by Δ_ϕ. The morphism extends to $\phi\colon F^e_* \mathcal{O}_X((p^e - 1)\Delta_\phi) \to \mathcal{O}_X$.

Other way round, given a $\mathbb{Q}$-divisor Δ such that $(p^e - 1)(K_X + \Delta) \sim 0$, the corresponding morphism will be denoted by $\mathrm{Tr}_{X,\Delta}\colon F^e_* \mathcal{O}_X \to \mathcal{O}_X$. As before, it extends to

$$\mathrm{Tr}_{X,\Delta}\colon F^e_* \mathcal{O}_X((p^e - 1)\Delta_\phi) \to \mathcal{O}_X.$$

A general definition of the trace map $\mathrm{Tr}_{X,\Delta}$ maybe be found in Section 2.2.

Lemma 2.1.4 ([31, Lemma 3.5])**.** *If (X, Δ) is globally F-regular (F-split), then $(X, \overline{\Delta})$ is globally F-regular (F-split, respectively) for any $\overline{\Delta} \leqslant \Delta$.*

Now, it is easy to see, that a log pair (X, Δ) is globally F-regular if and only if for every principal divisor $D \subseteq X$ there exists a splitting $\phi\colon F^e_* \mathcal{O}_X \to \mathcal{O}_X$ such that $\Delta_\phi \geqslant \Delta + \frac{1}{p^e - 1} D$.

We also consider local versions of these notions.

Definition 2.1.5. A log pair (X, Δ) is *strongly F-regular* (*F-pure*), if all of its local rings are globally F-regular (F-split, respectively).

Considering X to be a Spec of a local ring, all the above remarks hold true in the local setting. The local and global notions coincide for affine varieties.

Frobenius singularities are alleged to be the correct counterparts of birational singularities in positive characteristic (see Chapter 3 for a more detailed explanation). This supposition is propped up by the following theorems.

Theorem 2.1.6 (see [27])**.** *Let X be a normal variety. If (X, B) is strongly F-regular (F-pure), then (X, B) is kawamata log terminal (log canonical, respecitvely).*

Theorem 2.1.7 ([25])**.** *Let X be a log terminal projective surface defined over an algebraically closed field of characteristic $p > 5$. Then X is strongly F-regular.*

We sketch the proof of the erstwhile theorem. First, we need to understand how our theory behaves under taking resolutions of singularities. The following proposition should be clear given the discussion above.

Proposition 2.1.8 ([32, Proof of Theorem 6.7], [6, Exercise 4.17])**.** *Suppose that $\pi\colon \widetilde{X} \to X$ is a proper birational map of varieties, where X is normal, and take $\phi \in \mathrm{Hom}_X(F^e_*\mathcal{O}_X, \mathcal{O}_X)$. Let Δ_ϕ be as in Proposition 2.1.3 and let $\widetilde{\Delta}_\phi$ be such that*

$$K_{\widetilde{X}} + \widetilde{\Delta}_\phi = \pi^*(K_X + \Delta_\phi).$$

Then $\phi\colon F^e_*\mathcal{O}_X((p^e-1)\Delta_\phi) \to \mathcal{O}_X$ induces a map

$$\widetilde{\phi}\colon F^e_*\mathcal{O}_{\widetilde{X}}((p^e-1)\widetilde{\Delta}_\phi) \longrightarrow \mathcal{O}_{\widetilde{X}}$$

which agrees with ϕ, where π is an isomorphism.

Sketch of a proof of Theorem 2.1.6. We assume that (X, B) is F-pure, and show that it is log canonical. The strongly F-regular part is analogous.

We can assume that X is a Spec of a local ring. Take an enough divisible $e \gg 0$ and let $\phi\colon F^e_*\mathcal{O}_X((p^e-1)\Delta_\phi, \mathcal{O}_X)$ be an F-splitting of (X, B) (see Proposition 2.1.3). Recall that $\Delta_\phi \supseteq B$.

Let $\pi\colon \widetilde{X} \to X$ be a log resolution of singularities. By contradiction, assume that (X, B) is not log canonical. Then, for $e \gg 0$, there exists an integral divisor S such that

$$(p^e-1)\widetilde{\Delta}_\phi = nS + R,$$

where $n \geqslant p^e$, and $S \nsubseteq \mathrm{Supp}\, R$.

By Proposition 2.1.8, there exists a surjection

$$\widetilde{\phi}\colon F^e_*\mathcal{O}_{\widetilde{X}}((p^e-1)\widetilde{\Delta}_\phi) \longrightarrow \mathcal{O}_{\widetilde{X}}.$$

Replacing $\widetilde{X}$ by its localisation at the generic point of S, we get a morphism

$$\widetilde{\phi}\colon F^e_*\mathcal{O}_{\widetilde{X}}(nS) \longrightarrow \mathcal{O}_{\widetilde{X}}.$$

In particular, since $n \geqslant p^e$, we get $\widetilde{\phi}(F^e_* \mathcal{O}_{\widetilde{X}}) \subseteq \mathcal{O}_X(-S)$, and so

$$\phi(F^e_* \mathcal{O}_X) \subseteq m_q,$$

where m_q is the maximal ideal of a point $q \in \pi(S)$. This yields a contradiction, because ϕ was assumed to be an F-splitting. □

Although, taking localisation at the generic point of S is not necessary, it makes the statement clearer given that $\widetilde{\Delta}_\phi$ does not need to be effective.

2.2 Trace map

A key tool in the theory of Frobenius splittings is a trace map (see [3] and [16]). For an integral divisor D on a normal variety X, there is an isomorphism derived from the Grothendieck duality (see [22, Lemma 2.9]):

$$\mathcal{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(D), \mathcal{O}_X) \simeq \mathcal{O}_X(-(p^e-1)K_X - D). \tag{2.1}$$

Definition 2.2.1. Let B be a $\mathbb{Q}$-divisor such that $(p^e-1)(K_X+B)$ is Cartier. We call

$$\mathrm{Tr}^e_{X,B} \colon F^e_* \mathcal{O}_X(-(p^e-1)(K_X+B)) \to \mathcal{O}_X,$$

the *trace map*. It is constructed by applying the above isomorphism (2.1) to the map

$$\mathcal{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e-1)B), \mathcal{O}_X) \xrightarrow{\mathrm{ev}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \tag{2.2}$$

being the dual of the composition $\mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X((p^e-1)B)$.

The rank one sheaves in question are not necessary line bundles, but since X is normal, we can always restrict ourselves to the smooth locus. If $B = 0$, then we denote the trace map by $\mathrm{Tr}_X \colon F^e_* \mathcal{O}_X(-(p^e-1)K_X) \to \mathcal{O}_X$.

The following proposition reveals the significance of the trace map and unravels its a bit complicated definition.

Proposition 2.2.2 ([22, Proposition 2.10])**.** *Let (X, B) be a normal log pair. Then (X, B) if F-pure at a point $x \in X$ if and only if the trace map $\mathrm{Tr}^e_{X,B}$ is surjective at x for all enough divisible $e \gg 0$.*

Proof. The key point is that $\mathrm{Tr}^e_{X,B}$ is induced by the evaluation map (2.2). Replace (X, B) by the local ring at x. For $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e-1)B), \mathcal{O}_X)$, the image $\mathrm{ev}(\phi)$ is defined by the commutativity of the following diagram:

$$\mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X((p^e-1)B) \xrightarrow{\phi} \mathcal{O}_X, \qquad \mathrm{ev}(\phi)\colon \mathcal{O}_X \to \mathcal{O}_X.$$

Note that $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X$ is generated by the identity morphism id. In particular, ev is surjective if and only if there exists ϕ such that $\mathrm{ev}(\phi) = \mathrm{id}$, which is equivalent to ϕ being a splitting. □

Remark 2.2.3. As announced in Proposition 2.1.3, if Δ is a $\mathbb{Q}$-divisor such that $(p^e-1)(K_X+\Delta) \sim 0$, then

$$\mathrm{Tr}_{X,\Delta}\colon F^e_*\mathcal{O}_X \to \mathcal{O}_X$$

is the morphism corresponding to Δ. We sketch a proof below.

Consider the following commutative diagram.

$$\begin{array}{ccccc}
(p^e-1)\Delta & \longmapsto & & \longrightarrow & \mathrm{Tr}^e_{X,\Delta} \\
\rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\
H^0\big(X, F^e_*\omega_X^{1-p^e}\big) & \xrightarrow{\simeq} & \mathrm{Hom}\big(F^e_*\mathcal{O}_X, \mathcal{O}_X\big) & \xrightarrow{\simeq} & \mathrm{Hom}\big(F^e_*\mathcal{O}_X\big((1-p^e)(K_X+\Delta)\big), \mathcal{O}_X\big) \\
\Big\downarrow = & & \Big\downarrow \simeq & & \Big\downarrow \simeq \\
H^0\big(X, F^e_*\omega_X^{1-p^e}\big) & \xrightarrow{\simeq} & \mathrm{Hom}\big(\mathcal{H}om\big(F^e_*\omega_X^{1-p^e}, \mathcal{O}_X\big), \mathcal{O}_X\big) & \xrightarrow{\simeq} & \mathrm{Hom}\big(\mathcal{H}om\big(F^e_*\mathcal{O}_X\big((p^e-1)\Delta\big), \mathcal{O}_X\big), \mathcal{O}_X\big) \\
\rotatebox{90}{$\ni$} & & & & \rotatebox{90}{$\ni$} \\
(p^e-1)\Delta & \longmapsto & & \longrightarrow & \mathrm{ev}
\end{array}$$

The upper left horizontal arrow is constructed via Grothendieck duality. The lower left horizontal arrow follows from a natural isomorphism between a vector bundle and its double dual. The right horizontal arrows are constructed via the isomorphism

$$\mathcal{O}_X \simeq \mathcal{O}_X((1-p^e)(K_X+\Delta)),$$

and the two vertical isomorphism follows from Grothendieck duality. In particular, the diagram is commutative.

Our goal is to show that $(p^e-1)\Delta$ is sent to $\mathrm{Tr}^e_{X,\Delta}$ via those isomorphisms. By the construction of $\mathrm{Tr}^e_{X,\Delta}$, this is equivalent to showing that $(p^e-1)\Delta$ is sent to the evaluation map ev via the lower horizontal isomorphisms.

The last supposition follows by the following fact. Let $\mathcal{E}$ be a vector bundle and $s \in H^0(X, \mathcal{E}) \simeq \mathrm{Hom}(\mathcal{O}_X, \mathcal{E})$ a section of it. This section induces some vector bundle $\mathcal{E}_s$ and an isomorphism $\hat{s}\colon \mathcal{E}_s \simeq \mathcal{E}$. Then, via the following sequence of isomorphisms:

$$\begin{array}{ccccc}
H^0(X, \mathcal{E}) & \longrightarrow & \mathrm{Hom}(\mathcal{H}om(\mathcal{E}, \mathcal{O}_X), \mathcal{O}_X) & \xrightarrow{(\hat{s}^\wedge)^\wedge} & \mathrm{Hom}(\mathcal{H}om(\mathcal{E}_s, \mathcal{O}_X), \mathcal{O}_X), \\
\rotatebox{90}{$\ni$} & & & & \rotatebox{90}{$\ni$} \\
s & \longmapsto & & \longrightarrow & \mathrm{ev}
\end{array}$$

the section s is sent to the evaluation map $\mathrm{ev} \in \mathrm{Hom}(\mathcal{H}om(\mathcal{E}_s, \mathcal{O}_X), \mathcal{O}_X)$, which takes $\phi \in \mathcal{H}om(\mathcal{E}_s, \mathcal{O}_X)(U)$ on some open set U, composes it with $s \in \mathrm{Hom}(\mathcal{O}_X, \mathcal{E}_s)$, and evaluates it at $1 \in \mathcal{O}_X$.

$$\underbrace{\mathcal{O}_X \xrightarrow{\ s\ } \mathcal{E}_s \xrightarrow{\ \phi\ } \mathcal{O}_X}_{\mathrm{ev}(\phi)\cdot\mathrm{id}}.$$

We leave it to the reader to verify this fact.

Further, we consider another version of the trace map. Let D be a $\mathbb{Q}$-divisor such that $K_X + H + D$ is Cartier. Tensoring the trace map $\mathrm{Tr}^e_{X,B}$ by it, we obtain:

$$\mathrm{Tr}^e_{X,B}(D)\colon F^e_*\mathcal{O}_X(K_X + B + p^e D) \to \mathcal{O}_X(K_X + B + D).$$

By abuse of notation, both versions of the trace map are denoted in the same way.

2.2.1 Examples

All the proofs and examples in this subsection come from [3].

Let X be a smooth algebraic variety and D be an effective divisor. The following proposition exemplifies the importance of the trace maps.

Proposition 2.2.4. *If the action of the trace map on the i-th cohomology of X*

$$\mathrm{Tr}^e_X\colon H^i(X, \mathcal{O}_X(K_X + p^e D)) \to H^i(X, \mathcal{O}_X(K_X + D))$$

is surjective for $e \gg 0$, and D is ample, then the Kodaira vanishing holds, that is

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0.$$

Proof. By Serre vanishing we have $H^i(X, \omega_X(p^e D)) = 0$, and so $H^i(X, \omega_X(D)) = 0$ by the surjectivity of the trace map. □

We provide two examples of varieties, which always satisfy the assumptions of the above proposition: globally F-split varieties and abelian varieties.

Lemma 2.2.5 ([3, Proposition 2.11]). *If X is globally F-split, then the action of the trace map on cohomologies*

$$\mathrm{Tr}^e_X\colon H^i(X, \mathcal{O}_X(K_X + p^e D)) \to H^i(X, \mathcal{O}_X(K_X + D))$$

is surjective for any $i \geqslant 0$.

Proof. Let $\phi\colon F^e_*\mathcal{O}_X \to \mathcal{O}_X$ be a splitting of Frobenius. By applying $\mathcal{H}\mathrm{om}(-,\omega_X)$ and Grothendieck duality, we obtain a map $\psi\colon \omega_X \to F^e_*\omega_X$.

The composition

$$H^i(X,\omega_X(D)) \xrightarrow{\psi} H^i(X,\omega_X(p^eD)) \xrightarrow{\mathrm{Tr}^e_X} H^i(X,\omega_X(D))$$

is an identity, which concludes the proof of the lemma. □

Lemma 2.2.6 ([3, Proposition 2.9])**.** *If X is an abelian variety and D is ample, then the action of the trace map on cohomologies*

$$\mathrm{Tr}^e_X\colon H^i(X,\mathcal{O}_X(K_X+p^eD)) \to H^i(X,\mathcal{O}_X(K_X+D))$$

is surjective for any $i \geqslant 0$.

