MATH 218 PROBLEM SET 2

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Problem 1: Recall that for $s \in \mathbb{R}$ we have

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \, : \, (1 + |\xi|^2)^{s/2} \hat{u} \in L^2(\mathbb{R}^n) \right\}.$$

(a) Let $u \in \mathcal{E}'(\mathbb{R}^n)$, i.e. u is a compactly supported distribution. Show that there exists $s \in \mathbb{R}$ such that $u \in H^s(\mathbb{R}^n)$.
(b) Let $u \in H^s(\mathbb{R}^n)$ and $\chi \in C_c^\infty(\mathbb{R}^n)$. Show that $\chi u \in H^s(\mathbb{R}^n)$.
(c) Show that $\mathcal{S}(\mathbb{R}^n) \subsetneq \cap_{s \in \mathbb{R}} H^s(\mathbb{R}^n) \subsetneq C^\infty(\mathbb{R}^n)$.

Problem 2: For $u \in \mathcal{D}'(\mathbb{R}^n)$, we say that $u \in H^s_{loc}(\mathbb{R}^n)$ (for some $s \in \mathbb{R}$) if $\chi u \in H^s(\mathbb{R}^n)$ for all $\chi \in C_c^\infty(\mathbb{R}^n)$.

(a) Suppose $v \in \mathcal{E}'(\mathbb{R}^{n+1}_{t,x})$ satisfies the property that

$$(\partial_t - \Delta) v = v^0 + \sum_{i=1}^n \partial_{x_i} v^i$$

for some distributions $v^0, v^1, \ldots, v^n \in \mathcal{D}'(\mathbb{R}^{n+1}_{t,x})$, such that for some s we have

$$v^0 \in H^{s-1}(\mathbb{R}^{n+1}), \quad v^i \in H^{s-1/2}(\mathbb{R}^{n+1}) \text{ for } 1 \le i \le n.$$

Show that $v \in H^s(\mathbb{R}^{n+1})$. (*Hint*: because v is compactly supported, it suffices to make an estimate on $\hat{v}(\tau, \xi)$ when $|(\tau, \xi)|$ is large. You may find it useful to note that $||\xi|^2 + i\tau| \ge \max(|\xi|^2, |\tau|)$.)

(b) Suppose $u \in \mathcal{D}'(\mathbb{R}^{n+1}_{t,x})$ satisfies the property that

$$(\partial_t - \Delta) u \in H^{s_0}_{loc}(\mathbb{R}^{n+1}).$$

Show that $u \in H^{s_0+1}_{loc}(\mathbb{R}^{n+1})$. (*Hint*: For a fixed bounded set $U \subset \mathbb{R}^{n+1}$, first show, without knowing anything about $u \in \mathcal{D}'(\mathbb{R}^{n+1})$, that there exists some fixed s' such that $\chi u \in H^{s'}(\mathbb{R}^{n+1})$ *for all* $\chi \in C_c^\infty(U)$. Then, use the assumption on $(\partial_t - \Delta) u$ to show by induction that if

$$\chi u \in H^s(\mathbb{R}^{n+1}) \text{ for all } \chi \in C_c^\infty(U),$$

then

$$\chi u \in H^{\min(s+1/2, s_0+1)}(\mathbb{R}^{n+1}) \text{ for all } \chi \in C_c^\infty(U).$$

Do so by rewriting $(\partial_t - \Delta)(\chi u)$ for $\chi \in C_c^\infty(U)$ in terms of sums of derivatives of other functions in $C_c^\infty(U)$ times derivatives of u.)

(c) Suppose u is a distributional solution to the heat equation $(\partial_t - \Delta) u = 0$ on all of $\mathbb{R}^{n+1}_{t,x}$. Show that u must be smooth on $\mathbb{R}^{n+1}_{t,x}$.

Problem 3: Recall that given a constant coefficient operator P on $\mathbb{R}^{n+1}_{t,x}$, we say that $E \in \mathcal{D}'(\mathbb{R}^n)$ is a *backward fundamental solution* if $PE = \delta_{(0,0)}$ and

$$\operatorname{supp} E \subseteq \{(t,x) \in \mathbb{R}^{n+1} \,:\, t \le 0\}.$$

Show that there does not exist a backward fundamental solution for the heat operator $P = \partial_t - \Delta$. (*Hint*: Let E_+ be the *forward* fundamental solution derived in class. Show that any backward fundamental solution E must satisfy $E - E_+ \in C^\infty(\mathbb{R}^{n+1})$. Show that this is incompatible with the support condition on E.)

Problem 4: Verify directly that the fundamental solution E derived in class

$$E(t,x) = \begin{cases} (4\pi t)^{-n/2} e^{-|x|^2/(4t)} & t > 0 \\ 0 & t \le 0 \end{cases}$$

satisfies $(\partial_t - \Delta)E = \delta_{(0,0)}$. That is, verify that for any $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ we have

$$\int_{\mathbb{R}^{n+1}} E(t,x)(-\partial_t \phi - \Delta\phi)(t,x)\, dx\, dt = \phi(0,0).$$

Problem 5: Let $(a^{ij})_{i,j=1}^n$ be a positive-definite symmetric real-valued matrix, and consider the constant-coefficient differential operator on $\mathbb{R}^{n+1}_{t,x}$

$$P = \partial_t - \sum_{i,j=1}^n a^{ij} \partial_{x_i} \partial_{x_j}.$$

Find a formula for (a) forward fundamental solution of P.

