MATH 218 PROBLEM SET 3

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Problem 1:

(a) Suppose $u \in C(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}^n_x))$, i.e. for each $t \in \mathbb{R}$ we associate $u(t) \in \mathcal{D}'(\mathbb{R}^n_x)$ such that $t \mapsto u(t)$ is continuous (w.r.t the topology on $\mathcal{D}'(\mathbb{R}^n_x)$). Show that u defines a distribution on $\mathbb{R}^{n+1}_{t,x}$, via

$$\phi \mapsto \int_{\mathbb{R}} (u(t), \phi(t, \cdot))_{\mathbb{R}^n}\, dt \text{ for } \phi \in C_c^\infty(\mathbb{R}^{n+1}).$$

(b) For $t \in \mathbb{R}$, define $E_+(t) \in \mathcal{D}'(\mathbb{R}^n_x)$ by

$$E_+(t) = \begin{cases} \mathcal{F}^{-1}\left(\frac{\sin(t|\xi|)}{|\xi|}\right) & t > 0 \\ 0 & t \le 0 \end{cases}.$$

Show that $E_+ \in C(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}^n_x))$, and that the corresponding distribution on $\mathbb{R}^{n+1}$ is a fundamental solution to the wave operator $\partial_t^2 - \Delta$.

Problem 2: Let $(a^{ij})_{i,j=1}^n$ be a symmetric complex-valued matrix, and suppose $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfies the property that

$$\sum_{i,j=1}^n a^{ij} \partial_{x_i} u \partial_{x_j} u \in C^\infty(\mathbb{R}^n)$$

in the sense of distributions. Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ be a vector such that

$$\sum_{i,j=1}^n a^{ij} \xi_i \xi_j \neq 0.$$

Let U be a bounded open set in $\mathbb{R}^n$. Show that, for every $\chi \in C_c^\infty(U)$ and $N > 0$, there exists a constant $C_{\chi,N}$ such that

$$|\widehat{\chi u}(\lambda\xi)| \le C_N \lambda^{-N} \text{ as } \lambda \to +\infty.$$

In particular, if $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ solves the wave equation $(\partial_t^2 - c^2\Delta)u = 0$ in the sense of distributions on $\mathbb{R}^{n+1}$, show that

$$|\widehat{\chi u}(\lambda\tau, \lambda\xi)| \le C_N \lambda^{-N}$$

whenever (τ, ξ) lies outside the light cone $\{(\tau, \xi) \in \mathbb{R}^{n+1} : \tau^2 = c^2|\xi|^2\}$.

Hint: First, show that there exists $m \in \mathbb{R}$ such that for any $\chi \in C_c^\infty(U)$ and any multi-index α there exists some $C_{\chi,\alpha} > 0$ such that

$$|\widehat{\chi \partial^\alpha u}(\xi)| \le C_{\chi,\alpha}(1 + |\xi|)^{m+|\alpha|}$$

for all $\xi \in \mathbb{R}^n$. Next, show by induction that for any $\chi \in C_c^\infty(U)$ and any $N > 0$, there exist $\chi_{\alpha;N} \in C_c^\infty(U)$ (ranging over multi-indices α with $|\alpha| \le N$) and $v_N \in C_c^\infty(U)$ such that

$$L^N(\chi u) = \sum_{|\alpha| \le N} \chi_{\alpha;N} \partial^\alpha u + v_N, \quad L = \sum_{i,j=1}^n a^{ij} \partial_{x_i} \partial_{x_j}$$

(note that L^N is a differential operator of order $2N$, whereas the right-hand side involves terms taking at most N derivatives of u). Take the Fourier transform of both sides to conclude.