We follow a proof from [3].

Proof. Consider the morphism $n_X\colon X \to X$, which is the multiplication by an integer n. Assume that $p \nmid n$. We have the following diagram.

$$\begin{array}{ccc} H^i(X,F^e_*\omega_X(p^en^*_XD)) & \longrightarrow & H^i(X,F^e_*\omega_X(p^eD)) \\ \downarrow{\scriptstyle \mathrm{Tr}^e_X} & & \downarrow{\scriptstyle \mathrm{Tr}^e_X} \\ H^i(X,\omega_X(n^*_XD)) & \longrightarrow & H^i(X,\omega_X(D)) \end{array}$$

To obtain the horizontal arrows, we applied the functor $\mathcal{H}om(\cdot,\omega_X)$ to the natural morphism $\mathcal{O}_X(-D) \to (n_X)_*\mathcal{O}_X(-n^*_XD)$ and used Grothendieck duality.

It is a well known fact that the $\mathcal{O}_X$-morphism $\mathcal{O}_X \to (n_X)_*\mathcal{O}_X$ splits. Hence, the horizontal arrows in the diagram are surjective. Thus, to conclude the proof of the lemma, it is enough to show that

$$H^i(X,\omega_X(p^en^*_XD)) \xrightarrow{\mathrm{Tr}^e_X} H^i(X,\omega_X(n^*_XD))$$

is surjective. Set

$$B := \ker\left(\mathrm{Tr}^e_X\colon F^e_*\omega_X \to \omega_X\right).$$

By the long exact sequence of cohomologies, it is enough to show that

$$H^{i+1}(X,B(n^*_XD)) = 0.$$

No, we apply the following formula, well known in the theory of abelian varieties ([34]):

$$n^*_XD = \frac{n^2+n}{2}D + \frac{n^2-n}{2}(-1)^*_XD.$$

The line bundle $(-1)^*_XD$ is ample, and so we can conclude the proof by Theorem 1.0.12. □

We get a simple proof of the following fact.

Corollary 2.2.7. *Let L be an ample Cartier divisor on an abelian variety A. Then $H^0(A, L) \neq 0$.*

Proof. By the Kodaira vanishing for abelian varieties, $H^i(A, L) = 0$ for $i > 0$. Hence $H^0(A, L) = \chi(L) = L^{\dim A} \neq 0$ (see [34]). □

We finish this section, by providing a sketch of a simple example, how one can apply the trace map to show surjectivity of a restriction map.

Proposition 2.2.8 ([3, Corollary 4.3])**.** *Let C be a smooth curve on a smooth surface X. Fix an ample divisor A on X. If $H^0(C, K_C + A|_C) \neq 0$, then*

$$H^0(X, K_X + C + A) \to H^0(C, K_C + A|_C)$$

is a nonzero map.

Sketch. Consider the following diagram for $e \gg 0$ (see Lemma 2.2.9)

$$\begin{array}{ccccc} H^0(X, K_X + p^e(C + A)) & \xrightarrow{\phi} & H^0(C, K_C + p^e A|_C) & \longrightarrow & H^1(X, K_X + p^e(C + A) - C) \\ \downarrow{\scriptstyle \mathrm{Tr}^e_X} & & \downarrow{\scriptstyle \mathrm{Tr}^e_C} & & \\ H^0(X, K_X + C + A) & \xrightarrow{\psi} & H^0(C, K_C + A|_C). & & \end{array}$$

Since ϕ is surjective by Serre vanishing, it is enough to show that Tr^e_C is nonzero. This is proved in [3, Corollary 4.2]. □

In the above proposition we used the following lemma. We state it in a higher generality, as we will need this version in a later part of this article.

Lemma 2.2.9 (c.f. [3, Lemma 2.6])**.** *Let X be a projective variety. Let S be an irreducible smooth divisor on it such that* $\operatorname{Supp} S \cap \operatorname{Sing}(X) = \emptyset$*, and let B and D be $\mathbb{Q}$-divisors such that*

$$\begin{aligned} &(p^e - 1)(K_X + B), \ \text{and} \\ &K_X + B + D \end{aligned}$$

are Cartier for some $e > 0$. Then, the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^e_* \mathcal{O}_X(K_X + p^e D) & \longrightarrow & F^e_* \mathcal{O}_X(K_X + S + p^e D) & \longrightarrow & F^e_* \mathcal{O}_S(K_S + p^e D|_S) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Tr}^e_X(D)} & & \downarrow{\scriptstyle \mathrm{Tr}^e_{X,S}(D)} & & \downarrow{\scriptstyle \mathrm{Tr}^e_S(D|_S)} & & \\ 0 & \longrightarrow & \mathcal{O}_X(K_X + D) & \longrightarrow & \mathcal{O}_X(K_X + S + D) & \longrightarrow & \mathcal{O}_S(K_S + D|_S) & \longrightarrow & 0. \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^e_*(\mathcal{O}_X(-S)) & \longrightarrow & F^e_*\mathcal{O}_X & \longrightarrow & F^e_*\mathcal{O}_S & \longrightarrow & 0 \\ & & {\scriptstyle F^e_X}\uparrow & & {\scriptstyle F^e_X}\uparrow & & {\scriptstyle F^e_S}\uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(-S) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0. \end{array}$$

Since $\mathcal{E}\mathrm{xt}^i(\mathcal{O}_S, \omega_X) = \omega_S$ for $i = 1$, and it is zero otherwise, the lemma follows by applying the functor $\mathcal{H}\mathrm{om}(-, \omega_X)$ to the above diagram, using Grothendieck duality (see (2.1)) and tensoring everything by D.

Note, that after restricting to $X \setminus S$ the commutativity is trivial, and X is smooth along S, so there are no issues with singularities. □

2.3 Cartier isomorphism

A source of miracles in positive characteristic algebraic geometry is provided by the Cartier isomorphism ([35, Theorem 3.5]):

$$\Omega^i_X \simeq \mathcal{H}^i(X, F_*\Omega^\bullet_X),$$

where X is a smooth projective variety and $\Omega^\bullet_X$ denotes the de Rham complex with standard differentials.

To exemplify its significance let us give a sketch of the following celebrated result of Deligne, Illusie and Raynaud. The sketch comes from an article of Illusie in [35]. We refer to this source for a definition of the second Witt vectors $W_2(k)$.

Theorem 2.3.1 ([35, Theorem 5.8])**.** *Let X be a smooth projective variety defined over an algebraically closed field k of characteristic $p > \dim(X)$. Assume that X is liftable to $W_2(k)$. Then the Kodaira vanishing holds on X, that is*

$$H^i(X, L^{-1} \otimes \Omega^j_X) = 0$$

for any $i + j < \dim(X)$ and an ample divisor L.

They key component of the proof is the fact, that if $p > \dim(X)$ and X is liftable to $W_2(k)$, then $F_*\Omega^\bullet_X$ is decomposable inside the bounded derived category of coherent sheaves $D_b(X)$ of X ([35, Corollary 5.5]). In particular, by Cartier isomorphism, we get a quasi-isomorphism

$$F_*\Omega^\bullet_X \simeq \bigoplus \Omega^j_X[-j].$$

For simplicity, we omit the details related to the fact that the absolute Frobenius is not a morphism of $\mathcal{O}_X$-modules.

Proof. By Serre vanishing and, if necessary, repeatedly replacing L by L^p, it is enough to prove the theorem under the assumption that

$$H^i(X, L^{-p} \otimes \Omega_X^j) = 0$$

for any $i + j < \dim(X)$.

Consider the standard hypercohomology spectral sequence

$$E_1^{i,j} = H^i(X, L^{-1} \otimes F_*\Omega_X^j) \implies \mathbb{H}^{i+j}(X, L^{-1} \otimes F_*\Omega_X^\bullet).$$

Since by the above assumption

$$E_1^{i,j} = H^i(X, L^{-1} \otimes F_*\Omega_X^j) = H^i(X, L^{-p} \otimes \Omega_X^j) = 0,$$

we get $\mathbb{H}^k(X, L^{-1} \otimes F_*\Omega_X^\bullet) = 0$ for $k < \dim(X)$. But, by the decomposability of $F_*\Omega_X^\bullet$, we know that

$$\mathbb{H}^k(X, L^{-1} \otimes F_*\Omega_X^\bullet) = \bigoplus_{i+j=k} H^i(X, L^{-1} \otimes \Omega_X^j),$$

which concludes the proof. □

The second application of Cartier isomorphism provided in this subsection concerns surjectivity of trace maps for curves.

Proposition 2.3.2 ([30, Lemma 10], [16, Section 3.1])**.** *Let C be a smooth curve defined over an algebraically closed field of characteristic $p > 0$, and let D be an effective divisor on it such that* $\deg D \geqslant \frac{2g(C)-2}{p}$. *Then*

$$\mathrm{Tr}_C^e(D)\colon H^0(C, K_C + p^e D) \longrightarrow H^0(C, K_C + D)$$

is surjective.

Proof. It is enough to prove the proposition for $e = 1$. The Cartier isomorphism for curves is equivalent to the exactness of the following sequence

$$0 \to \mathcal{O}_C \to \overbrace{F_*\mathcal{O}_C \to F_*\mathcal{O}_C(K_C)}^{F_*\Omega_X^\bullet} \xrightarrow{\mathrm{Tr}_C(D)} \mathcal{O}_C(K_C) \to 0.$$

After tensoring it by D and splitting it in the middle, we get

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow F_*\mathcal{O}_C(pD) \longrightarrow K \longrightarrow 0$$
$$0 \longrightarrow K \longrightarrow F_*\mathcal{O}_C(K_C + pD) \longrightarrow \mathcal{O}_C(K_C) \longrightarrow 0$$

By chasing the corresponding long exact sequences, is is easy to see that the surjectivity of $\mathrm{Tr}_C(D)$ follows from the following vanishings

$$H^2(C, \mathcal{O}_C(D)) = 0, \text{ and}$$
$$H^1(C, \mathcal{O}_C(pD)) = 0.$$

The latter is a consequence of $\deg D \geqslant \frac{2g(C)-2}{p}$ and Serre duality.

□

In Chapter 2, we will need a more general version of this proposition, and for this we need a more general Cartier isomorphism.

For a reduced simple normal crossing divisor $E = \sum E_j$ on X,

$$\Omega^i_X(\log E) \simeq \mathcal{H}^i(X, F_*\Omega^\bullet_X(\log E))$$

holds (see [36]). Further, for $B = \sum r_j E_j$ such that $0 \leqslant r_j \leqslant p-1$, it is true that the following inclusion of complexes of $\mathcal{O}_X$-modules

$$F_*\Omega^\bullet_X(\log E) \to F_*(\Omega^\bullet_X(\log E)(B))$$

is a quasi-isomorphism (see [37, Lemma 3.3], [5, Lemma 23.4]). All the above implies:

Theorem 2.3.3 ([5], [36], [37])**.** *Let X be a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$ and let E be a reduced simple normal crossing divisor on X. Let $B = \sum r_j E_j$ be an effective integral divisor supported on E such that $0 \leqslant r_j \leqslant p-1$. Then, there exists a quasi-isomorphism*

$$\Omega^i_X(\log E) \simeq \mathcal{H}^i\big(X, F_*\big(\Omega^\bullet_X(\log E)(B)\big)\big).$$

Now, we can prove a more general version of Proposition 2.3.2.

Proposition 2.3.4. *Let C be a smooth curve defined over an algebraically closed field of characteristic $p > 0$. Let Δ be an effective, and D an arbitrary $\mathbb{Q}$-divisor such that*

$$\lfloor \Delta \rfloor = 0,$$
$$\Delta + D \text{ is Cartier, and}$$
$$\deg D \geqslant \frac{2g(C)-2}{p} + \deg \Delta,$$

where $g(C)$ is the genus of C. Then,

$$\mathrm{Tr}^e_{C,\Delta}(D)\colon H^0(C, K_C + \Delta + p^e D) \longrightarrow H^0(C, K_C + \Delta + D)$$

is surjective for all $e > 0$ for which $(p^e-1)D$, or equivalently $(p^e-1)\Delta$, is Cartier.

Proof. Let $e > 0$ be such that $(p^e - 1)D$ and $(p^e - 1)\Delta$ are Cartier. Fix $0 \leqslant k \leqslant e$ and consider the enhanced Cartier isomorphism (Theorem 2.3.3):

$$0 \to \mathcal{O}_C \to \overbrace{F_*\mathcal{O}_C(B) \to F_*\mathcal{O}_C(K_C + E + B)}^{F_*\Omega^\bullet_X(\log E)(B)} \to \mathcal{O}_C(K_C) \to 0,$$

where

$$\begin{aligned} B &= \lfloor p^{k+1}\Delta \rfloor - p\lfloor p^k\Delta \rfloor, \text{ and} \\ E &= \operatorname{Supp} B. \end{aligned}$$

Note that the sequence consists only of vector bundles, because C is smooth, and so $F_*\mathcal{O}_C$ is free.

Set $\Delta_k = p^k\Delta - \lfloor p^k\Delta \rfloor$. Observe that $\Delta_e = \Delta_0 = \Delta$. Applying the functor

$$\mathcal{H}\mathrm{om}\big(-, \mathcal{O}_C(K_C + \Delta_k + p^kD)\big)$$

to the exact sequence, and using Grothendieck duality (2.1), we get a sequence:

$$\begin{aligned} 0 \leftarrow \mathcal{O}_C\big(K_C + \Delta_k + p^kD\big) &\xleftarrow{\mathrm{Tr}_k} F_*\mathcal{O}_C\big(\, K_C + \Delta_{k+1} + p^{k+1}D\big) \\ &\longleftarrow F_*\mathcal{O}_C\big(-E + \Delta_{k+1} + p^{k+1}D\big) \leftarrow \mathcal{O}_C\big(\Delta_k + p^kD\big) \leftarrow 0, \end{aligned}$$

It is exact, because a dualization of an exact sequence consisting only of vector bundles is exact.