Problem 6: Let $g \in C^\infty(\mathbb{R})$, and consider the power series

$$u(t,x) = \sum_{n=0}^\infty g^n(t) \frac{x^{2n}}{(2n)!} \quad (x \in \mathbb{R}).$$

(a) Suppose for each t there exist constants $C_t, C_t' > 0$ such that

$$|g^n(t)| \le C_t (C_t')^k k!.$$

Show then that the power series defines a smooth function $u(t,x)$ which solves the heat equation.

(b) For $a > 0$, let $g(t) = \begin{cases} e^{-1/t^a} & t > 0 \\ 0 & t \le 0 \end{cases}$. Show that g satisfies the assumptions of part (a), so that in particular the power series defines a nonzero smooth solution $u(t,x)$ of the heat equation which nonetheless satisfies $u(0,x) = 0$ for all $x \in \mathbb{R}$. (This shows the Cauchy problem for the heat equation does not have a unique solution among the space of *all* smooth functions.)

(c) **Bonus**: Show that $|u(t,x)| \le Ce^{c|x|^{2a/(a-1)}}$ for some $C, c > 0$. (Note that the exponent of $|x|$ is always larger than 2 and tends towards 2 as $a \to +\infty$; this shows that the growth condition $|u(t,x)| \le Ce^{c|x|^2}$ needed to guarantee uniqueness is sharp in the exponent of $|x|$ in the exponential.)

Problem 7: Let U be either $\mathbb{R}^n$ or a bounded open set. Let $u \in H^2(U)$, and if U is a bounded open set, make the additional assumption that $u \in H^1_0(U)$ as well. Let

$$\|D^2 u\|^2_{L^2(U)} := \sum_{i,j=1}^n \|\partial_i \partial_j u\|^2_{L^2(U)}.$$

Show that

$$\|D^2 u\|_{L^2(U)} = \|\Delta u\|_{L^2(U)}.$$

Problem 8: Let $U \subset \mathbb{R}^n$ be a bounded open set. For $u, v \in H^1_0(U)$, and $a^{ij}, b^i, c \in L^\infty(U)$ ($1 \le i \le n$, $1 \le j \le n$), with a^{ij} satisfying

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n$$

for some $\theta > 0$ (with the estimate holding uniformly for all x), let

$$B[u,v] = \int_U \sum_{i,j=1}^n a^{ij}(x)\partial_{x_i}u(x)\partial_{x_i}v(x) + \sum_{i=1}^n b^i(x)\partial_{x_i}u(x)v(x) + c(x)u(x)v(x)\,dx.$$

Show the upper and lower bounds

$$|B[u,v]| \le \alpha\|u\|_{H^1_0(U)}\|v\|_{H^1_0(U)}$$

and

$$B[u,u] \ge \beta\|u\|^2_{H^1_0(U)} - \gamma\|u\|^2_{L^2(U)}$$

for some $\alpha, \beta > 0$ and $\gamma \ge 0$. Express α, β, and γ in terms of relevant estimates on the coefficients a^{ij}, b^i, and c.

Problem 9: Let

$$L = -\sum_{i,j=1}^n a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^n b^i(x)\partial_{x_i} + c(x)$$

where $a^{ij}, b^i, c \in C^\infty(\mathbb{R}^n)$, and a^{ij} satisfies elliptic estimates

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2, \quad \theta > 0$$

uniformly on $\mathbb{R}^n$. Suppose $u \in L^2(\mathbb{R}^{n+1})$ is compactly supported, say with support in $(0,T) \times U$ for some bounded open set U. Let

$$f = \partial_t u + Lu.$$

Suppose f, initially well-defined as a distribution in $\mathcal{D}'((0,T)\times U)$, is in fact in $L^2((0,T)\times U)$.

Fix $\rho \in C_c^\infty(\mathbb{R})$ with $\rho \geq 0$ and $\int_{\mathbb{R}} \rho = 1$, and let $\rho_\epsilon(s) = \epsilon^{-1}\rho(s/\epsilon)$. Viewing u as a function in $L^2((0,T);L^2(U))$, let $u_\epsilon = u *_t \rho_\epsilon$ be the time convolution of u with ρ_ϵ, i.e.

$$u_\epsilon(t) = \int_{\mathbb{R}} \rho_\epsilon(s)u(t-s)\,ds.$$

Note that u_ϵ is also supported in $(0,T)\times U$ if ϵ is sufficiently small; furthermore $u_\epsilon \to u$ as $\epsilon \to 0^+$ in $L^2((0,T)\times U)$.

