Asymptotic Behavior of Solutions of Oblique Derivative Boundary Value Problems

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1. Problems and Main Results

In the Euclidean space R^n let $L=(1/2)a^{ij}\partial_i\partial_j+b^i\partial_i$ be a uniformly elliptic operator and let $V=(V^1,\ldots,V^n)$ be a vector field. Let q be a bounded nonnegative continuous function. Let D be a bounded domain and f a bounded measurable function on ∂D. Finally, let γ be a nontangential vector field on ∂D. Consider the solution u_f^ϵ of the boundary value problem $(\epsilon^2 L+V)u_f^\epsilon-qu_f^\epsilon=0$ on D, $\partial u_f^\epsilon/\partial\gamma=f$ on ∂D. In this paper, we study the asymptotic behavior of the solution u_f^ϵ as the parameter $\epsilon\to 0$. To guarantee the existence of a unique solution, we assume that q is not identically equal to zero on D. Under this condition, the solution tends to zero as ϵ goes to zero; the question is to find the appropriate exponential rate. This exponential rate depends on the behavior of the dynamical system $\dot\phi_s=V(\phi_s)$. We will discuss two typical cases: (1) the dynamical system has a unique equilibrium point in D; and (2) $V\equiv 0$. In the first case, we prove that $\lim_{\epsilon\to 0}\epsilon^2\log u_f^\epsilon$ exists and is equal to $-\inf_y I^+(x,y)$, where I^+ is the quasipotential function for the oblique derivative boundary value problem, and the infimum is taken over the essential support of the boundary value function f. In the second case, under the stronger condition that q is strictly positive on $\bar D$ and f is continuous, $\lim_{\epsilon\to 0}\epsilon\log u_f^\epsilon$ exists and the limit can also be explicitly identified.

The key to our discussion is a probabilistic representation of the solution u_f^ϵ. Let $\sigma=(\sigma^{ij})$ be a square root of the matrix $a=(a^{ij})$. Let $X=X^{x,\epsilon}$ be the solution of the stochastic differential equation with oblique reflection:

$$dX_t=\epsilon\sigma(X_t)dB_t+\epsilon^2 b(X_t)dt+V(X_t)dt-\gamma(X_t)\phi(dt),\quad X_0=x, \tag{1.1}$$

where B is a standard n-dimensional Brownian motion and ϕ is the boundary local time of the process X. Introduce the Feynman–Kac functional

$$e_q(t)=\exp\left\{-\int_0^t q(X_s)\,ds\right\}.$$

The solution u_f^ϵ can be represented explicitly as

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$$u_f^\epsilon(x) = E_x^\epsilon\left[\int_0^\infty e_q(t) f(X_t)\,\phi(dt)\right] \tag{1.2}$$

(see [3] and [7]).

Let T_A denote the first hitting time of set A by the process X; that is,

$$T_A = \inf\{t \geq 0 : X_t \in A\}.$$

Let $S_f = \text{ess.supp}\, f$. The additive functional

$$\phi_f(t) = \int_0^t f(X_s)\,\phi(ds)$$

does not increase until time T_{S_f}. Using the Markov property in (1.2) at time T_{S_f}, we have

$$u_f^\epsilon(x) = E_x^\epsilon[e_q(T_{S_f}) u_f^\epsilon(X_{T_{S_f}})]. \tag{1.3}$$

It is thus clear from this representation that the magnitude of u_f^ϵ depends on how soon the process X can reach the set S_f, and how fast u_f^ϵ vanishes on S_f as $\epsilon \to 0$. These two properties in turn depend on the path-space large deviation properties of the associated diffusion process and the behavior of the transition density function near the boundary.

Let us now state our assumptions and main results. Throughout the paper, the following assumptions will be in force:

(i) $a = (a^{ij})$, $b = (b^i)$, and $V = (V^i)$ are uniformly bounded and with uniformly Hölder continuous second derivatives on R^n; $\sigma = (\sigma_{ij})$ is a square root of the matrix a with the same property.

(ii) a is uniformly elliptic on R^n; that is, there exists a constant λ such that, for all $x \in R^n$ and $\xi \in R^n$,

$$\lambda^{-1}|\xi|^2 \leq \xi^T a(x)\xi \leq \lambda|\xi|^2$$

(ξ^T denotes the transpose of ξ).

