MATH 218 PROBLEM SET 1

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The exercises in this problem set regard distribution theory and Fourier theory. They can be completed with material covered up to the end of Week 2 (Lecture 4).

Problem 1: Let $\phi \in C_c^\infty(U)$ where $U \subset \mathbb{R}^n$ is open. For $h \in \mathbb{R}^n$, let

$$\tau_h\phi(x) := \phi(x-h). \tag{1}$$

Note that $\tau_h\phi \in C_c^\infty(U)$ as well if h is sufficiently small.

(a) For $t > 0$, let

$$\phi_{h,t} = \frac{\phi - \tau_{th}\phi}{t}.$$

Show, if $t > 0$ is sufficiently small, that $\phi_{h,t} \in C_c^\infty(U)$ as well, and that

$$\phi_{h,t} \to h \cdot \nabla\phi$$

as $t \to 0^+$ in the topology of $C_c^\infty(U)$.

(b) Let $u \in \mathcal{D}'(\mathbb{R}^n)$. For $h \in \mathbb{R}^n$, let $\tau_h u$ be the distribution defined by

$$(\tau_h u, \phi) := (u, \tau_{-h}\phi) \text{ for } \phi \in C_c^\infty(\mathbb{R}^n).$$

Show that if $u \in C_c^\infty(\mathbb{R}^n)$, then this definition agrees with the definition in (1). Moreover, for any $u \in \mathcal{D}'(\mathbb{R}^n)$, if we let

$$u_{h,t} = \frac{u - \tau_{th}u}{t}$$

for $h \in \mathbb{R}^n$ and $t > 0$, then show that $u_{h,t} \to h \cdot \nabla u$ in the sense of distributions (i.e. in the topology of $\mathcal{D}'(\mathbb{R}^n)$) as $t \to 0^+$.

Problem 2: For $a \in \mathbb{C}$ with Re $a > -1$, define $\chi_+^a : \mathbb{R} \to \mathbb{R}$ by

$$\chi_+^a(x) = \begin{cases} \frac{x^a}{\Gamma(a+1)} & x > 0 \\ 0 & x \le 0 \end{cases}$$

where Γ is the gamma function (this is a meromorphic function whose poles are at $0, -1, -2, \ldots$, and it satisfies $\Gamma(1) = 1$ and $\Gamma(z+1) = z\Gamma(z)$ for z away from the poles). Recall as well that x^a is defined as $e^{a \ln(x)}$ for $x > 0$ and $a \in \mathbb{C}$, where $\ln(x)$ is real-valued for $x > 0$.

(a) Show that $\chi_+^a \in L^1_{loc}(\mathbb{R})$ and $\chi_+^a = \frac{d}{dx}\chi_+^{a+1}$ for any Re $a > -1$.

(b) For $n \in \mathbb{N}$, define $\chi^a_{+,n}$ for Re $a > -n$ as follows: define $\chi^a_{+,1} = \chi^a_+$ as above, and if $\chi^a_{+,n-1}$ is defined for all Re $a > -n+1$, define

$$\chi^a_{+,n} := \frac{d}{dx}\chi^{a+1}_{+,n-1} \quad \text{for } a > -n.$$

For any $a \in \mathbb{C}$, show that $\chi^a_{+,n} = \chi^a_{+,n'}$ for any $n, n' > -\text{Re } a$. Thus, for any $a \in \mathbb{C}$, we can define

$$\chi^a_+ := \chi^a_{+,n} \text{ for } \textit{any } n > -\text{Re } a. \tag{2}$$

Note for a with Re $a > -1$ that this agrees with the original definition.

(c) Let $a \in \mathbb{R}$. Show that χ^a_+ (as defined in (2)) is homogeneous of degree a.

(d) For any $\phi \in C_c^\infty(\mathbb{R})$, let $f_\phi : \mathbb{C} \to \mathbb{C}$ be the function

$$f_\phi(a) = (\chi^a_+, \phi)$$

where χ^a_+ is defined in (2). Show that f_ϕ is an entire analytic function.

(e) For any $k \in \mathbb{N}$, show that

$$\chi^{-k-1}_+ = \delta^{(k)},$$

where $\delta^{(k)}$ is the kth distributional derivative of the Dirac delta.

(f) If $0 \notin \text{supp } \phi$ and a is not a negative integer, show that

$$(\chi^a_+, \phi) = \int_0^\infty \frac{1}{\Gamma(a+1)} x^a \phi(x)\, dx.$$

Here, we interpret $\frac{1}{\Gamma(a+1)}$ as the entire analytic function which agrees with $1/\Gamma$ away from the poles of Γ (note in particular that this number equals 0 when a is a negative integer). In particular, if a is not a negative integer, then away from 0 we have that χ^a_+ is given by the same formula as before, while if a is a negative integer the pairing gives zero, i.e. χ^a_+ is supported only at 0 when a is a negative integer.