Problem 3: Let n be odd, and let $u \in C^\infty(\mathbb{R}^{1+n})$ solve the linear wave equation $(\partial_t^2 - \Delta)u = 0$, with $f_0(x) = u(0,x)$ and $f_1(x) = \partial_t u(0,x)$ both in $\mathcal{S}(\mathbb{R}^n)$. Show that there exists a constant C such that

$$\sup_{x \in \mathbb{R}^n} |u(t,x)| \le \frac{C}{t^{(n-1)/2}} \text{ for all } t \ge 1.$$

Hint: It suffices (by rotation) to prove such an estimate for x of the form $x = (x_1, 0, \dots, 0)$. Writing $\xi = (\xi_1, \xi')$ with $\xi' \in \mathbb{R}^{n-1}$, we have

$$u(t,x) = (2\pi)^{-n} \int_{\mathbb{R}} e^{ix_1\xi_1} \int_{\mathbb{R}^{n-1}} e^{it|\xi|} \frac{\hat{f}_1(\xi)}{2i|\xi|} \, d\xi' \, d\xi_1 + \dots$$

Now introduce polar coordinates for ξ', and integrate by parts in the radial variable.

Problem 4: Let $\phi(r) \in C_c^\infty(\mathbb{R})$ be nonzero and supported in $\{\frac{1}{2} < r < 2\}$, and let $u \in C^\infty([0,\infty) \times \mathbb{R}^3)$ be given by

$$u(t,x) = \frac{\phi(|x| - t)}{|x|}.$$

Show that u solves the linear wave equation on $(0,\infty) \times \mathbb{R}^3$, with C_c^∞ initial data, and that there exists $c > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |u(t,x)| \ge \frac{c}{t} \text{ for all } t \ge 1.$$

Problem 5: Let U be a bounded open set in $\mathbb{R}^n$, and let $L = \sum_{j,k=1}^n g^{jk}(x) \partial_{x_j} \partial_{x_k} + \sum_{k=1}^n b^k(x) \partial_{x_k} + q(x)$, where (g^{jk}) is uniformly elliptic on U, and all coefficients have bounded derivatives of all orders. Suppose $x_0 \in U$, and that there exists a continuous function $d : U \to \mathbb{R}_{\ge 0}$ which is smooth on $U \backslash \{x_0\}$ such that

$$\sum_{j,k=1}^n g^{ij}(x) \partial_{x_j} d(x) \partial_{x_k} d(x) = 1 \text{ on } U \backslash \{x_0\}, \quad d(x_0) = 0.$$

(a) Show that $d(x) = \operatorname{dist}_g(x, x_0)$, where dist_g is the geodesic distance with respect to the Riemannian metric g where in coordinates we have $(g_{ij}(x)) = (g^{ij}(x))^{-1}$.

(b) Suppose T satisfies that $d^{-1}([0,T])$ is a compact subset of U. Let u be a smooth solution to $(\partial_t^2 - L)u = 0$ in $(0,T)\times U$, and for $0 \le t \le T$ let

$$E(t) = \frac{1}{2}\int_{\{x\in U\,:\,d(x)<T-t\}} \left(|\partial_t u(t,x)|^2 + \sum_{j,k=1}^{n} g^{jk}(x)\partial_{x_j}u(t,x)\partial_{x_k}u(t,x) + |u(t,x)|^2 \right) dx.$$

Show that there exists a constant C, depending on the coefficients g^{jk}, b^k and q (but not on the solution u) such that

$$\dot{E}(t) \le CE(t).$$

Conclude that if $u(0,x) = \partial_t u(0,x) = 0$ for all x satisfying $d(x) < T$, then $u(T,x_0) = 0$.

Problem 6: Show that the limits

$$\lim_{\epsilon\to 0^+} \frac{1}{-\tau^2 + |\xi|^2 \pm i\epsilon}$$

exist in $\mathcal{D}'(\mathbb{R}^{1+n}_{\tau,\xi})$, and that the distributions

$$F_\pm = \mathcal{F}^{-1}\left(\lim_{\epsilon\to 0^+} \frac{1}{-\tau^2 + |\xi|^2 \pm i\epsilon} \right)$$

are fundamental solutions of the wave operator.

Is either $F_\pm$ equal to the forward fundamental solution of the wave operator?