(a) Show that u_ϵ is a weak solution (as defined in class) to the problem

$$\partial_t u_\epsilon - Lu_\epsilon = f_\epsilon \text{ in } (0,T)\times U, \quad u_\epsilon(0,x) = 0 \text{ on } U$$

for all sufficiently small $\epsilon > 0$, where $f_\epsilon = f *_t \rho_\epsilon$. (*Hint*: the main difficulty is showing that $u_\epsilon \in L^2([0,T];H_0^1(U))$, in particular that $u_\epsilon(t)$ has H^1 regularity for (almost) every t. To do so, you may use elliptic estimates, such as those in [Eva10] Section 6.3; you may take for granted that all constants in the elliptic estimates are continuous with respect to the C^∞ topology on the coefficients in question.)

(b) Show that

$$\|u_\epsilon\|_{H^1((0,T)\times U)}$$

is uniformly bounded as $\epsilon \to 0^+$.

(c) Conclude that u is also in $H^1((0,T)\times U)$, and that u is a weak solution to

$$\partial_t u - Lu = f \text{ in } (0,T)\times U, \quad u(0,x) = 0 \text{ on } U.$$

(In particular, u enjoys all of the regularity estimates derived in class.)

Problem 10: This problem regards deriving the expansion for heat kernels on compact manifolds M by finding a sequence $u_j \in C^\infty(M\times M)$ such that

$$(\partial_t - (\Delta_g)_x)\left(p_0(t,x,y)\sum_{j=0}^k t^j u_j(x,y)\right) \in p_0 t^k C^\infty([0,\infty)\times M\times M).$$

You may take for granted that there exists $\epsilon > 0$ (known as the "injectivity radius") such that, for every $y \in M$, the map

$$(0,\epsilon)\times \mathbb{S}^{n-1} \to M, \quad (r,\omega) \mapsto \exp_y(r\omega)$$

is a diffeomorphism between $(0,\epsilon)\times\mathbb{S}^{n-1}$ and a punctured neighborhood of y in M; in particular (r,ω) provide "geodesic polar coordinates" on M near y. Furthermore, we have $d_g(\exp_y(r\omega), y) = r$, and under these geodesic polar coordinates we have

$$\det g(r,\theta) = r^{2(n-1)} D(\exp_y(r,\omega), y)$$

for all $0 < r < \epsilon$, where $D(x,y) \in C^\infty(M \times M)$ and $D(y,y) = 1$ for all $y \in M$, with $(\Delta_g)_x D(y,y) = -\frac{1}{3}S(y)$ where S is the scalar curvature at y, and the Laplace-Beltrami operator takes the form

$$\Delta_g = \partial_r^2 + \left(\frac{\partial_r(\sqrt{D})}{D} + \frac{n-1}{r} \right) \partial_r + \Delta_{g_{S_y^{n-1}(r)}},$$

where $\Delta_{g_{S_y^{n-1}(r)}}$ is the Laplace-Beltrami operator corresponding to the metric induced on the geodesic sphere of radius r centered at y (in particular it annihilates any function depending on r only). (If you have some background in differential geometry, you are welcome to verify these facts as well.)

Let

$$p_0(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-d_g(x,y)^2/(4t)}.$$

Note that under geodesic polar coordinates we have $p_0 = (4\pi t)^{-n/2} e^{-r^2/(4t)}$.

(a) Show that if $v \in C^\infty(M \times M)$, then for any $j \geq 0$ we have[1]

$$(\partial_t - (\Delta_g)_x)\left(p_0 t^j v(x,y)\right) = \left[t^{j-1} \left(\left(j + \frac{r}{2}\frac{\partial_r D}{D} \right) v + r\partial_r v \right) - t^j (\Delta_g)_x v \right] p_0.$$

Conclude that if $u_j \in C^\infty(M \times M)$ satisfy the recursive equations

$$\left(j + \frac{r}{2}\frac{\partial_r D}{D} \right) u_j + r\partial_r u_j = \Delta u_{j-1}$$

for $j \geq 0$ (where by convention we set $u_{-1} \equiv 0$), then

$$(\partial_t - (\Delta_g)_x)\left(p_0(t,x,y)\alpha(d_g(x,y)) \sum_{j=0}^k t^j u_j(x,y) \right) \in p_0 t^k C^\infty([0,\infty) \times M \times M),$$

where $\alpha \in C_c^\infty(\mathbb{R})$ is supported in $r < \epsilon$ and is identically 1 on $r \leq \epsilon/2$.

(b) By solving the recursive equation above for $j = 0$, show that

$$u_0(x,y) = C D^{-1/2}(x,y)$$

for some constant C for all (x,y) with $d_g(x,y) < \epsilon$. Show that if we insist on the property

$$p_0(t,x,y)\alpha(d_g(x,y)) \sum_{j=0}^k t^j u_j(x,y) \to \delta_y(x)$$

as $t \to 0^+$, then (regardless of the choice of the other u_j) we must have $C = 1$.

(c) By solving the recursive equation above for $j = 1$, show that

$$u_1(y,y) = -\frac{1}{2}(\Delta_g)_x D(y,y) = \frac{1}{6}S(y).$$

[1] In this expression, the ∂_r derivatives are interpreted as acting on the left factor x, where the left factor is given geodesic polar coordinates centered at y.

References

[Eva10] Lawrence C. Evans. *Partial Differential Equations*. American Mathematical Society, 2nd edition, 2010.