Since $\Delta_e = \Delta_0 = \Delta$, we can decompose $\mathrm{Tr}^e_{C,\Delta}(D)$ into a composition of the maps Tr_k:

$$H^0(C, K_C + \Delta_e + p^eD) \xrightarrow{\mathrm{Tr}_{e-1}} H^0(C, K_C + \Delta_{e-1} + p^{e-1}D) \to \ldots \to H^0(C, K_C + \Delta_0 + D).$$

Hence, we are left to show that Tr_k is surjective. By chasing the corresponding long exact sequences, is is easy to see that the surjectivity of Tr_k follows from the following vanishings

$$\begin{aligned} &H^2(C, \mathcal{O}_C(\Delta_k + p^kD)) = 0, \text{ and} \\ &H^1(C, \mathcal{O}_C(-E + \Delta_{k+1} + p^{k+1}D)) = 0. \end{aligned}$$

The former is a consequence of $\dim C = 1$. Applying Serre duality to the latter cohomology, we see that to show its vanishing it is enough to prove that

$$2g(C) - 2 + \deg E - \deg \Delta_{k+1} - p^{k+1}\deg D < 0$$

This follows from the assumptions, given that $\deg \Delta_{k+1} \geqslant 0$ and $\deg E \leqslant \deg \lfloor p^{k+1}\Delta \rfloor$. □

This train of ideas is developed later in Chapter 3.

2.4 Evaporation technique

Except of Proposition 1.0.1, this section is based on [38].

Definition 2.4.1. Let X be a scheme, and let $\mathcal{F}$ be a sheaf on it. We say that a morphism $f\colon Y \to X$, where Y is a scheme, *evaporates* a cohomology class $\alpha \in H^i(X, \mathcal{F})$, if $f^*\alpha = 0$.

The idea of evaporation sprouts out from the following observation. Let D be an ample Cartier divisor on a projective scheme X defined over a field of characteristic $p > 0$. Then e-times iterated Frobenius F^e evaporates all clases in $H^i(X, \mathcal{O}_X(D)) = 0$ for $i > 0$ and $e \gg 0$. Indeed, it follows by Serre vanishing, since

$$(F^e)^* H^i(X, \mathcal{O}_X(D)) \subseteq H^i(X, \mathcal{O}_X(p^e D)) = 0.$$

The theorem below embraces the full strength of evaporation.

Theorem 2.4.2 (Hochster, Huneke, see [38, Theorem 5.19])**.** *Let X be a projective scheme over a field of characteristic $p > 0$ and let D be an ample divisor on it. Then, there exists a finite map $f\colon Y \to X$ of degree p^k such that*

$$f^* H^i(X, \mathcal{O}_X(mD)) = 0$$

for all $i > 0$ and $m \geqslant 0$,

For $m > 0$, the theorem follows by the above observation for f being a power of Frobenius. The main difficulty lies in the case $m = 0$, where one needs to consider more complicated purely inseparable morphisms than the Frobenius.

As an example, we will use this theorem to show the semiampleness of rational and elliptic nef curves on smooth surfaces defined over $\overline{\mathbb{F}}_p$, following an exercise in [38]. Note that, by [39], nef curves of genus two do not need to be semiample.

Proposition 2.4.3. *Let C be a smooth curve on a smooth surface S defined over $\overline{\mathbb{F}}_p$. Assume that C is nef and $g(C) \leqslant 1$. Then C is semiample.*

Proof. If $C^2 > 0$, then C is big and nef, and so it is semiample ([40, Theorem 2.9]). Hence, we may assume that $C^2 = 0$. Since we are working over $\overline{\mathbb{F}}_p$, all points of the abelian variety $\mathrm{Pic}^0(C)$ are torsion, and so there exists $m \in \mathbb{Z}_{\geqslant 0}$ such that $mC|_C$ is trivial. Let m be the smallest such positive integer.

We have the following exact sequence

$$H^0(S, kC) \to H^0(C, kC|_C) \to H^1(S, (k-1)C) \to H^1(S, kC) \to H^1(C, kC|_C).$$

Given, that for $0 < k < m$

$$H^0(C, kC|_C) = 0, \text{ and}$$
$$H^1(C, kC|_C) = 0, \text{ by } g(C) \leqslant 1,$$

we get

$$H^1(S, \mathcal{O}_S) = H^1(S, C) = \ldots = H^1(S, (m-1)C).$$

By Theorem 2.4.2, there exists a purely inseparable morphism $f\colon \widehat{S} \to S$ such that $f^*H^1(S, \mathcal{O}_S) = 0$. Set $\widehat{C} := f^*(C)$. Note that $\widehat{C}$ is not reduced.

Consider the following commutative diagram, where the rows come from the above long exact sequence for $k = m$:

$$\begin{array}{ccccc} H^0(\hat{S}, \mathcal{O}_{\hat{S}}(m\widehat{C})) & \longrightarrow & H^0(\widehat{C}, \mathcal{O}_{\widehat{C}}) & \xrightarrow{f^*\phi} & H^1(\widehat{S}, \mathcal{O}_{\hat{S}}((m-1)\widehat{C})) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ H^0(S, \mathcal{O}_S(mC)) & \longrightarrow & H^0(C, \mathcal{O}_C) & \xrightarrow{\phi} & H^1(S, \mathcal{O}_S((m-1)C)). \end{array}$$

Take $1 \in H^0(C, \mathcal{O}_C)$. Since $f^*(1) = 1$ and $f^*(\phi(1)) = 0$, we get that $1 \in H^0(\widehat{C}, \mathcal{O}_{\widehat{C}})$ lifts, and so there exists $D \in |m\widehat{C}|$ such that $D \cap \widehat{C} = \varnothing$. Hence, $m\widehat{C}$ is base point free.

Since f factors through Frobenius F^k for some $k \geqslant 0$, we get that mp^kC is base point free (see [41, Lemma 1.4]). ☐

The majority of cases in the proof of Proposition 1.0.1 are covered by [3, Theorem 0.4]. The new part is very similar to the above proof of the semiampleness of nef rational and elliptic curves.

Proof of Proposition 1.0.1. First, assume that $\kappa(S, K_S + A|_S) \neq 0$. Then, by [3, Theorem 0.4], the restriction map

$$H^0(X, k(K_X + S + A)) \to H^0(S, k(K_S + A|_S))$$

is surjective for $k \gg 0$. By the base point free theorem for surfaces [26, Theorem A.4], $K_S + A|_S$ is semiample, and so is $K_X + S + A \sim 2(K_X + A)$.

Thus, we can assume that $\kappa(S, K_S + A|_S) = 0$. Since we are working over $\overline{\mathbb{F}}_p$, all points of the abelian variety $\mathrm{Pic}^0(S)$ are torsion, and so $K_S + A|_S$ is torsion. Since $K_S + A|_S \sim 2S|_S$, there exists $m \in \mathbb{Z}_{\geqslant 0}$ such that $mS|_S$ is trivial. Let m be the smallest such positive integer.

We have the following exact sequence

$$H^0(X, kS) \to H^0(S, kS|_S)) \to H^1(X, (k-1)S) \to H^1(X, kS) \to H^1(S, kS|_S).$$

Note that $K_S \equiv -A|_S$, and so S is Fano. Given, that for $0 < k < m$

$$H^0(S, kS|_S) = 0, \text{ and}$$
$$H^1(S, kS|_S) = H^1(S, K_S - K_S + kS_S) = 0, \text{ by Kodaira vanishing,}$$

we get

$$H^1(X, \mathcal{O}_X) = H^1(X, S) = \ldots = H^1(X, (m-1)S).$$

By Theorem 2.4.2, there exists a purely inseparable morphism $f \colon \widehat{X} \to X$ such that $f^* H^1(X, \mathcal{O}_X) = 0$. Set $\widehat{S} := f^*(S)$. Note that $\widehat{S}$ is not reduced.

Consider the following commutative diagram, where the rows come from the above long exact sequence for $k = m$:

$$\begin{array}{ccccc} H^0(\hat{X}, \mathcal{O}_{\hat{X}}(m\widehat{S})) & \longrightarrow & H^0(\widehat{X}, \mathcal{O}_{\hat{X}}) & \xrightarrow{f^*\phi} & H^1(\widehat{X}, \mathcal{O}_{\hat{X}}((m-1)\widehat{S})) \\ \uparrow f^* & & \uparrow f^* & & \uparrow f^* \\ H^0(X, \mathcal{O}_X(mS)) & \longrightarrow & H^0(X, \mathcal{O}_X) & \xrightarrow{\phi} & H^1(X, \mathcal{O}_X((m-1)S)). \end{array}$$

Take $1 \in H^0(X, \mathcal{O}_X)$. Since $f^*(1) = 1$ and $f^*(\phi(1)) = 0$, we get that $1 \in H^0(\widehat{X}, \mathcal{O}_{\hat{X}})$ lifts, and so there exists $D \in |m\widehat{S}|$ such that $D \cap \widehat{S} = \varnothing$. Hence, $m\widehat{S}$ is base point free.

Since f factors through Frobenius F^k for some $k \geqslant 0$, we get that mp^kS is base point free (see [41, Lemma 1.4]). □

The proof of [3, Theorem 0.4] is based on the surjectivity of the trace map, which fails when $\kappa(S, K_S + A|_S) = 0$. It is astonishing, that the evaporation technique shines exactly when the trace map fails. We believe that this complementarity is worth future investigation.

2.5 Very ampleness

The goal of this section is to prove Theorem 1.0.3. We start with the following example as a warm-up.

Example 2.5.1 (c.f. [16, Theorem 3.8])**.** Let X be an n-dimensional smooth projective variety defined over an algebraically closed field of characteristic $p > 0$ and let L be a Cartier divisor on it. The goal of this example is to prove that:

if L is ample and base point free, then $K_X + (n+1)L$ is base point free.

First, notice that by Theorem 1.0.11, $F^e_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_X((n+1)L)$ is a globally generated coherent sheaf for $e \gg 0$, given that for $0 < i \leqslant n$

$$H^i\big(X, F^e_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_X((n+1-i)L)\big) = H^i\big(X, \mathcal{O}_X(K_X + p^e(n+1-i)L)\big) = 0,$$

where the last equality follows from Serre vanishing.

Since X is smooth, the trace map

$$\mathrm{Tr}^e_X(L) \colon F^e_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_X((n+1)L) \twoheadrightarrow \mathcal{O}_X(K_X) \otimes \mathcal{O}_X((n+1)L).$$

is surjective (Proposition 2.2.2). As the former sheaf is globally generated, so is the later one.

Now, we proceed with the proof of Theorem 1.0.3. A similiar strategy was considered in [13]. We also apply Mumford regularity, but for a different version of the trace map. Further, since we work on singular varieties, we cannot use that $F^e_*\mathcal{O}_X$ is flat.

The following proposition is based on a train of ideas from [19], but we need to perform a careful analysis of Tor modules.

Proposition 2.5.2 (c.f. [19, Examples 1.8.18 and 1.8.22])**.** *Let X be a normal irreducible projective variety of dimension n. Consider a coherent sheaf $\mathcal{F}$ and a point $x \in X$. Let B be a globally generated ample line bundle. If*

$$H^{i+k-1}\big(X, \mathcal{F} \otimes B^{-(i+k)}\big) = 0,$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n$, then $\mathcal{F} \otimes m_x$ is globally generated.

Proof. Set $\mathcal{F}(-i) := F \otimes B^{-i}$. Our goal is to prove that

$$H^i(X, \mathcal{F}(-i) \otimes m_x) = 0$$

for all $i > 0$. Then, Theorem 1.0.11 would imply the global generatedness of $F \otimes m_x$.

Since B is ample and globally generated, it defines a finite map and so there exist sections $s_1, s_2, \dots, s_n \in H^0(X, B)$ intersecting in a zero dimensional scheme W containing x. We claim the following.

Claim $H^i(X, \mathcal{F}(-i) \otimes I_W) = 0$.

Assuming the claim, we prove the proposition. Consider the following short exact sequence

$$0 \longrightarrow I_W \longrightarrow m_x \longrightarrow m_x/I_W \longrightarrow 0,$$

and tensor it by $\mathcal{F}(-i)$, to get a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}(-i) \otimes I_W \longrightarrow \mathcal{F}(-i) \otimes m_x \longrightarrow \mathcal{F}(-i) \otimes \big(m_x/I_W\big) \longrightarrow 0,$$

where the term
$$\mathcal{G} := \ker\big(\mathcal{F}(-i) \otimes I_W \longrightarrow \mathcal{F}(-i) \otimes m_x\big)$$
comes from the fact that $\mathcal{F}$ may not be flat. Since m_x/I_W is flat off W, we have that
$$\dim \operatorname{Supp}\big(\operatorname{Tor}^1\big(\mathcal{F}(-i), m_x/I_W\big)\big) = 0,$$
and so $\dim \operatorname{Supp}(\mathcal{G}) = 0$. Simple diagram chasing (similiar to the proof of Lemma 2.5.3), shows that $H^i(X, \mathcal{F}(-i) \otimes m_x) = 0$ for $i > 0$, and so we are done.

We are left to show the claim. In order to do this, we take $U := B^{\oplus n}$ and consider a Koszul complex induced by the map $U^* \to I_W$ coming from the sections $s_1, s_2, \ldots, s_n$:
$$0 \longrightarrow \bigwedge^n U^* \longrightarrow \ldots \longrightarrow \bigwedge^1 U^* \longrightarrow I_W \longrightarrow 0.$$
By [19, Appendix B2], the complex is exact off W. We tensor it by $\mathcal{F}(-i)$. Since on $X \setminus \operatorname{Supp}(W)$ the sequence consists of free objects, tensoring by $\mathcal{F}(-i)$ is exact there, and so the homologies of the sequence $\mathcal{F}(-i) \otimes \bigwedge^\bullet U^*$ are supported on W. Hence, we can apply Lemma 2.5.3, and so to show that
$$H^i(X, \mathcal{F}(-i) \otimes I_W) = 0,$$
it is enough to know that
$$\begin{aligned} 0 &= H^{i+k-1}\Big(X, \mathcal{F}(-i) \otimes \bigwedge^k U^*\Big) \\ &= H^{i+k-1}\Big(X, \mathcal{F}(-i-k)\Big)^{\oplus\binom{n}{k}}. \end{aligned}$$
□

Proof of Theorem 1.0.3. Choose a point $q \in X$. To prove the theorem, it is enough to show that $\mathcal{O}_X\big((n+2)H + N\big) \otimes m_x$ is globally generated at q for all $x \in X$.