(iii) D is a bounded domain in R^n with C^3 boundary.

(iv) q is a bounded nonnegative continuous function on R^n which is not identically equal to zero on the domain D.

(v) γ is a C^2 nontangential vector field on ∂D which points *outwards*; that is, $\langle \gamma(x), n(x)\rangle > 0$ on ∂D, where $n(x)$ is the outward unit normal vector of D at $x \in \partial D$.

(vi) f is a nonnegative bounded measurable function on ∂D.

Any additional assumptions will be stated explicitly.

The following notations will be used:

$|x-y|$ = Euclidean distance between x and y;
$B_\epsilon(x)$ = ball of radius ϵ centered at x;
$\Delta(x,\epsilon) = B_\epsilon(x) \cap \partial D$;
σ = surface measure on ∂D;
$\langle \xi, \eta\rangle_\alpha = \xi^T \alpha \eta$ if α is a nonnegative symmetric matrix;
$|\xi|_\alpha^2 = \langle \xi, \xi\rangle_\alpha$;
ess.supp f = essential support of f
$= \{x \in \partial D : \sigma[\Delta(x,\epsilon) \cap \{y : f(y) > 0\}] > 0$ for all positive $\epsilon\}$.

We need various rate functions (I-functions) from the large deviation theory. For a set F on R^n, let $\Omega_T(F)$ denote the space of continuous paths $\phi: [0,T] \to F$. The path space $\Omega_T(F)$ is a topological space equipped with the metric of uniform convergence $|\psi-\phi|_T = \sup_{0\le s\le T}|\psi_s-\phi_s|$. We will only use the cases $F=\bar{D}$ and $F=R^n$. In the former case we abbreviate $\Omega_T(\bar{D})$ as Ω_T. For $\phi\in\Omega_T(R^n)$ and a function α from R^n to the space of nonnegative symmetric matrices, we set

$$I_T(\alpha;\phi)=\frac{1}{2}\int_0^T|\dot{\phi}_s-V(\phi_s)|^2_{\alpha(\phi_s)}\,ds.$$

[It is always understood that $I_T(\alpha;\phi)$ and its likes are set equal to infinity if $\dot{\phi}$ is not L^2 integrable.] It is a well-known fact [4, p. 155] that $I_T(a^{-1};\phi)$ is the large deviation rate function in the path space $\Omega_T(R^n)$ for the unrestricted (without boundary condition) stochastic differential equation

$$dZ_t=\epsilon\sigma(Z_t)dB_t+\epsilon^2b(Z_t)dt+V(Z_t)dt. \tag{1.4}$$

According to [1] (see also [2]), the appropriate large deviation rate function on $\Omega_T(\bar{D})$ for the oblique reflecting diffusion (1.1) is $I_T^+(a^{-1};\phi)$, where

$$I_T^+(\alpha;\phi)=\frac{1}{2}\int_0^T|\dot{\phi}_s-\chi_{\partial D}(\phi_s)w(s)\gamma(\phi_s)-V(\phi_s)|^2_{\alpha(\phi_s)}\,ds, \tag{1.5}$$

and

$$w(s)=\frac{\langle\dot{\phi}_s-V(\phi_s),\gamma(\phi_s)\rangle_{\alpha(\phi_s)}}{|\gamma(\phi_s)|_{\alpha(\phi_s)}}\vee 0.$$

For any two points x,y in R^n, we define

$$I_T(\alpha;x,y)=\inf_{\substack{\phi\in\Omega_T\\ \phi(0)=x,\,\phi(T)=y}} I_T(\alpha;\phi),$$

$$I_T(\alpha;x,A)=\inf_{y\in A} I_T(\alpha;x,y),$$

$$I(\alpha;x,y)=\inf_{T>0} I_T(\alpha;x,y),$$

$$I(\alpha;x,A)=\inf_{y\in A} I(\alpha;x,y).$$

Notations $I_T^+(\alpha;x,y)$, $I_T^+(\alpha;x,A)$, $I^+(\alpha;x,y)$, and $I^+(\alpha;x,A)$ are defined similarly. The only two cases we will use are $\alpha=a^{-1}$ and $\alpha=qa^{-1}$. In the former case, we often suppress a^{-1} from the notation. We recall that all these I-functions are continuous in the space variables.

We also need the following fundamental assumptions on the vector field V which will be collectively referred to as Condition (C):

(C1) $\langle V(x),n(x)\rangle<0$ for all $x\in\partial D$.