Problem 3: This problem concerns extending homogeneous distributions on $\mathbb{R}^n\backslash\{0\}$ to distributions on $\mathbb{R}^n$. See Section 3.2, particularly Theorem 3.2.3, of Hörmander's *The Analysis of Linear Partial Differential Operators* Vol. 1 [Hör90].

In this problem, a denotes a real number.

(a) Fix $u \in \mathcal{D}'(\mathbb{R}^n\backslash\{0\})$ and $\phi \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$. Consider the function $f(t) = (u, \phi(tx))$ for $t \in \mathbb{R}$. Show that

$$f'(1) = \left(u, \sum_{i=1}^n x_i \partial_i \phi\right).$$

(b) Suppose that $u \in \mathcal{D}'(\mathbb{R}^n\backslash\{0\})$ is homogeneous of degree a. Show that

$$\left(u, \sum_{i=1}^n x_i \partial_i \phi\right) + (a+n)(u, \phi) = 0$$

for any $\phi \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$.

(c) Suppose $\psi \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$ satisfies

$$\int_0^\infty r^{a+n-1}\psi(r\omega)\,dr = 0 \quad \text{for all } \omega \in \mathbb{S}^{n-1}.$$

Let $\phi : \mathbb{R}^n\backslash\{0\} \to \mathbb{C}$ be defined in polar coordinates by

$$\phi(r\omega) = r^{-(a+n)}\int_0^r s^{a+n-1}\psi(s\omega)\,ds.$$

Show that $\phi \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$, and $\sum_{i=1}^n x_i\partial_i\phi + (a+n)\phi = \psi$. Conclude that $(u,\psi) = 0$ for all distributions $u \in \mathcal{D}'(\mathbb{R}^n\backslash\{0\})$ homogeneous of degree a.

We now assume either $a > -n$ or $a \notin \mathbb{Z}$. We now show that distributions on $\mathbb{R}^n\backslash\{0\}$ which are homogeneous of degree a where can be extended to $\mathbb{R}^n$. Let $x_+^a = \Gamma(a+1)\chi_+^a$, where χ_+^a is defined in (2). Note that x_+^a restricted to $\mathbb{R}\backslash\{0\}$ equals x^a for $x > 0$ and 0 for $x < 0$.

(d) For $\phi \in C_c^\infty(\mathbb{R}^n)$, define

$$(R_a\phi)(y) = (x_+^{a+n-1}, \phi(xy))_{\mathbb{R}} \qquad \text{for } y \in \mathbb{R}^n\backslash\{0\}.$$

Show that $R_a\phi \in C^\infty(\mathbb{R}^n\backslash\{0\})$ and that $R_a\phi$ is homogeneous of degree $-n-a$.

(e) Suppose $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$, and $R_a\phi_1 = R_a\phi_2$ on $\mathbb{R}^n\backslash\{0\}$. Show that $(u,\phi_1) = (u,\phi_2)$ for all distributions $u \in \mathcal{D}'(\mathbb{R}^n\backslash\{0\})$ homogeneous of degree a.

(f) Let $\psi \in C_c^\infty((0,\infty))$ satisfy

$$\int_0^\infty \psi(t)\frac{dt}{t} = 1.$$

For $\phi \in C_c^\infty(\mathbb{R}^n)$ (not necessarily supported away from zero), show that $\psi(|x|)R_a\phi(x) \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$, and $R_a(\psi(|x|)R_a\phi)(x) = (R_a\phi)(x)$ for all $x \in \mathbb{R}^n\backslash\{0\}$. Conclude that if ϕ is supported away from zero, then

$$(u, \psi(|x|)R_a\phi) = (u,\phi)$$

for all distributions $u \in \mathcal{D}'(\mathbb{R}^n\backslash\{0\})$ homogeneous of degree a.

(g) Given $u \in \mathcal{D}'(\mathbb{R}^n\backslash\{0\})$ homogeneous of degree a, let $\tilde{u}$ be the distribution on $\mathbb{R}^n$ defined by

$$(\tilde{u},\phi) := (u, \psi(|x|)R_a\phi).$$

Show that $\tilde{u}$ is indeed a distribution on $\mathbb{R}^n$ whose restriction to $\mathbb{R}^n\backslash\{0\}$ is u.

Problem 4: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n}\varphi(x)\,dx = 1$, and for $\epsilon > 0$ let $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$.