By F-purity we know that there exists a $\mathbb{Q}$-divisor B such that
$$(p^e - 1)(K_X + B)$$
is Cartier, and
$$F^e_* \mathcal{O}_X(-(p^e-1)(K_X + B)) \longrightarrow \mathcal{O}_X$$
is surjective at q, for enough divisible $e \gg 0$ (Proposition 2.2.2 and Proposition 2.1.3). If the $\mathbb{Q}$-Gorenstein index of X is indivisible by p, then we can take $B = 0$. By increasing e and decreasing coefficients of B, we may assume that $H - K_X - 2B$ is ample.

By the above, the morphism

$$F^e_*\mathcal{O}_X\big(-(p^e-1)(K_X+B)+p^e\big((n+2)H+N\big)\big)\otimes m_x \longrightarrow \mathcal{O}_X\big((n+2)H+N\big)\otimes m_x$$

is surjective at q. It implies that in order to show global generatedness of

$$\mathcal{O}_X\big((n+2)H+N\big)\otimes m_x,$$

at q, it is enough to show that

$$F^e_*\mathcal{O}_X\big(-(p^e-1)(K_X+B)+p^e\big((n+2)H+N\big)\big)\otimes m_x$$

is globally generated. And this follows from Proposition 2.5.2, because by Fujita vanishing (Theorem 1.0.12),

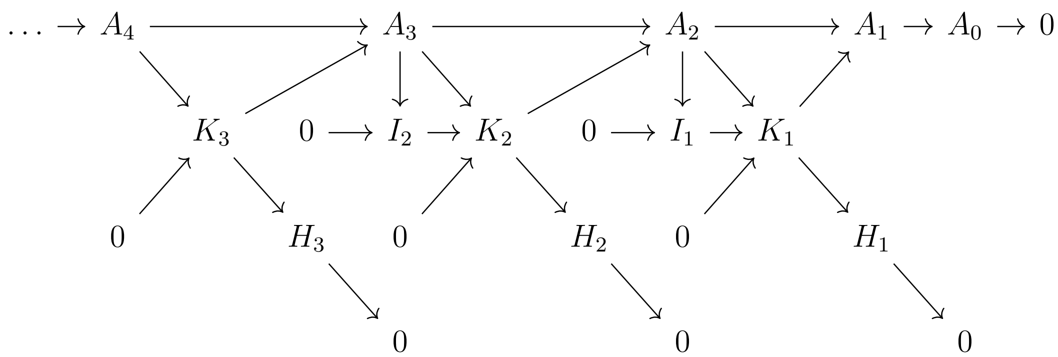
$$\begin{aligned} &H^{i+k-1}\big(X, F^e_*\mathcal{O}_X\big(-(p^e-1)(K_X+B)+p^e\big((n+2)H+N\big)\big)\otimes\mathcal{O}_X(-(k+i)H)\big)\\ &\quad= H^{i+k-1}\big(X,\mathcal{O}_X\big((p^e-1)(H-K_X-B)+H+p^e\big((n+1-k-i)H+N\big)\big)\\ &\quad= 0, \end{aligned}$$

for $e \gg 0$ and all $i,k>0$ such that $i+k-1\leqslant n$. □

In the proof we used the following lemma.

Lemma 2.5.3. *Let $A^\bullet$ be a bounded complex of sheaves on a scheme X such that $A^{-i}=0$ for $i>0$ and $A^1\to A^0$ is surjecitve. Assume that $\mathcal{H}^i(A^\bullet)$ are supported on a zero dimensional scheme for each $i\geqslant 0$ and $H^{i+k-1}(X,A_k)=0$ for $i,k>0$. Then $H^i(X,A_0)=0$.*

Proof. Consider the following diagram:



Here, K_i is a kernel of $A_i\to A_{i-1}$. Further, I_i is the image of $A_{i+1}\to K_i$, and H_i is the cohomology of $A^\bullet$ at the i-th position.

Note that $H^i(X,H_j)=0$ for $i,j>0$, since H_j is supported on a zero dimensional scheme.

Now, to show the vanishing of $H^i(X, A_0)$ it is enough to show that $H^{i+1}(X, K_1) = 0$, as we have the short exact sequence

$$0 \longrightarrow K_1 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$$

and we know that $H^i(X, A_1) = 0$.

Using the short exact sequence

$$0 \longrightarrow I_1 \longrightarrow K_1 \longrightarrow H_1 \longrightarrow 0$$

we reduce to showing that $H^{i+1}(X, I_1) = 0$. The same way, using

$$0 \longrightarrow K_2 \longrightarrow A_2 \longrightarrow I_1 \longrightarrow 0,$$

we reduce to showing that $H^{i+2}(X, K_2) = 0$. Now, the argument goes in exactly the same manner. $\square$

CHAPTER 3

Effective bounds on surfaces

The readers interested in Matsusaka theorem and Fujita-type theorems are encouraged to consult [20, Section 10.2 and 10.4]. Certain Fujita-type bounds for singular surfaces in characteristic zero are obtained in [18].

3.1 Reider's analysis

Reider's analysis is a method of showing that divisors of the form $K_X + L$ are globally generated or very ample, where L is a big and nef divisor on a smooth surface X.

The idea is that a base point of $K_X + L$ provides us with a rank two vector bundle $\mathcal{E}$ which does not satisfy Bogomolov inequality $c_1(\mathcal{E})^2 > 4c_2(\mathcal{E})$. In characteristic zero, such vector bundles must split. Using such a splitting one can deduce a contradiction, when L is "numerically-ample enough".

In positive characteristic, the aforementioned fact about unstable vector bundles, has been proved by Shepherd-Barron ([10]) for surfaces which are neither of general type nor quasi-elliptic of Kodaira dimension one. This leads to the following.

Proposition 3.1.1 ([12, Theorem])**.** *Let X be a smooth projective surface neither of general type nor quasi-elliptic with $\kappa(X) = 1$, and let D be a nef divisor such that $D^2 \geqslant 5$. Assume that $q \in X$ is a base-point of $K_X + D$. Then, there exists a divisor C containing q, such that $D \cdot C \leqslant 3$.*

In particular, for such surfaces, $K_X + 4A$ is base point free for an ample divisor A.

3.1.1 A proof of Fujita conjecture using Reider's analysis

In this subsection, we present a proof of Proposition 3.1.1. We follow closely [12].

Let X be a smooth projective surface defined over an algebraically closed field.

Definition 3.1.2. A vector bundle E of rank two is called *unstable* if there exist a short exact sequence

$$0 \longrightarrow \mathcal{O}_S(A) \longrightarrow E \longrightarrow \mathcal{O}_S(C) \otimes I_Z \longrightarrow 0,$$

where A and C are Cartier divisors, I_Z is the ideal sheaf of a 0-dimensional subscheme Z on S and

- $(A - C)^2 > 0$, and
- $(A - C) \cdot H > 0$ for any ample divisor H on S.

One can show that this definition is equivalent to the standard definition of unstability.

One of the most striking results of Bogomolov is that when a vector bundle E does not satisfy Bogomolov inequality $c_1(E)^2 \leqslant 4c_2(E)$, then E is unstable. In positive characteristic we have the following result of Shepherd-Barron.

Theorem 3.1.3 ([10])**.** *Let X be a smooth projective surface defined over an algebraically closed field of characteristic $p > 0$ which is neither of general type nor quasi-elliptic with $\kappa(X) = 1$. Further, let E be a rank two vector bundle on X such that $c_1(E)^2 > 4c_2(E)$. Then E is unstable.*

Now, we can proceed with the proof of Proposition 3.1.1.

Proof of Propostion 3.1.1. Assume that $q \in X$ is a base point of $K_X + D$. Let I_q be the ideal sheaf corresponding to q. The long exact sequence associated to

$$0 \longrightarrow \mathcal{O}_X(K_X + D) \otimes I_q \longrightarrow \mathcal{O}_X(K_X + D) \longrightarrow \mathcal{O}_q(K_X + D) \longrightarrow 0$$

gives us that

$$\mathrm{Ext}^1(\mathcal{O}_X(K_X + D) \otimes I_q, \mathcal{O}_X(K_X)) = H^1(X, \mathcal{O}_S(K_X + D) \otimes I_q)^\wedge \neq 0,$$

where the first equality follows by Serre duality. In particular, we get a rank two vector bundle E, which sits in the long exact sequence:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{O}_X(D) \otimes I_q \longrightarrow 0.$$

Since $c_1(E) = [D]$ and $c_2(E) = [q]$, Theorem 3.1.3 implies that E is unstable. In particular, we have the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{O}_X(A) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & E & \longrightarrow & \mathcal{O}_X(D) \otimes I_q & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathcal{O}_X(C) \otimes I_Z & & & & \\ & & & & \downarrow & & & & \\ & & & & 0, & & & & \end{array}$$

where A, C and I_Z are as in Definition 3.1.2. By conducting a comparison between Chern classes, we get that $D \sim A + C$.

Step 1 In this step we show that $H^0(X, \mathcal{O}_X(C) \otimes I_q) \neq 0$.

First, notice that $H^0(X, \mathcal{O}_X(-A)) = 0$. Indeed, if $-A$ was effective, then for any ample divisor H, we would have $(A - C) \cdot H = (2A - D) \cdot H < 0$, because D is nef. This is a contradiction with the assumptions of Definition 3.1.2.

Consider the map $\mathcal{O}_X(A) \to \mathcal{O}_X(D) \otimes I_q$ induced by the above diagram. It must be non-zero, because otherwise $\mathcal{O}_X(A) \to E$ would factor through some map $\mathcal{O}_X(A) \to \mathcal{O}_X$, which contradicts $H^0(X, \mathcal{O}_X(-A)) = 0$. Since $D \sim A + C$, the claim of this step follows.

Now, we can assume that C is effective and passes through q.

Step 2 We show that $D^2 > 2D \cdot C$.

Since D is nef and big, we have by the Hodge index theorem

$$0 < (A - C)^2 \cdot D^2 \leqslant ((A - C) \cdot D)^2 = ((D - 2C) \cdot D)^2,$$

and so $(D - 2C) \cdot D > 0$, which implies $D^2 > 2D \cdot C$. Here, we used that $(A - C) \cdot D \geqslant 0$ by one of the assumptions of Definition 3.1.2.

Step 3 We finish the proof.

By comparison of second Chern classes, we get that

$$1 = A \cdot C + \deg Z,$$

and in particular $A \cdot C = (D - C) \cdot C \leqslant 1$. Hence $D \cdot C - 1 \leqslant C^2$.

On the other hand, by Step 2 and the Hodge index theorem

$$2(D \cdot C) \cdot C^2 < D^2 \cdot C^2 \leqslant (D \cdot C)^2,$$

and this implies that $2C^2 < D \cdot C$.

Combining inequalities from the above two paragraphs, we get

$$2D \cdot C - 2 < 2C^2 \leqslant D \cdot C,$$

which concludes the proof.

□

To show that Reider's analysis is useful not only for the Fujita conjecture, let us also present the following theorem of Mumford.

Theorem 3.1.4 ([12, Theorem 1.6])**.** *Let X be a smooth projective surface defined over an algebraically closed field, which is neither of general type nor quasi-elliptic with $\kappa(X) = 1$. Then, the Kodaira vanishing holds, that is $H^1(X, \mathcal{O}_X(K_X + L)) = 0$ for any ample divisor L.*

Proof. Assume by contradiction that $H^1(X, \mathcal{O}_X(K_X + L)) \neq 0$. Since by Serre duality

$$H^1(X, \mathcal{O}_X(K_X + L)) = \mathrm{Ext}^1(\mathcal{O}_X(K_X + L), \mathcal{O}_X(K_X))^\wedge,$$

we get a rank two vector bundle E and a non-split sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{O}_X(L) \longrightarrow 0.$$

Since $c_1(E) = [L]$ and $c_2(E) = 0$, Theorem 3.1.3 implies that E is unstable. In particular, we have the following diagram:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & \mathcal{O}_X(A) & & & \\ & & & \downarrow & & & \\ 0 \rightarrow & \mathcal{O}_X & \longrightarrow & E & \longrightarrow & \mathcal{O}_X(L) & \rightarrow 0 \\ & & & \downarrow & & & \\ & & & \mathcal{O}_X(C) \otimes I_Z & & & \\ & & & \downarrow & & & \\ & & & 0, & & & \end{array}$$

where A, C and I_Z are as in Definition 3.1.2.

As before, we get that $L \sim A + C$, and C is effective. Comparing second Chern classes, we obtain

$$0 = A \cdot C + \deg Z,$$

and so $A \cdot C \leqslant 0$. In particular, $L \cdot C \leqslant C^2$.

Further, by assumptions in Definition 3.1.2, we have

$$(L - 2C) \cdot L = (A - C) \cdot L \geqslant 0,$$

and so $L^2 \geqslant 2L \cdot C$.

Mixing inequalities obtained above with the Hodge index theorem, we get

$$(L \cdot C)^2 \geqslant L^2 \cdot C^2 \geqslant 2(L \cdot C)^2.$$

In particular, $L \cdot C = C^2 = 0$. Since C is effective, we get $C = 0$ and $L \sim A$. In this case, the vertical arrow $\mathcal{O}_X(A) \to E$ provides a splitting of

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{O}_X(L),$$

which is a contradiction. □

3.1.2 Effective bounds for smooth surfaces

The goal of this subsection is to prove the following proposition, which covers all types of surfaces.

Proposition 3.1.5. *Let X be a smooth projective surfaces defined over an algebraically closed field of characteristic $p > 0$, and let D be a nef and big line bundle on it. Assume that $q \in S$ is a base point of $K_X + D$. Further, suppose that*

1. *$D^2 > 4$, if X is quasi-elliptic with $\kappa(X) = 1$,*
2. *$D^2 > \mathrm{vol}(K_X) + 4$, if X is of general type and $p \geqslant 3$,*
3. *$D^2 > \mathrm{vol}(K_X) + 6$, if X is of general type and $p = 2$, or*
4. *$D^2 > 8$, otherwise.*

Then, there exists a curve C containing q such that

(1a) $D \cdot C \leqslant 5$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 3$,

(1b) $D \cdot C \leqslant 7$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 2$,

(2) $D \cdot C \leqslant 1$, if X is of general type and $p \geqslant 3$,

(3) $D \cdot C \leqslant 7$, *if* X *is of general type and* $p = 2$, *or*

(4) $D \cdot C \leqslant 3$, *otherwise.*

Note that the case (4) is nothing else but Proposition 3.1.1. The proof follows step by step the proof by Di Cerbo and Fanelli ([8]). The only addition, is that the curve C must contain q. This chain of ideas traces back to Sakai ([23]).