(C2) The dynamical system determined by the vector field V has a unique equilibrium point $x_0\in D$. Namely, for every $x\in\bar{D}$, the trajectory ϕ^x defined by $\dot{\phi}^x_s=V(\phi_s)$ and $\phi^x(0)=x$ has the property that

$$\lim_{t\to\infty}\phi^x(t)=x_0.$$

We are now in a position to state our main results.

THEOREM 1. *Suppose that Condition (C) is satisfied and that $q(x_0)>0$. Let u_f^ϵ be the unique solution of the boundary value problem $[\epsilon^2 L+V]u_f^\epsilon - qu_f^\epsilon = 0$ on D, $\partial u_f^\epsilon/\partial\gamma = f$ on ∂D. Then we have, uniformly on $\bar{D}$,*

$$\lim_{\epsilon\to 0}\epsilon^2\log u_f^\epsilon(x) = -I^+(a^{-1};x,S_f),$$

and $S_f = \text{ess.supp}\, f$.

THEOREM 2 [Case $V\equiv 0$]. *Suppose that q is strictly positive on $\bar{D}$ and f is continuous on ∂D. Let u_f^ϵ be the unique solution of the boundary value problem $\epsilon^2 Lu_f^\epsilon - qu_f^\epsilon = 0$ on D, $\partial u_f^\epsilon/\partial\gamma = f$ on ∂D. Then we have, uniformly on $\bar{D}$,*

$$\lim_{\epsilon\to 0}\epsilon\log u_f^\epsilon(x) = -2\sqrt{I_1^+(qa^{-1};x,S_f)}\,,$$

and $S_f = \text{supp}\, f$.

REMARK. Our method applies to vector fields V with more complicated ω-limit sets. In particular, if the ω-limit set of V satisfies the condition specified in [4, p. 169] then Theorem 1 holds as stated. The proof of this case only requires some minor modifications to the proof we will present.

As we mentioned earlier, the starting point of our method is the probabilistic representation (1.2) of the solution u_f^ϵ. We naturally divide the proof of Theorem 1 or 2 into two parts—that is, estimating the exponential decay rate from above and estimating it from below. An estimate from above can be obtained via (1.3) from a crude upper bound of u_f^ϵ on the boundary and an exponential upper bound of $E_x^\epsilon[e_q(T_{S_f})]$. (See Propositions 3 and 8 below.)

To get a lower bound of the exponential rate from (1.3), we have two difficulties if f is merely assumed to be measurable (as in Theorem 1). First, since S_f is a closed set and may not have a dense open set, the lower bound of $E_x^\epsilon[e_q(T_{S_f})]$ is not easy to obtain. Second, $\inf_{y\in S_f} u_f^\epsilon(y)$ may decrease too fast to zero to be useful. To overcome these difficulties, we replace the set S_f in (1.3) by an open set of the form $\Delta(x,\epsilon^\alpha)$ (with $x\in\partial D$), whose diameter is shrinking at the rate ϵ^α. The point x is chosen so that

$$\int_{\Delta(x,\lambda)} f(y)\frac{\sigma(dy)}{\sigma(\Delta(x,\lambda))}\to l_f(x)>0 \quad \text{as } \lambda\to 0.$$

Such points are dense in S_f. We have

$$\begin{aligned} u_f^\epsilon(x) &= E_x^\epsilon\left[\int_0^{T_\epsilon} e_q(s)f(X_s)\,\phi(ds)\right] + E_x^\epsilon[e_q(T_\epsilon)u_f^\epsilon(X_{T_\epsilon})] \\ &\ge E_x^\epsilon[e_q(T_\epsilon)]\inf_{y\in\Delta(x,\epsilon^\alpha)} u_f^\epsilon(y), \end{aligned}$$

where $T_\epsilon = T_{\Delta(x,\epsilon^\alpha)}$, the first hitting time of $\Delta(x,\epsilon^\alpha)$. It turns out that if $\alpha > 1$ we can get an effective lower bound of u_f^ϵ over the set $\Delta(x,\epsilon^\alpha)$ (Proposition 6). On the other hand, for $\alpha < 2$ the exponential decay rate of $E_x^\epsilon[e_q(T_\epsilon)]$ can be obtained by proving a version of the large deviation lower bound for a shrinking open set in the path space Ω_T (Lemma 9 and Proposition 10). The lower bound in Theorem 2 is relatively easy because S_f does contain a dense open subset.