(a) Show that φ_ϵ converges to δ in the sense of distributions as $\epsilon \to 0^+$.

(b) Show that $|\varphi_\epsilon|^2$ does **not** converge in the sense of distributions.

Upshot: There does not exist an operator $\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ which extends the notion of pointwise multiplication (say defined initially on $C_c^\infty(\mathbb{R}^n)\times C_c^\infty(\mathbb{R}^n)$) which is also continuous on $\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}'(\mathbb{R}^n)$, since we can find a sequence φ_n in $C_c^\infty(\mathbb{R}^n)$ converging to δ in $\mathcal{D}'(\mathbb{R}^n)$ such that $\varphi_n \cdot \overline{\varphi_n}$ does not converge.

Problem 5:

(a) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function such that $\nabla f(x) \neq 0$ when $f(x) = 0$. Note then that $f^{-1}(0)$ is a smooth codimension 1 submanifold of $\mathbb{R}^n$, i.e. a hypersurface. Show that

$$\delta(f) = \frac{dS}{|\nabla f|},$$

where dS is the Euclidean surface measure on the $f^{-1}(0)$.

(b) With f as in part (a), let $\Omega = \{f > 0\}$, and let $u = \mathbb{1}_\Omega$ be the indicator function of Ω. Let $\phi_j \in C_c^\infty(\mathbb{R}^n)$, $1 \leq j \leq n$, and $\phi = (\phi_1, \dots, \phi_n)$. Show that

$$\sum_{j=1}^{n} (\partial_j u, \phi_j) = (\delta(f), \phi \cdot \nabla f).$$

Problem 6: Let $a : \mathbb{R}^{n+1}_{\xi,\tau} \backslash \{0\} \to \mathbb{C}$ and $H : \mathbb{R}^{n+1}_{x,t} \to \mathbb{R}$ be defined by

$$a(\xi, \tau) = \frac{1}{|\xi|^2 + i\tau}$$

and

$$H(x,t) = \begin{cases} (4\pi t)^{-n/2} e^{-|x|^2/4t} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

Here, $|.|$ is the norm on $\mathbb{R}^n$.

(a) Show that a and H are both locally integrable on $\mathbb{R}^{n+1}$, and furthermore that they define tempered distributions, i.e. $a \in \mathcal{S}'(\mathbb{R}^{n+1}_{\xi,\tau})$ and $H \in \mathcal{S}'(\mathbb{R}^{n+1}_{x,t})$.

(b) Show that the Fourier transform of H equals a.

Problem 7:

(a) Let $H(x) = \mathbb{1}_{\{x>0\}}$ be the Heaviside function. Show that the Fourier transform of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-\epsilon x} H(x)$ (where $\epsilon > 0$) is given by

$$\hat{f}(\xi) = \frac{1}{i\xi + \epsilon}.$$

(b) Conversely, show by direct computation (i.e. without invoking the Fourier Inversion formula and part (a)) that the inverse Fourier transform of $g(\xi) = \frac{1}{i\xi+\epsilon}$ is $f(x)$ defined in part (a). (*Hint*: First show that if $g_R(\xi) = \mathbb{1}_{[-R,R]}(\xi) g(\xi)$, then $g_R \to g$ in $\mathcal{S}'(\mathbb{R})$. To compute the inverse Fourier transform of g_R, you need to evaluate the integral

$$\frac{1}{2\pi} \int_{-R}^{R} \frac{e^{ix\xi}}{i\xi + \epsilon} \, d\xi.$$

This integral, or rather its limit/asymptotics as $R \to \infty$, can be evaluated using contour integration, by taking a semicircular contour with the semicircle lying either in the upper or lower half plane, chosen appropriately so that the exponential factor $e^{ix\xi}$ is exponentially decaying and not growing in that half plane.)

Problem 8: Recall that the distributions $(x \pm i0)^{-1}$ on $\mathbb{R}$ are defined by

$$\phi \mapsto \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{\phi(x)}{x \pm i\epsilon}\, dx.$$

(a) Show that the limit on the RHS does indeed exist for any $\phi \in C_c^\infty(\mathbb{R})$.
(b) Show that

$$(x - i0)^{-1} - (x + i0)^{-1} = 2\pi i \delta_0.$$

(c) Compute the Fourier transforms of $(x \pm i0)^{-1}$.

Problem 9: For $n \geq 3$, consider the distribution

$$u = \delta\left(x_n^2 - \sum_{i=1}^{n-1} x_i^2\right).$$

This is a distribution of order -2 well-defined on $\mathbb{R}^n \backslash \{0\}$, so by Problem 3 there is a unique way to extend this to a homogeneous distribution of order -2 defined on $\mathbb{R}^n$ when $n \geq 3$.