The following is crucial in the proof of Proposition 3.1.5.

Proposition 3.1.6 ([8])**.** *Consider a birational morphism* $\pi\colon Y \to X$ *between smooth projective surfaces* X *and* Y*. Let* $\overline{D}$ *be a big divisor on* Y *such that* $H^1(Y, -\overline{D}) \neq 0$ *and* $\overline{D}^2 > 0$*. Further, suppose that*

1. $\overline{D}^2 > \operatorname{vol}(K_X)$, *if* X *is of general type and* $p \geqslant 3$, *or*

2. $\overline{D}^2 > \operatorname{vol}(K_X) + 2$, *if* X *is of general type and* $p = 2$.

Then, there exists a non-zero non-exceptional effective divisor $\overline{E}$ *on* Y*, such that*

$$\begin{aligned} &0 \leqslant D \cdot E < \frac{k\alpha}{2}, \\ &\overline{D} - 2\overline{E} \text{ is big}, \\ &(\overline{D} - \overline{E}) \cdot \overline{E} \leqslant 0, \end{aligned}$$

where $D = \pi_*\overline{D}$, $E = \pi_*\overline{E}$, $\alpha = D^2 - \overline{D}^2$, *and*

- $k = 3$, *if* X *is quasi-elliptic with* $\kappa(X) = 1$ *and* $p = 3$,
- $k = 4$, *if* X *is quasi-elliptic with* $\kappa(X) = 1$ *and* $p = 2$,
- $k = 1$, *if* X *is of general type and* $p \geqslant 3$, *or*
- $k = 4$, *if* X *is of general type and* $p = 2$.

Proof. It follows directly from [8, Proposition 4.3], [8, Theorem 4.4], [8, Proposition 4.6] and [8, Corollary 4.8]. □

Further, we need the following lemma.

Lemma 3.1.7 ([23, Lemma 2])**.** *Let* D *be a nef and big divisor on a smooth surface* S*. If*

$$D \equiv D_1 + D_2$$

for numerically non-trivial pseudo-effective divisors D_1 *and* D_2*, then* $D_1 \cdot D_2 > 0$.

Now, we can proceed with the proof of the main proposition in this subsection.

Proof of Proposition 3.1.5. The first case is covered by Proposition 3.1.1, so we may assume that X is of general type or quasi-elliptic with $\kappa(X) = 1$.

Let $\pi\colon Y \to X$ be a blow-up at $q \in X$ with the exceptional curve F. Given that q is a base point of $K_X + D$, we obtain that

$$H^1(Y, \mathcal{O}_Y(K_Y + \pi^* D - 2F) = H^1(Y, \mathcal{O}_Y(-(\pi^* D - 2F))) \neq 0.$$

Set $\overline{D} := \pi^* D - 2F$. Since

$$\overline{D}^2 = D^2 - 4,$$

we have $\overline{D}^2 > 0$, and the assertions (1) and (2) in Proposition 3.1.6 are satisfied. Hence, by this proposition, there exists a non-zero non-exceptional effective divisor $\overline{E}$ on Y, such that

$$\begin{aligned} &0 \leqslant D \cdot E \leqslant 2k - 1, \text{ and} \\ &\overline{D} - 2\overline{E} \text{ is big, and} \\ &(\overline{D} - \overline{E}) \cdot \overline{E} \leqslant 0, \end{aligned}$$

where $E = \pi_* \overline{E}$.

To finish the proof, it is enough to show that $\overline{E}$ contains a component, which intersects F properly. Its pushforward onto X would be the sought-for curve C.

Assume that the claim is not true, i.e. $\overline{E} = \mu^* E + aF$ for $a \geqslant 0$. We have that

$$0 \geqslant (\overline{D} - \overline{E}) \cdot \overline{E} = (D - E) \cdot E + (2 + a)a.$$

It implies $D \cdot E \leqslant E^2$. Since $D \cdot E \geqslant 0$, it holds that $E^2 \geqslant 0$. Given $D - 2E$ is big, we may apply Lemma 3.1.7 with $D = (D - 2E) + 2E$, and obtain that $D \cdot E > 2E^2$. This is a contradiction with the other inequalities in this paragraph. □

3.2 Effective bounds for singular surfaces

The goal of this section is to prove Theorem 1.0.4.

The following lemma is a crucial component, without which we would not be able to apply our strategy.

Lemma 3.2.1. *Let L be an ample Cartier divisor on a normal projective surface X. Let $\pi\colon \widetilde{X} \to X$ be the minimal resolution of singularities. Then $K_{\widetilde{X}} + 3\pi^* L$ is nef, and $K_{\widetilde{X}} + n\pi^* L$ is nef and big for $n \geqslant 4$.*

Proof. Take an effective curve C. We need to show that $(K_{\tilde{X}} + 3\pi^*L) \cdot C \geqslant 0$. If $K_{\tilde{X}} \cdot C \geqslant 0$, then the inequality clearly holds. Thus, by cone theorem ([26, Theorem 3.13] and [26, Remark 3.14]), we need to prove it, when C is an extremal ray satisfying $K_{\tilde{X}} \cdot C < 0$. In such a case, we have that $K_{\tilde{X}} \cdot C \geqslant -3$.

If C is not an exceptional curve, then $3\pi^*L \cdot C \geqslant 3$, and so the inequality holds. But C cannot be exceptional, because then its contraction would give a smooth surface (see [21, Theorem 1.28]), and so $\widetilde{X}$ would not be a minimal resolution. This concludes the first part of the lemma.

As for the second part, $K_{\tilde{X}} + n\pi^*L$ is big and nef for $n \geqslant 4$, since adding a nef divisor to a big and nef divisor gives a big and nef divisor. □

The following proposition yields the first step in the proof.

Proposition 3.2.2. *Let X be a projective surface defined over an algebraically closed field of characteristic $p > 3$. Assume that mK_X is Cartier for some $m \in \mathbb{N}$. Let A be an ample Cartier divisor on X. Then*

$$\mathbb{B}(m(aK_X + bA)) \subseteq \mathrm{Sing}(X),$$

where $a = 2$ and $b = 7$.

Proof. Let $\pi\colon \overline{X} \to X$ be a minimal resolution of singularities with the exceptional locus E. First, we prove that

$$\mathbb{B}(2K_{\overline{X}} + 7\pi^*A) \subseteq E.$$

This would imply that $\mathbb{B}(m(2K_X + 7A)) \subseteq \pi(E)$, which would conclude the proof. Assume it is not true, and so there exists a base point $q \in \overline{X}$ of $2K_{\overline{X}} + 7\pi^*A$ such that $q \notin E$.

We apply Proposition 3.1.5 for $D = K_{\overline{X}} + 7\pi^*A$. The assumptions are satisfied, because, by Lemma 3.2.1, D is big and nef, and, by Theorem 1.0.13,

$$\mathrm{vol}(D) \geqslant \mathrm{vol}(K_{\overline{X}}) + 49.$$

Henceforth, there exists a curve C containing q such that

$$C \cdot D \leqslant 3.$$

We can write $D = (K_{\overline{X}} + 3\pi^*A) + 4\pi^*A$. By Lemma 3.2.1, the first summand is nef. Fruther, $C \cdot \pi^*A > 0$, as C is not exceptional. Thus, we obtain a contradiction. □

Applying above Proposition 3.2.2 and Theorem 2.1.7, the base point free part of Theorem 1.0.4 follows from the following proposition by taking $L := m(aK_X + bA) - K_X$.

Proposition 3.2.3. *Let X be an F-pure projective surface defined over an algebraically closed field of characteristic p. Let L be an ample $\mathbb{Q}$-divisor on X such that $K_X + L$ is a nef Cartier divisor and*

$$\mathbb{B}(K_X + L) \subseteq \operatorname{Sing}(X),$$

Then $2(K_X + L) + N$ is base point free for any nef Cartier divisor N.

If we just assume that $\dim \mathbb{B}(K_X + L) = 0$, then the same proof will give us that $3(K_X + L) + N$ is base point free.

Before proceeding with the proof, we would like to give an example explaining our strategy in characteristic zero.

Example 3.2.4. Here, X is a normal Gorenstein surface defined over an algebraically closed field k of characteristic zero, and L is an ample Cartier divisor on it. The goal of this example is to prove:

> *if $K_X + L$ is ample and $\dim \mathbb{B}(K_X + L) = 0$, then $3(K_X + L)$ is base point free.*

Take any point $q \in \mathbb{B}(K_X + L)$. It is enough to show that $3(K_X + L)$ is base point free at q. By assumptions, $K_X + L$ defines a finite map outside of its zero dimensional base locus, and so there exist divisors $D_1, D_2, \ldots, D_n \in |K_X + L|$ intersecting properly such that the multiplier ideal sheaf $\mathcal{I}(X, \Delta)$ for $\Delta = \frac{2}{n}(D_1 + \ldots D_n)$ satisfies

$$\dim \mathcal{I}(X, \Delta) = 0, \text{ and}$$
$$q \in \mathcal{I}(X, \Delta).$$

Note that $\Delta \sim 2(K_X + L)$.

Let W be a zero-dimensional subscheme defined by $\mathcal{I}(X, \Delta)$. We have the following exact sequence

$$0 \to \mathcal{O}_X(K_X + \Delta + L) \otimes \mathcal{I}(X, \Delta) \to \mathcal{O}_X(K_X + \Delta + L) \to \mathcal{O}_W(K_X + \Delta + L) \to 0.$$

By Nadel vanishing theorem ([20, Theorem 9.4.17])

$$H^1(X, \mathcal{O}_X(K_X + \Delta + L) \otimes \mathcal{I}(X, \Delta)) = 0,$$

and so

$$H^0(X, \mathcal{O}_X(K_X + \Delta + L)) \longrightarrow H^0(W, \mathcal{O}_W(K_X + \Delta + L))$$

is surjective. Since $\dim W = 0$, we get that $K_X + \Delta + L \sim 3(K_X + L)$ is base point free along W, and so it is base point free at q.

Proof of Proposition 3.2.3. Take an arbitrary closed point $q \in W$. We need to show that $q \notin \mathbb{B}(K_X + L)$.

By assumptions, $K_X + L$ defines a finite map outside of its zero dimensional base locus, so there exist divisors $D_1, D_2 \in |K_X + L|$ such that $\dim(D_1 \cap D_2) = 0$. We can assume that $q \in D_1 \cap D_2$ and $q \in \mathrm{Sing}(X)$. Note, that $I_W = I_{D_1} + I_{D_2}$, and D_1, D_2 are Cartier.

By Theorem 1.0.12, we can choose $e > 0$ such that

$$H^1\big(X, \mathcal{O}_X\big((p^e-1)L + M\big) \otimes I_W\big) = 0$$

for any nef Cartier divisor M.

By F-purity (see Proposition 2.1.3 and Proposition 2.2.2), we know that there exists a $\mathbb{Q}$-divisor B such that

$$(p^e - 1)(K_X + B)$$

is Cartier, and

$$\mathrm{Tr}_{X,B} \colon F^e_* \mathcal{O}_X(-(p^e-1)(K_X+B)) \longrightarrow \mathcal{O}_X$$

is surjective at q, for enough divisible $e \gg 0$. If the $\mathbb{Q}$-Gorenstein index of X is indivisible by p, then we can take $B = 0$. By increasing e and decreasing coefficients of B, we may assume that $\frac{1}{2}L - B$ is ample.

Now, take maximal $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geqslant 0}$ such that

$$\mathrm{Tr}_{X,\Delta} \colon F^e_* \mathcal{L} \to \mathcal{O}_X$$

is surjective at the stalk $\mathcal{O}_{X,q}$, where

$$\mathcal{L} := \mathcal{O}_X(-(p^e-1)(K_X+B) - \lambda_1 D_1 - \lambda_2 D_2), \text{ and}$$
$$\Delta := B + \frac{\lambda_1}{p^e-1}D_1 + \frac{\lambda_2}{p^e-1}D_2.$$

We want to show existence of the following diagram:

$$\begin{array}{ccc} F^e_*\mathcal{L} & \longrightarrow & F^e_*\big(\mathcal{L}|_W\big) \\ \downarrow{\scriptstyle \mathrm{Tr}_{X,\Delta}} & & \downarrow \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X,q}/m_q \end{array}$$

To show that such a diagram exists we need to prove that the image of $F^e_*(\mathcal{L} \otimes I_W)$ under $\mathrm{Tr}_{X,\Delta}$ is contained in m_q. This follows from the fact that $I_W = \mathcal{O}(-D_1) + \mathcal{O}(-D_2)$ and from the maximality of λ_1, λ_2. More precisely the image of

$$F^e_*\mathcal{O}_X(-(p^e-1)(K_X+B) - (\lambda_1+1)D_1 - \lambda_2 D_2)$$

must be contained in m_q (analogously for $\lambda_2 + 1$).

So, we tensor this diagram by the line bundle $\mathcal{O}_X(K_X + \Delta + H)$, where

$$H := \Big(2 - \frac{\lambda_1}{p^e - 1} - \frac{\lambda_2}{p^e - 1}\Big)(K_X + L) - K_X - B + N,$$

and take H^0 to obtain the diagram

$$\begin{array}{ccc} H^0\big(X, F^e_* \mathcal{O}_X(K_X + \Delta + p^e H)\big) & \longrightarrow & H^0\big(W, \mathcal{O}_W\big) \\ \downarrow & & \downarrow \\ H^0\big(X, \mathcal{O}_X(K_X + \Delta + H)\big) & \longrightarrow & H^0\big(q, \mathcal{O}_{X,q}/m_q\big), \end{array}$$

Note that $K_X + \Delta + H = 2(K_X + L) + N$ and $K_X + \Delta + p^e H = (p^e - 1)L + M$ for some nef Cartier divisor M. Further, by Theorem 2.1.6, (X, Δ) is log canonical at $q \in X$, and so by Lemma 3.2.5 we get

$$\frac{\lambda_1}{p^e - 1} + \frac{\lambda_2}{p^e - 1} \leqslant 1.$$

Therefore, H is ample.

The right vertical arrow is surjective, since $\mathrm{Tr}_{X,\Delta} \colon F^e_* \mathcal{L} \to \mathcal{O}_X$ is surjective, and $\dim W = 0$. The upper horizontal arrow is surjective by

$$H^1(X, \mathcal{O}_X((p^e - 1)L + M) \otimes I_W) = 0.$$

Thus, the lower horizontal arrow is surjective, and so the proof of base point freeness is completed. □

The following lemma was used in the proof.