The plan of the paper is as follows. Section 2 is devoted to the statement of various intermediate results and the proof of the main Theorems 1 and 2. The proof of these intermediate results are taken up in Sections 3 and 4. In Section 3, we discuss the bounds for the solution u_f^ϵ on the boundary. In Section 4 we discuss the exponential decay rate of the Feynman–Kac functional $E_x^\epsilon[e_q(T_A)]$ for various sets A on the boundary.

2. Outline of the Proof of the Main Theorems

Throughout the paper, $\epsilon_0, \epsilon_1, \ldots, t_0, t_1, \ldots, c_0, c_1, \ldots$ denote constants which depend only on L, D, V, γ, q; otherwise the dependence will be explicitly indicated. u_f^ϵ will denote the solution of the boundary value problem

$$(\epsilon^2 L + V)u_f^\epsilon - qu_f^\epsilon = 0 \text{ on } D, \quad \partial u_f^\epsilon/\partial\gamma = f \text{ on } \partial D,$$

with data L, D, V, γ, q subject to conditions (i)–(vi) stated in the preceding section. The propositions stated below will be proved in the next two sections.

First we give a global upper bound of the solution.

PROPOSITION 3. *Suppose that Condition (C) is satisfied and that $q(x_0) > 0$. Then there exist positive constants c_1 and ϵ_0 such that*

$$\sup_{x\in\bar{D}} u_f^\epsilon(x) \le c_1 \|f\|_\infty \epsilon$$

for all $\epsilon \le \epsilon_0$.

PROPOSITION 4. *Suppose that $V \equiv 0$. Then u_f^ϵ tends to zero uniformly as $\epsilon \to 0$.*

We now turn to the lower bounds of the solution on the boundary. Define

$$f_\epsilon(x) = \frac{1}{\sigma[\Delta(x,\epsilon)]} \int_{\Delta(x,\epsilon)} f(y)\sigma(dy).$$

By the Lebesgue differentiation theorem [11, p. 100], we have $f_\epsilon(x) \to f(x)$, σ-a.e. on ∂D as $\epsilon \to 0$. Let

$$Q_f = \left\{x \in \partial D : \lim_{\epsilon\to 0} f_\epsilon(x) = l_f(x) \text{ exists and is positive}\right\}.$$

Obviously, Q_f is dense in ess.supp$f$. We have the following lower bound for u_f^ϵ.

PROPOSITION 6. *Suppose that Condition (C) is satisfied. Fix an* $\alpha > 1$ *and* $x \in Q_f$. *Then for each* $h > 0$ *there exist* $\epsilon_0(x, f, h) > 0$ *such that*

$$\inf_{z \in \Delta(x, \epsilon^\alpha)} u_f^\epsilon(z) \geq e^{-h\epsilon^{-2}}$$

for all $\epsilon \leq \epsilon_0(x, f, h)$.

PROPOSITION 7. *Suppose that* $V \equiv 0$ *and that* f *is continuous on* ∂D. *Then for each* x *with* $f(x) > 0$ *there exist* $\delta = \delta(x, f)$ *and* $\epsilon_0(x, f) > 0$ *such that*

$$\inf_{z \in \Delta(x, \delta)} u_f^\epsilon(z) \geq c_1 \epsilon f(x)$$

for all $\epsilon \leq \epsilon_0(x, f)$.

Recall that

$$e_q(t) \overset{\text{def}}{=} \exp\left\{-\int_0^t q(X_s)\, ds\right\}.$$

The first hitting time of a set F by the process X is denoted by T_F. We thus have the following.

PROPOSITION 8. *Suppose that Condition (C) is satisfied and that* $q(x_0) > 0$. *Then for each closed subset* F *of the boundary* ∂D *and each* $h > 0$, *there exists* $\epsilon_0(F, h) > 0$ *such that*

$$\epsilon^2 \log E_x^\epsilon[e_q(T_F)] \leq -I^+(a^{-1}; x, F) + h$$

for all $x \in \bar{D}$ *and all* $\epsilon \leq \epsilon_0(F, h)$.