Show that $u \in \mathcal{S}'(\mathbb{R}^n)$, and compute its Fourier transform in the case $n = 4$.

Problem 10: The purpose of this problem is to derive the *stationary phase lemma* for the particular quadratic phase function $(x, y) \mapsto x \cdot y$ on $\mathbb{R}^{2n}$ using the Fourier inversion formula. See Theorem 7.7.3 of Hörmander's *The Analysis of Linear Partial Differential Operators* Vol. 1 [Hör90] for the most general version of quadratic stationary phase, as well as generalizations to non-quadratic phase functions.

For $n \in \mathbb{N}$, we view $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, with coordinates (x, y) where $x, y \in \mathbb{R}^n$.

(a) Let $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$. For $\lambda > 0$, consider the integral

$$I(\lambda) = \int_{\mathbb{R}^{2n}} e^{i\lambda x \cdot y} \phi(x)\psi(y)\, dx\, dy.$$

Show that $\lambda^n I(\lambda) \to (2\pi)^n \phi(0)\psi(0)$ as $\lambda \to +\infty$. Furthermore, show that if $\phi \in C_c^\infty(\mathbb{R}^n)$ is constant in a neighborhood of the origin, then

$$\lambda^n I(\lambda) - (2\pi)^n \phi(0)\psi(0) = O(\lambda^{-N})$$

as $\lambda \to +\infty$ for any $N > 0$. (*Hint*: show that $I(\lambda)$ equals

$$\frac{1}{\lambda^n} \int_{\mathbb{R}^n} \phi(-\xi/\lambda)\hat{\psi}(\xi)\, d\xi$$

by making the substitution $\xi = -\lambda x$.)

(b) Suppose $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ are constant in a neighborhood of the origin. For multi-indices α, β, let

$$I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}^{2n}} e^{i\lambda xy} x^\alpha y^\beta \phi(x)\psi(y)\, dx\, dy.$$

Show that if $\alpha \neq \beta$, then $I_{\alpha,\beta}(\lambda) = O(\lambda^{-N})$ for all $N > 0$. In addition, show that

$$i^{-|\alpha|}\lambda^{|\alpha|+n} I_{\alpha,\alpha}(\lambda) - (2\pi)^n \alpha! \phi(0)\psi(0) = O(\lambda^{-N})$$

for all $N > 0$, where $\alpha! = \alpha_1!\alpha_2!\dots\alpha_n!$ if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

(c) Suppose that $f(x,y) = x^\alpha y^\beta g(x,y)$ for some multi-indices α and β and for some $g \in C_c^\infty(\mathbb{R}^{2n})$. Show that

$$\int_{\mathbb{R}^{2n}} e^{i\lambda x\cdot y} f(x,y)\,dx\,dy = O(\lambda^{-n-\max(|\alpha|,|\beta|)})$$

as $\lambda \to \infty$.

(d) Now let f be *any* function in $C_c^\infty(\mathbb{R}^{2n})$, and let

$$I_f(\lambda) = \int_{\mathbb{R}^{2n}} e^{i\lambda x\cdot y} f(x,y)\,dx\,dy.$$

Show that $I_f(\lambda)$ admits the *asymptotic expansion*

$$I_f(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^n \sum_\alpha \frac{i^{|\alpha|}\partial_x^\alpha \partial_y^\alpha f(0)}{\alpha!}\lambda^{-|\alpha|},$$

in the sense that for any integer $N > 0$ we have

$$\left| I_f(\lambda) - \left(\frac{2\pi}{\lambda}\right)^n \sum_{|\alpha|<N} \frac{i^{|\alpha|}\partial_x^\alpha \partial_y^\alpha f(0)}{\alpha!}\lambda^{-|\alpha|} \right| = O(\lambda^{-n-N}).$$

Hint: Use Taylor's theorem, which states that for any $f \in C^\infty(\mathbb{R}^m)$ and any $N > 0$ we have

$$f(z) = \sum_{|\gamma|<N} \frac{\partial^\gamma f(0)}{\gamma!} z^\gamma + \sum_{|\gamma|=N} R_\gamma f(z) z^\gamma$$

for some choice of functions $R_\gamma f$ which are *also smooth*. Also note that since in our case we have $f \in C_c^\infty(\mathbb{R}^{2n})$, there exist $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ which are identically one near the origin such that $f(x,y) = f(x,y)\phi(x)\psi(y)$.

References

[Hör90] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Springer-Verlag, Berlin, 2nd edition, 1990.