Lemma 3.2.5. *Let $(X, a_1 D_1 + a_2 D_2)$ be a log canonical two dimensional pair, such that $a_1, a_2 \in \mathbb{R}_{\geqslant 0}$ and D_1 together with D_2 are Cartier divisors, intersecting at a singular point $x \in X$. Then $a_1 + a_2 \leqslant 1$.*

Of course, the lemma is not true, when x is a smooth point.

Proof. Consider a minimal resolution of singularities $\pi \colon \widetilde{X} \to X$. Write

$$K_{\widetilde{X}} + \Delta_{\widetilde{X}} + \pi^*(a_1 D_1 + a_2 D_2) = \pi^*(K_X + a_1 D_1 + a_2 D_2).$$

Since π is a minimal resolution, we have that $\Delta_{\widetilde{X}} \geqslant 0$.

Take an exceptional curve C over x. Since D_1 and D_2 are Cartier, the coefficient of C in $\Delta_{\widetilde{X}} + \pi^*(a_1 D_1 + a_2 D_2)$ is greater or equal $a_1 + a_2$. Since $(X, \Delta_{\widetilde{X}} + \pi^*(a_1 D_1 + a_2 D_2))$ is log canonical, this concludes the proof of the lemma. □

Now, the proof of the Theorem 1.0.4 is straightforward.

Proof of Theorem 1.0.4. It follows directly from Theorem 2.1.7, Proposition 3.2.2, Proposition 3.2.3 and Theorem 1.0.3. □

3.3 Generalization of the main theorem

In this section we present a technical generalisation of Theorem 1.0.4.

Theorem 3.3.1. *Let X be an F-pure projective surface defined over an algebraically closed field of characteristic $p > 0$. Assume that mK_X is Cartier for some $m \in \mathbb{N}$. Let L be an ample Cartier divisor on X and let N be any nef Cartier divisor. The following holds.*

- *If X is neither of general type nor quasi-elliptic with $\kappa(X) = 1$, then*

  $2mK_X + 8mL + N$ *is base point free, and*

  $8mK_X + 32mL + N$ *is very ample.*

- *If $p = 3$ and X is quasi-elliptic with $\kappa(X) = 1$, then*

  $2mK_X + 12mL + N$ *is base point free, and*

  $8mK_X + 48mL + N$ *is very ample.*

- *If $p = 2$ and X is quasi-elliptic with $\kappa(X) = 1$, then*

  $2mK_X + 16mL + N$ *is base point free, and*

  $8mK_X + 64mL + N$ *is very ample.*

- *If $p \geqslant 3$ and X is of general type, then*

  $4mK_X + 12mL + N$ *is base point free, and*

  $16mK_X + 48mL + N$ *is very ample.*

- *If $p = 2$ and X is of general type, then*

  $4mK_X + 22mL + N$ *is base point free, and*

  $16mK_X + 88mL + N$ *is very ample.*

The bounds are rough. The theorem is a direct consequence of the following proposition.

Proposition 3.3.2. *Let X be a projective surface defined over an algebraically closed field of characteristic $p > 0$. Let L be an ample Cartier divisor on X. Then*

$$\mathbb{B}(aK_X + bL) \subseteq \mathrm{Sing}(X),$$

where

- $a = 1, b = 4$, *if* X *is neither of general type nor quasi-elliptic,*
- $a = 1, b = 6$, *if* X *is quasi-elliptic with* $\kappa(X) = 1$ *and* $p = 3$,
- $a = 1, b = 8$, *if* X *is quasi-elliptic with* $\kappa(X) = 1$ *and* $p = 2$,
- $a = 2, b = 5$, *if* X *is of general type and* $p \geqslant 3$,
- $a = 2, b = 11$, *if* X *is of general type and* $p = 2$.

Proof. It follows from Proposition 3.1.5, by exactly the same proof as of Proposition 3.2.2. □

Proof of Theorem 3.3.1. It follows directly from Theorem 2.1.7, Proposition 3.3.2, Proposition 3.2.3 and Theorem 1.0.3.

□

3.4 Matsusaka-type bounds

The goal of this section is to prove Corollary 1.0.5. The key part of the proof is the following proposition.

Proposition 3.4.1. *Let* A *be an ample Cartier divisor and let* N *be a nef Cartier divisor on a normal projective surface* X. *Then* $kA - N$ *is nef for any*

$$k \geqslant \frac{2A \cdot (H + N)}{A^2}((K_X + 3A) \cdot A + 1) + 1.$$

Proof. The proof is exactly the same as [8, Theorem 3.3]. The only difference is that for singular surfaces, the cone theorem is weaker, so we have $K_X + 3D$ in the statement, instead of $K_X + 2D$. □

The following proof is exactly the same as of [8, Theorem 1.2].

Proof of Proposition 1.0.5. By Theorem 1.0.4, we know that H is very ample. By the above proposition, we know that $kA - (H + N)$ is a nef Cartier divisor. Thus, by Theorem 1.0.4

$$H + (kA - (H + N)) = kA - N$$

is very ample. □

Applying Theorem 3.3.1, we obtain the following.

Corollary 3.4.2. *Let A and N be respectively an ample and a nef Cartier divisor on an F-split projective surface defined over an algebraically closed field of characteristic $p > 0$. Let $m \in \mathbb{N}$ be such that mK_X is Cartier. Then $kA - N$ is very ample for any*

$$k > \frac{2A \cdot (H + N)}{A^2}((K_X + 2A) \cdot A + 1),$$

where

- $H := 8m(K_X + 4A)$, *if X is neither quasi-elliptic with $\kappa(X) = 1$, nor of general type,*
- $H := 8m(K_X + 8A)$, *if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 3$,*
- $H := 8m(K_X + 19A)$, *if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 2$,*
- $H := 8m(2K_X + 8A)$, *if X is of general type and $p \geqslant 3$,*
- $H := 8m(2K_X + 19A)$, *if X is of general type and $p = 2$.*

3.5 Effective vanishing of H^1

The goal of this section is to present the proof of Theorem 1.0.6. It is an unpublished result of Hiromu Tanaka. We are grateful to him for allowing us to attach his proof in our paper.

First, we need the following lemma.

Lemma 3.5.1 ([30, Lemma 10],[16, Section 3.1])**.** *Let C be a smooth projective curve defined over an algebraically closed field of characteristic $p > 0$, let B be an effective $\mathbb{Q}$-divisor, and let A be an ample $\mathbb{Q}$-divisor on C, such that both $K_C + B + A$ and $(p^e - 1)B$ are Cartier for some natural number $e > 0$. Further, suppose that* $\deg B \leqslant 1$. *Then*

$$\mathrm{Tr}^e_{C,B}(K_C + A)\colon H^0\big(C, K_C + B + p^e(K_C + A)\big) \to H^0\big(C, K_C + B + (K_C + A)\big)$$

is surjective.

Proof. This is an immediate consequence of Proposition 2.3.4. □

Proof of Theorem 1.0.6. Assume that $m \neq 1$. The proof for $m = 1$ is analogous. We claim that

$$H^1(X, mK_X + kH + A) \simeq H^1(X, mK_X + (k+1)H + A)$$

for all $k \geqslant 1$. This claim, together with Serre vanishing, concludes the proof.

Since H is very ample, we can assume that it is a smooth irreducible curve such that $\operatorname{Supp} H \cap \operatorname{Sing}(X) = \varnothing$. Take a $\mathbb{Q}$-divisor B such that $(p^e-1)(K_X+B)$ is Cartier for divisible enough $e \gg 0$ and $(m-2)K_X + A - B$ is ample. We can assume, that $\deg B$ is negligibly small, in particular $\deg B \leqslant 1$.

Set $D_k := K_X + H + ((k-1)H + (m-2)K_X + A - B)$. Since, $(m-1)K_X + A$ is nef, D_K is ample. By Lemma 2.2.9, we have the following commutative diagram:

$$\begin{array}{ccccc} H^0\big(X, K_X{+}H{+}B + p^e D_k\big) & \longrightarrow & H^0\big(H, K_H{+}B|_H + p^e D_k|_H\big) & \longrightarrow & H^1\big(X, K_X{+}B + p^e D_k\big) \\ \big\downarrow \mathrm{Tr}_{X,H+B}(D_k) & & \big\downarrow \mathrm{Tr}_{H,B}(D_k) & & \\ H^0\big(X, K_X{+}H{+}B + D_k\big) & \longrightarrow & H^0\big(H, K_H{+}B|_H + D_k|_H\big), & & \end{array}$$

where $K_X + H + B + D_K = mK_X + (k+1)H + A$.

By Serre vanising, $H^1(X, K_X + B + p^e D_k) = 0$ for $e \gg 0$, and so the upper horizontal arrow is surjective. Futher, by the fact that

$$D_k|_H = K_H + ((k-1)H + (m-2)K_X + A - B)|_H,$$

the middle vertical arrow $\mathrm{Tr}_{H,B}(D_k)$ is surjective as well, by Lemma 3.5.1.

Henceforth, the lower horizontal arrow is surjective. Since

$$H^1(H, K_H + D_k|_H) = 0,$$

the claim holds by considering the long exact sequence of cohomologies. □

CHAPTER 4

Globally F-regular varieties

In this chapter we consider globally F-regular varieties. First, we will try to explain that globally F-regular varieties are positive characteristic counterparts of klt log Fano varieties. Second, we will discuss F-spliteness in the case of surfaces.

We start by recalling basic properties of globally F-regular varieties. The theorem below underpins the theory.

Theorem 4.0.1 ([31, Theorem 4.3])**.** *Let (X, Δ) be a globally F-regular variety. Then, there exists an effective divisor $\widehat{\Delta}$ such that $\Delta \leqslant \widehat{\Delta}$ and $(X, \widehat{\Delta})$ is a klt log Fano pair.*

Further, there is the following correspondence between local and global F-regularity.

Proposition 4.0.2 ([31, Proposition 5.3])**.** *Let (X, Δ) be a log pair with a fixed embedding $X \subseteq \mathbb{P}^n$. Then (X, Δ) is globally F-regular (F-split) if and only if $(\mathrm{Cone}(X), \mathrm{Cone}(\Delta))$ is strongly F-regular (F-pure, respectively).*

Here, $\mathrm{Cone}(X)$ denotes the cone of X inside $\mathbb{A}^{n+1}$.

Proposition 4.0.3 ([43])**.** *Let (X, Δ) be a log Fano pair, and let $m \in \mathbb{N}$ be such that $-m(K_X + \Delta)$ is very ample. Consider an embedding $X \subseteq \mathbb{P}^n$ induced by $-m(K_X + \Delta)$. Then $(\mathrm{Cone}(X), \mathrm{Cone}(\Delta))$ is kawamata log terminal.*

The following two proofs are copied from the soon-to-be-published paper of Cascini, Tanaka and Witaszek.

Lemma 4.0.4. *Let (Y, Δ) be a globally F-regular log pair, let X be a normal variety, and let $f\colon Y \to X$ be a proper birational morphism. Then, $(X, f_*\Delta)$ is globally F-regular.*

Proof. For every divisor D on X and a sufficiently divisible $e > 0$, the Frobenius morphism

$$\mathcal{O}_Y \longrightarrow \mathcal{O}_Y(\ulcorner (p^e - 1)\Delta \urcorner + \pi^* D)$$

splits. Let $U \subseteq X$ be a subscheme of codimension at least two, such that $f\colon f^*U \to U$ is an isomorphism. By the above, we get that the Frobenius morphism

$$\mathcal{O}_U \longrightarrow \mathcal{O}_U(\ulcorner (p^e - 1) f_*\Delta \urcorner + D)$$

splits. Since U is of codimension at least two, the lemma follows. □

Lemma 4.0.5 (cf. [22, Propostion 2.11])**.** *Let (Y, Δ_Y) and (X, Δ_X) be normal log pairs. Suppose we have a proper birational morphism $\pi\colon Y \to X$, such that*

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta_X).$$

Then (X, Δ_X) is globally F-regular if and only if (Y, Δ_Y) is globally F-regular.

Proof. If (X, Δ_X) is globally F-regular, then (Y, Δ_Y) is globally F-regular by [22, Propostion 2.11]. The other direction follows by Lemma 4.0.4. □

4.1 Reduction modulo p of a log Fano pair

In this section, we consider the following theorem.

Theorem 4.1.1 ([31, Theorem 5.1])**.** *Let (X, Δ) be a log Fano pair defined over an algebraically closed field of characteristic zero. Then, for $p \gg 0$, the reduction of (X, Δ) modulo p is globally F-regular.*

First, we present a proof of this theorem from [31]. After that, we give a new proof of it, made up by the author, in the case when X is smooth and Δ is a simple normal crossing divisor.

4.1.1 Reduction modulo p in the local setting

By Proposition 4.0.2 and Proposition 4.0.3, the proof of Theorem 4.1.1, follows from the following.

Theorem 4.1.2. *Let (X, Δ) be a klt pair defined over an algebraically closed field of characteristic zero. Then, for $p \gg 0$, the reduction of (X, Δ) modulo p is strongly F-regular.*

Before starting the proof, we need to state Hara's surjectivity lemma.

Theorem 4.1.3 (Hara's surjectivity lemma [5, Lemma 23.1])**.** *Suppose that R is a ring of characteristic zero, $\pi\colon \widetilde{X} \to \operatorname{Spec} R$ is a log resolution of singularities, D is a π-ample $\mathbb{Q}$-divisor with simple normal crossing support. We reduce this setup to characteristic $p \gg 0$. Then, the natural map*

$$(F^e)^\wedge\colon H^0(\widetilde{X}, F^e_*\omega_{\widetilde{X}}(\lceil p^e D\rceil)) = \operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}(F^e_*\mathcal{O}_{\widetilde{X}}(\lfloor -p^e D\rfloor), \omega_{\widetilde{X}}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}(\lfloor -D\rfloor), \omega_{\widetilde{X}}) = H^0(\widetilde{X}, \omega_{\widetilde{X}}(\lceil D\rceil))$$

surjects.

Note, that $(F^e)^\wedge$ is obtained by taking the dual of the Frobenius $F^e\colon \mathcal{O}_{\widetilde{X}} \to F^e_*\mathcal{O}_{\widetilde{X}}$, tensoring by $\omega_{\widetilde{X}}$ and restricting to certain subsheaves of considered sheaves.

Proof of Theorem 4.1.2. For simplicity, we will show only that (X, Δ) is F-pure.

