THE FLOW ASSOCIATED TO WEAKLY DIFFERENTIABLE VECTOR FIELDS

GIANLUCA CRIPPA
NORTHWESTERN UNIVERSITY, MAY 14–18, 2007

In these seminars we will try to give an account of some recent literature on the *ordinary differential equation*

$$\begin{cases} \dot\gamma(t) = b(t,\gamma(t)) \\ \gamma(0) = x \end{cases} \qquad \gamma : [0,T] \to \mathbb{R}^d$$

and the closely related *transport equation*

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot \nabla_x u(t,x) = 0 \\ u(0,\cdot) = \bar u \end{cases} \qquad u : [0,T]\times \mathbb{R}^d \to \mathbb{R}$$

under weak differentiability assumptions on the vector field $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$. We will start by recalling the classical results of the Cauchy-Lipschitz theory (in which Lipschitz regularity of b is assumed). Then we will tackle the question of the well-posedness of the PDE, illustrating the importance of the notion of renormalized solution. The well-posedness of the ODE will be first presented in an abstract setting, relying on the connection with the well-posedness of the PDE. Finally we will show with some detail a new and simpler a-priori estimates approach to the Lagrangian problem.

A tentative plan of the lectures follows. Various reference material, including lecture notes, can be found starting from the web-page `http://cvgmt.sns.it`.

First talk. (1 hour) Recalls on the theory in the smooth framework. The renormalization property. How this property implies well-posedness of the PDE, heuristically and rigorously.

Second talk. (2 hours) Depauw's counterexample. An abstract characterization of the renormalization property. DiPerna–Lions regularization scheme. Various renormalization results, in particular DiPerna–Lions and Ambrosio.

Third talk. (1 hour) The link between PDE and ODE out of the smooth framework. The concept of regular Lagrangian flow.

Fourth and fifth talk. (2 hours + 1 hour) Estimates for the ODE with Sobolev vector field and a purely Lagrangian approach to the theory of regular Lagrangian flows.

References

[1] L. Ambrosio: *Transport equation and Cauchy problem for BV vector fields.* Invent. Math., **158** (2004), 227–260.

[2] L. Ambrosio & G. Crippa: *Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields.* To appear in the series UMI Lecture Notes (available at http://cvgmt.sns.it).

[3] F. Bouchut & G. Crippa: *Uniqueness, Renormalization, and Smooth Approximations for Linear Transport Equations.* SIAM J. Math. Anal., **38** (2006), 1316–1328.

[4] G. Crippa & C. De Lellis: *Estimates and regularity results for the DiPerna–Lions flow.* Preprint, 2006 (available at http://cvgmt.sns.it). Accepted by J. Reine Angew. Math.

[5] C. De Lellis: *Ordinary differential equations with rough coefficients and the renormalization theorem of Ambrosio.* Sèminaire Bourbaki, mars 2007.

[6] N. Depauw: *Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan.* C.R. Math. Sci. Acad. Paris, **337** (2003), 249–252.

[7] R. J. DiPerna & P. L. Lions: *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math., **98** (1989), 511–547.

Gianluca Crippa, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail address: g.crippa@sns.it