Since X is local affine, by perturbing Δ we may assume that $(p^e-1)(K_X+\Delta) \sim 0$ for divisible enough $e > 0$.

Replace (X, Δ) by its reduction modulo p for some $p \gg 0$. We need to show that $\operatorname{Tr}^e_{X,\Delta}$ is surjective (see Proposition 2.2.2). Let $\pi\colon \widetilde{X} \to X$ be a log resolution of singularities and let $\widetilde{\Delta}$ be such that

$$K_{\widetilde{X}} + \widetilde{\Delta} = \pi^*(K_X + \Delta).$$

Consider the following commutative diagram (c.f. Section 2.2):

$$\begin{array}{ccc} \pi_*F^e_*\mathcal{O}_{\widetilde{X}}((p^e-1)\widetilde{\Delta}) & \xrightarrow{\operatorname{Tr}^e_{\widetilde{X},\widetilde{\Delta}}} & \pi_*\mathcal{O}_{\widetilde{X}} \\ \downarrow \simeq & & \downarrow \simeq \\ F^e_*\mathcal{O}_X((p^e-1)\Delta) & \xrightarrow{\operatorname{Tr}^e_{X,\Delta}} & \mathcal{O}_X. \end{array}$$

Since $\lfloor \Delta \rfloor = 0$, this induces:

$$\begin{array}{ccc} \pi_*F^e_*\mathcal{O}_{\widetilde{X}}(\lceil -\widetilde{\Delta}\rceil) & \xrightarrow{\operatorname{Tr}^e_{\widetilde{X},\widetilde{\Delta}}} & \pi_*\mathcal{O}_{\widetilde{X}}(\lceil -\widetilde{\Delta}\rceil) \\ \downarrow \subseteq & & \downarrow \simeq \\ F^e_*\mathcal{O}_X & \xrightarrow{\operatorname{Tr}^e_{X,\Delta}} & \mathcal{O}_X. \end{array}$$

Since (X, Δ) is klt, $\lceil -\widetilde{\Delta}\rceil$ is effective, and so the right vertical arrow is an isomorphism.

Take a π-anti-ample effective exceptional $\mathbb{Q}$-divisor E such that $\lceil -\widetilde{\Delta} - E\rceil = \lceil -\widetilde{\Delta}\rceil$. Since E is exceptional, the diagram above induces

$$\begin{array}{ccc} \pi_* F^e_* \mathcal{O}_{\widetilde{X}}(\lceil -\widetilde{\Delta} - p^e E \rceil) & \xrightarrow{\mathrm{Tr}^e_{\widetilde{X},\widetilde{\Delta}}} & \pi_* \mathcal{O}_{\widetilde{X}}(\lceil -\widetilde{\Delta} - E \rceil) \\ \downarrow \subseteq & & \downarrow \simeq \\ F^e_* \mathcal{O}_X & \xrightarrow{\mathrm{Tr}^e_{X,\Delta}} & \mathcal{O}_X. \end{array}$$

In order to show surjectivity of the trace map $\mathrm{Tr}^e_{X,\Delta}$, it is enough to show that

$$\mathrm{Tr}^e_{\widetilde{X},\widetilde{\Delta}} \colon \pi_* F^e_* \mathcal{O}_{\widetilde{X}}(\lceil -\widetilde{\Delta} - p^e E \rceil) \to \pi_* \mathcal{O}_{\widetilde{X}}(\lceil -\widetilde{\Delta} - E \rceil)$$

is surjective. Since $-\widetilde{\Delta} = K_{\widetilde{X}} - \pi^*(K_X + \Delta)$ and $(p^e - 1)(K_X + \Delta) \sim 0$, we get isomorphisms

$$\begin{array}{ccc} \pi_* F^e_* \mathcal{O}_{\widetilde{X}}(\lceil -\widetilde{\Delta} - p^e E \rceil) & \xrightarrow{\mathrm{Tr}^e_{\widetilde{X},\widetilde{\Delta}}} & \pi_* \mathcal{O}_{\widetilde{X}}(\lceil -\widetilde{\Delta} - E \rceil) \\ \downarrow \simeq & & \downarrow \simeq \\ \pi_* F^e_* \omega_{\widetilde{X}}(\lceil -p^e \pi^*(K_X + \Delta) - p^e E \rceil) & \xrightarrow{(F^e)^\wedge} & \pi_* \omega_{\widetilde{X}}(\lceil -\pi^*(K_X + \Delta) - E \rceil), \end{array}$$

where $(F^e)^\wedge$ is as in Theorem 4.1.3. The diagram commutes, because both $(F^e)^\wedge$ and $\mathrm{Tr}^e_{\widetilde{X},\widetilde{\Delta}}$ are obtained by taking the dual of the Frobenius $F^e \colon \mathcal{O}_{\widetilde{X}} \to F^e_* \mathcal{O}_{\widetilde{X}}$, tensoring by $\omega_{\widetilde{X}}$ and restricting to certain subsheaves of considered sheaves.

Applying Theorem 4.1.3 with $D = -\pi^*(K_X + \Delta) - E$, we get that $(F^e)^\wedge$ is surjective, which concludes the proof. □

4.1.2 Reduction modulo p in the global setting

Here, we give a new proof of Theorem 4.1.1 in the case when X is smooth and Δ is a simple normal crossing divisor. The advantage of this proof is that it does not involve any transition to a local problem. As before, for simplicity, we only show global F-spliteness.

In [44], one can find a proof of Theorem 4.1.1 when X is a smooth Fano variety ($\Delta = 0$). The following proposition is motivated by their chain of ideas.

Proposition 4.1.4. *Let (X, Δ) be a smooth klt n-dimensional pair over an algebraically closed field of characteristic $p > 0$. Assume that Δ is an simple normal crossing divisor whose coefficients have denominators indivisible by p. Set $\Delta_k = \frac{1}{p^k}\lfloor p^k \Delta \rfloor$.*

If for all $k \geqslant 0$, we have

- $H^i(X, \Omega^{n-i+1}(\log E)(p^k(K_X + \Delta_k))) = 0$ *for* $0 \leqslant i \leqslant n-2$, *and*
- $H^i(X, \Omega^{n-i}(\log E)(p^{k+1}(K_X + \Delta_{k+1}))) = 0$ *for* $0 \leqslant i \leqslant n-1$

then (X, Δ) is globally F-split.

Observe that for $e > 0$ such that $(p^e - 1)\Delta$ is a Weil divisor, we have $p^e\Delta_e = (p^e - 1)\Delta$.

Remark 4.1.5. We do not assume that (X, Δ) is a log Fano variety. In particular, we get, that if those cohomological criterions are satisfied, then (X, Δ) is a log Fano variety.

Remark 4.1.6. The first condition holds by Kodaira-Akizuki-Nakano, and the second one by Serre vanishing if X is a reduction mod $p \gg 0$ of a log Fano variety of characteristic zero. In particular, this shows Theorem 4.1.1, when X is smooth and Δ is a simple normal crossing divisor.

Proof of Proposition 4.1.4. Take $0 \leqslant k \leqslant e - 1$. By the enhanced Cartier isomorphism (see Theorem 2.3.3), with $B = p^{k+1}(\Delta_{k+1} - \Delta_k)$, we get the following exact sequences

$$0 \longrightarrow F_*Z^i \longrightarrow F_*\Omega^i_X(\log E)(p^{k+1}(\Delta_{k+1} - \Delta_k)) \longrightarrow F_*B^{i+1} \longrightarrow 0$$
$$0 \longrightarrow F_*B^i \longrightarrow F_*Z^i \longrightarrow \Omega^i_X(\log E) \longrightarrow 0,$$

where $Z^\bullet$ and $B^\bullet$ are complexes of cycles and boundaries, respectively.

We tensor it by $p^k(K_X + \Delta_k)$, and get:

$$0 \to F_*Z^i(p^{k+1}(K_X + \Delta_k) \to F_*\Omega^i_X(\log E)(p^{k+1}(K_X + \Delta_{k+1})) \to F_*B^{i+1}(p^{k+1}(K_X + \Delta_k)) \to 0$$
$$0 \to F_*B^i(p^{k+1}(K_X + \Delta_k)) \to F_*Z^i(p^{k+1}(K_X + \Delta_k)) \to \Omega^i_X(\log E)(p^k(K_X + \Delta_k)) \to 0.$$

For $i = 0$, the first sequence gives us:

$$0 \longrightarrow \mathcal{O}_X(p^k(K_X+\Delta_k)) \longrightarrow F_*\mathcal{O}_X(p^{k+1}(K_X+\Delta_{k+1})) \longrightarrow F_*(B^1(p^{k+1}(K_X+\Delta_k))) \longrightarrow 0.$$

By Lemma 4.1.7, to prove that (X, Δ) is globally F-split, it is enough to show that

$$H^{n-1}(X, F_*B^1(p^{k+1}(K_X + \Delta_k))) = 0$$

for all $k \geqslant 0$. By the second exact sequence, this follows from the vanishing of:

$$H^{n-2}(X, \Omega^1_X(\log E)(p^k(K_X + \Delta_k))) = 0, \text{ and}$$
$$H^{n-1}(X, F_*Z^1(p^{k+1}(K_X + \Delta_k))) = 0.$$

Now, by the first sequence, the vanishing of the latter group follows from the vanishing of:

$$H^{n-1}(X, F_*(\Omega^1_X(\log E)(p^{k+1}(K_X + \Delta_{k+1})))) = 0, \text{ and}$$
$$H^{n-2}(X, F_*B^2(p^{k+1}(K_X + \Delta_k))) = 0.$$

Recursively proceeding in same way concludes the proof. □

Lemma 4.1.7. *Let (X, Δ) be a log pair of dimension n. Assume that*

$$H^n(X, \omega_X) \to H^n(X, F^e_* \omega_X^{p^k}(\lfloor (p^e - 1)\Delta \rfloor))$$

is injective for enough divisible $e \gg 0$, where the map is induced by the Frobenius. Then (X, Δ) is globally F-split.

Proof. It is enough to show that the evaluation by the Frobenius map

$$\mathrm{Hom}(F^e_* \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor), \mathcal{O}_X) \longrightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X)$$

is surjective. By Serre duality, it is equivalent to the injectivity of

$$H^n(X, \omega_X) \to H^n(X, \omega_X \otimes F^e_* \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor)).$$

□

Since the exceptional locus of a minimal resolution of a log terminal singularity is a simple normal crossing divisor, the above proposition and Lemma 4.0.5 imply Proposition 1.0.7.

4.2 Log del Pezzo pairs

4.2.1 Bounds on log del Pezzo pairs

The goal of this subsection is to prove Theorem 1.0.8. We need the following facts.

Proposition 4.2.1. *Let (X, Δ) be a weak ϵ-klt log del Pezzo pair for $\epsilon > 0$. Let $\pi \colon \overline{X} \to X$ be the minimal resolution. Then*

- *(a)* $0 \leqslant (K_X + \Delta)^2 \leqslant \max(64, 8/\epsilon + 4)$,
- *(b)* $\mathrm{rk}\,\mathrm{Pic}(\overline{X}) \leqslant 128(1/\epsilon)^5$
- *(c)* $2 \leqslant -E^2 \leqslant 2/\epsilon$ *for any exceptional curve E of $\pi \colon \overline{X} \to X$*
- *(d) If m is a $\mathbb{Q}$-factorial index at some point $x \in X$, then*

$$m \leqslant 2(2/\epsilon)^{128/\epsilon^5}.$$

Proof. Point (a) follows from [7, Theorem 4.4]. Points (b) and (c) follow from [28, Theorem 1.8] and [28, Lemma 1.2], respectively. Last, (d) follows from the fact that the Cartier index of a divisor divides the determinant of the intersection matrix of the minimal resolution of a singularity (see also the paragraph below [7, Theorem A]). □

Further, we need to prove the following:

Lemma 4.2.2. *Let (X, Δ) be a weak klt log del Pezzo pair of Cartier index m. Then*

1. $0 \leqslant (K_X + \Delta)K_X \leqslant 3m \max(64, 8m + 4)$, *and*
2. $|K_X^2| \leqslant 128m^5(2m - 1)$.

The *Cartier index* of (X, Δ) is the minimal number $m \in \mathbb{N}$ such that $m(K_X + \Delta)$ is Cartier. If (X, Δ) is klt, then it must be $1/m$-klt.

Proof. The non-negativity in (1) is clear, since

$$\big(K_X + \Delta\big)K_X = \big(K_X + \Delta\big)^2 - \big(K_X + \Delta\big)\Delta \geqslant 0.$$

Let $\pi \colon \overline{X} \to X$ be the minimal resolution of singularities of X. Let $\Delta_{\overline{X}}$ be such that $K_{\overline{X}} + \Delta_{\overline{X}} = \pi^*\big(K_X + \Delta\big)$. Note that

$$\big(K_{\overline{X}} + \Delta_{\overline{X}}\big)K_{\overline{X}} = \big(K_X + \Delta\big)K_X.$$

By Lemma 3.2.1, $K_{\overline{X}} - 3m\big(K_{\overline{X}} + \Delta_{\overline{X}}\big)$ is nef, and so

$$\big(K_{\overline{X}} + \Delta_{\overline{X}}\big) \cdot \big(K_{\overline{X}} - 3m\big(K_{\overline{X}} + \Delta_{\overline{X}}\big)\big) \leqslant 0.$$

This, together with (a) in Proposition 4.2.1, implies (1).

To prove (2), we proceed as follows. First, by (b) in Proposition 4.2.1, we have $\operatorname{rk}\operatorname{Pic}(\overline{X}) \leqslant 128m^5$, and so $-9 \leqslant -K_{\overline{X}}^2 \leqslant 128m^5$. Indeed, the self intesection of the canonical bundle on a minimal model of a rational surface is 8 or 9, and each blow-up decreases it by one.

Write

$$K_{\overline{X}} + \sum a_i E_i = \pi^* K_X,$$

where E_i are the exceptional divisors of π. Notice, that since $\overline{X} \to X$ is minimal and X is klt, we have $0 \leqslant a_i < 1$. By applying (b) and (c) from Proposition 4.2.1, we obtain

$$\begin{aligned} |K_X^2| &= \big|\big(K_{\overline{X}} + \sum a_i E_i\big) \cdot K_{\overline{X}}\big| \\ &\leqslant \big|K_{\overline{X}}\big|^2 + 128m^5\big(|E_i|^2 - 2\big) \\ &\leqslant 128m^5(2m - 1). \end{aligned}$$

□

Proof of Theorem 1.0.8. By Theorem 1.0.4 and Theorem 1.0.6, the divisor $H := aK_X - b(K_X + \Delta)$ is very ample, and $H^i(X, H) = 0$ for $i > 0$. The bounds on H^2, $H \cdot K_X$ and $H \cdot \Delta$ follow from Proposition 4.2.1 and Lemma 4.2.2.

Since $H^0(X, H) = \chi(H)$, the last part follows from the Riemann-Roch theorem.

□

4.2.2 Proof of Theorem 1.0.8

The goal of this section is to prove Theorem 1.0.8. The idea of the proof is to construct a $\mathbb{Z}$-bounded family of ϵ-klt log del Pezzo pairs, and use Theorem 4.1.1.

The construction of the bounded family is standard, but we review it for the convenience of the reader.

Proposition 4.2.3 (cf. [2, Lemma 1.2]). *Let $I \subseteq [0,1] \cap \mathbb{Q}$ be a finite set. Take $\epsilon > 0$. Then there exists a bounded family $\mathcal{X} \to Z$, where Z is a scheme of finite type over* $\operatorname{Spec}\mathbb{Z}$, *together with a $\mathbb{Q}$-divisor $\Xi \subseteq \mathcal{X}$ satisfying the following property.*

If (X,B) is an ϵ-klt log del Pezzo pair defined over an algebraically closed field K and such that the coefficients of B are contained in I, then it is a fiber of $(\mathcal{X},\Xi) \to Z$ up to base change of the field. More precisely, there exists a, nonnecessarily closed, point $q \in Z$ such that

$$X \simeq \mathcal{X}_q \otimes_{\mathcal{O}_{Z,q}/q} K, \text{ and}$$
$$\Delta \simeq \Xi_q \otimes_{\mathcal{O}_{Z,q}/q} K,$$

where the two isomorphisms are compatible. By abuse of notation, q also denotes the prime ideal corresponding to the chosen point.

Proof. Let (X,B) be an ϵ-klt log del Pezzo pair defined over an algebraically closed field K such that the coefficients of B are contained in I. Take $b(I,\epsilon)$ and H as in Theorem 1.0.8.

By representability of the Hilbert functor over $\mathbb{Z}$, there exists $n \in \mathbb{N}$ and a finitely generated subscheme $\mathcal{H} \subseteq \operatorname{Hilb}_{\mathbb{Z}} \mathbb{P}^n$ together with a universal family

$$\begin{array}{ccc} \mathcal{U} & \subseteq & \operatorname{Hilb}_{\mathbb{Z}}\mathbb{P}^n \times \mathbb{P}^n \\ \downarrow & & \\ \mathcal{H} & \subseteq & \operatorname{Hilb}_{\mathbb{Z}}\mathbb{P}^n \end{array}$$

such that X is a fiber of it up to base change of the field. Note that the choice of the scheme $\mathcal{H}$ depends only on $b(I,\epsilon)$. Now, we need to enlarge the family to be able to construct the divisor Ξ.

Let m be least common multiple of denominators of the elements in I. In particular, mB is Weil. Since, it is easier to work with Weil divisors, we focus on constructing $m\Xi$.

Let $\operatorname{Hilb}_{\mathcal{H}}(\mathcal{U})$ be the relative Hilbert scheme of subschemes of fibers in $\mathcal{U} \to \mathcal{H}$, and let $\widetilde{\mathcal{H}} \subseteq \operatorname{Hilb}_{\mathcal{H}}(\mathcal{U})$ be a finite type subscheme parametrizing Weil divisors D in those fibers such that

$$\deg \mathcal{O}_{\mathbb{P}^n}(1)|_D \leqslant mb(I,\epsilon).$$

Let $\widetilde{U}_{\mathrm{div}} \to \widetilde{\mathcal{H}}$ be the universal family over $\widetilde{\mathcal{H}}$. Its fibers are exactly one-dimensional schemes in $\mathbb{P}^n$ lying on appropriate surfaces, and satisfying the above inequality. Consider the following diagram:

$$\begin{array}{ccccc} \widetilde{U}_{\mathrm{div}} & \xrightarrow{\ \phi\ } & \mathcal{U} \times_{\mathcal{H}} \widetilde{\mathcal{H}} & \longrightarrow & \mathcal{U} \\ & \searrow & \downarrow & & \downarrow \\ & & \widetilde{\mathcal{H}} & \longrightarrow & \mathcal{H} \end{array}$$

where ϕ is the natural map coming from the construction of the relative Hilbert scheme. We take $\mathcal{X} := \mathcal{U} \times_{\mathcal{H}} \overline{\mathcal{H}}$ and $\Xi := \frac{1}{m}\phi(\overline{U}_{\mathrm{div}})$. By restricting to a suitable open subscheme of $\mathcal{H}$, we may assume that Ξ is a divisor.

□

Since ampleness of line bundles and having klt singularties are both open conditions, we can assume that all fibers of the constructed family $(\mathcal{X}, \Xi) \to \operatorname{Spec} R$ are klt log del Pezzo pairs.

Proof of Theorem 1.0.9. Let $(\mathcal{X}, \Xi) \to Z$ be the bounded family as above. By covering Z by a finite number of open sets, it is enough to show the proposition only for those log del Pezzo pairs which are isomorphic to fibers of $\mathcal{X}$ over some open set $\operatorname{Spec} R \subseteq Z$ up to base change of the field.

The ring R is a finitely generated $\mathbb{Z}$-module, and so $\mathcal{X}|_{\operatorname{Spec} R} \to \operatorname{Spec} R$ is a model of the log del Pezzo pair $(\mathcal{X}_\zeta, \Xi_\zeta)$ where ζ is the generic point of $\operatorname{Spec} R$. In particular, by Theorem 4.1.1 there exists an open dense subset $U \subseteq \operatorname{Spec} R$ such that the closed fibers of $\mathcal{X}|_U \to U$ are globally F-regular after a base change to an algebraically closed field. Further, by applying an induction on the components of $\operatorname{Spec} R \backslash U$ which are dominant over $\operatorname{Spec} \mathbb{Z}$, we may assume that $\operatorname{Spec} R \backslash U$ is not dominant over $\operatorname{Spec} \mathbb{Z}$.

Thus for $p \gg 0$, the closed fibers over $\operatorname{Spec} R$ are globally F-regular. The global F-regularity of fibers over nonclosed points of $\operatorname{Spec} R$ follows by openness of global F-regularity in equi-characteristic families. □

Notice, that existence of the $\mathbb{Z}$-bounded family has a lot of consequences. For example, for $p_0 \gg 0$ depending on m, all log del Pezzo pairs (X, Δ) of Cartier index m defined over an algebraically closed field of characteristic $p > p_0$ are good reductions from characteristic zero, and in particular they satisfy

$$H^1(X, L) = 0,$$

for any nef divisor L. In the last section of this article, we show that one can take $p_0 = 2m^2$.

4.3 Effective bounds for vanishing

The goal of this section is to prove Theorem 1.0.10.

First, we state the following effective bound for Kodaira vanishing by Langer.

Theorem 4.3.1 ([45, Theorem 2.22])**.** *Let L be a big and nef Cartier divisor on a smooth projective variety X. If $H^0(X, \Omega_X(-p^m L)) = 0$ for all $m \geqslant 1$, then $H^1(X, -L) = 0$.*

Langer also proved:

Proposition 4.3.2 ([45, Theorem 2.22])**.** *Let L be a big and nef Cartier divisor on a smooth projective variety X. Then $H^1(X, -mL) = 0$ for $m \gg 0$.*

We need the following generalization of Theorem 4.3.1 to the log case.

Proposition 4.3.3. *Let (X, Δ) be log pair such that X is smooth, $\lfloor \Delta \rfloor = 0$ and $E := \mathrm{Supp}(\Delta)$ is a simple normal crossings divisor. Let $K_X + \Delta + L$ be a Cartier divisor on X, where L is a big and nef $\mathbb{Q}$-divisor. If*

$$H^0(X, \Omega^1_X(\log E)(-p^k L - \Delta')) = 0$$

for all $k \geqslant 1$ and all $\mathbb{Q}$-divisors $\Delta' \geqslant 0$ such that

- $\lfloor \Delta' \rfloor = 0$,
- $\mathrm{Supp}(\Delta') \subseteq E$, *and*
- $p^k L + \Delta'$ *is Cartier,*

then

$$H^1(X, K_X + \Delta + L) = 0.$$

Proof. By the generalized Cartier isomorphism (Theorem 2.3.3), we get the following exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X(B) \to F_*(\Omega_X(\log E)(B)).$$

Set $B := \lfloor p\Delta \rfloor$. After tensoring this sequence by $-L - \Delta$, we get

$$0 \longrightarrow \mathcal{O}_X(-L - \Delta) \longrightarrow F_*\mathcal{O}_X(-pL - \Delta_1) \to F_*(\Omega_X(\log E)(-pL - \Delta_1)),$$

where $\Delta_1 = \{p\Delta\} := p\Delta - \lfloor p\Delta \rfloor$.

By Serre duality $H^1(X, K_X + \Delta + L) = H^1(-L - \Delta)$. By the above sequence, in order to show the vanishing of this cohomology group, it is sufficient to prove that

$$H^1(X, F_*\mathcal{O}_X(-pL - \Delta_1)) = 0, \text{ and,}$$
$$H^0(X, F_*(\Omega_X(\log E)(-pL - \Delta_1))) = 0.$$

The latter holds by our assumption. To prove the former, we repeat the same argument with L, Δ replaced by pL, Δ_1.

Set recursively $\Delta_k = \{p\Delta_{k-1}\}$ for $k \geqslant 2$. By repeating this procedure many times, we see that it is enough to show

$$H^1(X, -p^k L - \Delta_k) = 0$$

for some $k \geqslant 1$. Notice that $\Delta_k = 0$ for $k \gg 0$. Indeed, for $a \in \mathbb{Q} \cap (0, 1)$, taking $\{pa\}$ removes the first digit in the base-p expansion, and shifts the expansion "left". Since any rational number has a finite base-p expansion, this algorithm must terminate after a finite number of steps.

Now, the vanishing of $H^1(X, -p^k L)$ for $k \gg 0$ follows from Proposition 4.3.2. $\square$

Now, we can prove Theorem 1.0.10.

Proof of Theorem 1.0.10. Let $\pi\colon \widetilde{X} \to X$ be a minimal resolution of singularities. Take a divisor Δ so that

$$K_{\widetilde{X}} + \Delta = \pi^* K_X.$$

By the classification of klt singularities, Δ is a simple normal crossing divisor. Let $E = \mathrm{Supp}(\Delta)$.

By [26], klt surface singularities are rational, and hence it is enough to show that $H^1(\widetilde{X}, \pi^* L) = 0$. Set $M = \pi^* L - (K_{\widetilde{X}} + \Delta)$. Note that M is big and nef. By Proposition 4.3.3, in order to prove our vanishing, it is enough to verify that

$$H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log E)(-p^k M - \Delta')) = 0$$

for all $k \geqslant 1$ and all $\mathbb{Q}$-divisors $\Delta' \geqslant 0$ such that

- $\lfloor \Delta' \rfloor = 0$,
- $\mathrm{Supp}(\Delta') \subseteq E$, and
- $p^k L + \Delta'$ is Cartier.

Set $N = p^k M + \Delta'$. By [19, Lemma 4.2.4], we have an inclusion:

$$\Omega^1_{\widetilde{X}}(\log E) \subseteq \Omega^1_{\widetilde{X}}(E),$$

so it is enough to show that $H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}(-N + E)) = 0$.

Let $\rho\colon \widetilde{X} \to \widetilde{X}_{\min}$ be a minimal model map. It is enough to show that $H^0(\widetilde{X}_{\min}, \Omega^1_{\widetilde{X}_{\min}}(-\rho_* N + \rho_* E)) = 0$, where the pushforward is ment as an operation on divisors.

We need to consider two cases.

Case 1 $\widetilde{X}_{\min} \simeq \mathbb{P}^2$.

By the Euler exact sequence, we have a natural inclusion

$$\Omega^1_{\mathbb{P}^2}(-\rho_* N + \rho_* E) \subseteq \mathcal{O}_{\mathbb{P}^2}(-\rho_* N + \rho_* E - H)^{\oplus 3},$$

where H is a divisor of a line on $\mathbb{P}^2$. To get a desired vanishing, it is enough to show that $(-\rho_* N + \rho_* E - H) \cdot H < 0$.

We have that

$$-(K_{\widetilde{X}} + \Delta) \cdot \rho^* H \geqslant 0,$$

and so

$$2 = -K_{\widetilde{X}} \cdot \rho^* H \geqslant \Delta \cdot \rho^* H \geqslant \frac{1}{m} E \cdot \rho^* H,$$

since $m\Delta$ is a $\mathbb{Z}$-divisor supported on E. Now,

$$(-\rho_* N + \rho_* E - H) \cdot H < -p^k M \cdot \rho^* H + E \cdot \rho^* H \leqslant \frac{-p^k}{m} + 2m \leqslant 0,$$

where the sublast inequality followed from the fact that mM is Cartier.

Case 2 $\widetilde{X}_{\min} \simeq \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ for $n \leqslant 0$.

Let f be the fiber of the natural projection to $\mathbb{P}^1$, and let C be the normalized section. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}_{\min}}(-2f) \longrightarrow \Omega^1_{\widetilde{X}_{\min}} \longrightarrow \Omega^1_{\widetilde{X}_{\min}/\mathbb{P}^1} \longrightarrow 0,$$

and a natural inclusion coming from the Euler exact sequence

$$\Omega^1_{\widetilde{X}_{\min}/\mathbb{P}^1} \subseteq \mathcal{O}_{\widetilde{X}_{\min}}(-C_0) \oplus \mathcal{O}_{\widetilde{X}_{\min}}(-C_0 + nf).$$

Since an effective divisor has a nonnegative intersection with f, mixing the above, we see that it is enough to show

$$(-N + E) \cdot \rho^* f < 0.$$

As before, we have

$$2 = -K_{\tilde{X}} \cdot \rho^* f \geqslant \Delta \cdot \rho^* f \geqslant \frac{1}{m} E \cdot \rho^* f.$$

Hence

$$(-N + E) \cdot \rho^* f < -p^k M \cdot \rho^* f + E \cdot \rho^* f \leqslant \frac{-p^k}{m} + 2m \leqslant 0.$$

$\square$

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