PROPOSITION 10. *Suppose that Condition (C) is satisfied. Fix* $\alpha < 2$ *and* $z \in \partial D$. *Then for each* $h > 0$, *there exists* $\epsilon_0(z, h) > 0$ *such that*

$$\epsilon^2 \log E_x^\epsilon[e_q(T_{\Delta(z, \epsilon^\alpha)})] \geq -I^+(a^{-1}; x, z) - h$$

for all $\epsilon \leq \epsilon_0(z, h)$.

PROPOSITION 12. *Suppose that* $V \equiv 0$ *and* q *is strictly positive on* $\bar{D}$.

(a) *For each closed subset* F *of* ∂D *and for each* $h > 0$, *there exists*

$$\epsilon_0(F, h) > 0$$

such that

$$\epsilon \log E_x^\epsilon[e_q(T_F)] \leq -2\sqrt{I_1^+(qa^{-1}; x, F)} + h$$

for all $\epsilon \leq \epsilon_0(F, h)$.

(b) *Let* $z \in \partial D$. *Then for each open subset* G *of* ∂D *containing* z *and for each* $h > 0$, *there exists* $\epsilon_1(G, z, h) > 0$ *such that*

$$\epsilon \log E_x^\epsilon[e_q(T_G)] \geq -2\sqrt{I_1^+(qa^{-1};x,z)} - h$$

for all $\epsilon \leq \epsilon_1(G,z,h)$.

Note that $\sqrt{2I_1^+(qa^{-1};x,\partial D)}$ is simply the distance from x to ∂D in the Riemannian metric determined by qa^{-1}. We now turn to the proofs of our main theorems.

Proof of Theorem 1. Fix $\alpha \in (1,2)$. Let h be an arbitrary small positive number. Since Q_f is dense in ess.supp$f$, and $I^+(x,z)$ is uniformly continuous in x, z, there exist a finite set of points $\{z_1, \dots, z_l\} \subset Q_f$ such that

$$I^+(x,S_f) \geq \min_{1\leq i\leq l} I^+(x,z_i) - h$$

for all $x \in \bar{D}$. As we have argued in Section 1, the representation (1.2) and the Markov property give, for $i = 1, \dots, l$,

$$u_f^\epsilon(x) \geq E_x^\epsilon[e_q(T_\epsilon^i)] \cdot \inf_{y\in\Delta(z_i,\epsilon^\alpha)} u_f^\epsilon(y),$$

where $T_\epsilon^i = T_{\Delta(z_i,\epsilon^\alpha)}$. Using Propositions 6 and 10, we have

$$\epsilon^2 \log u_f^\epsilon(x) \geq -I^+(a^{-1};x,z_i) - 2h$$

for all $\epsilon \leq \epsilon_0(z_i, f, h)$. Hence, there exists a positive number $\epsilon_2(f,h)$ such that

$$\epsilon^2 \log u_f^\epsilon(x) \geq -\min_{1\leq i\leq l} I^+(a^{-1};x,z_i) - 2h \geq -I^+(x,S_f) - 3h \tag{2.1}$$

for all $\epsilon \leq \epsilon_2(f,h)$. On the other hand, since

$$u_f^\epsilon(x) \leq E_x^\epsilon[e_q(T_{S_f})] \sup_{z\in\partial D} u_f^\epsilon(z), \tag{2.2}$$

Propositions 3 and 8 immediately give

$$\epsilon^2 \log u_f^\epsilon(x) \leq -I^+(a^{-1};x,S_f) + 2h \tag{2.3}$$

for $\epsilon \leq \epsilon_3(f,h)$. (2.1) and (2.3) imply the conclusion of the theorem. □

Proof of Theorem 2. The upper bound can be obtained from (2.2) by using Propositions 4 and 12(i). For the lower bound, we choose a finite set of points $\{z_1, \dots, z_l\}$ such that $f(z_i) > 0$, $(i = 1, \dots, l)$ and

$$I_1^+(qa^{-1};x,S_f) \geq \min_{1\leq i\leq l} I_1^+(qa^{-1};x,z_i) - h$$

for all $x \in \bar{D}$. Let $\Delta_i = \Delta(z_i, \delta_i)$ be a neighborhood of z_i such that Proposition 7 holds. Using the inequality

$$u_f^\epsilon(x) \geq E_x^\epsilon[e_q(T_{\Delta_i})] \cdot \inf_{y\in\Delta_i} u_f^\epsilon(y)$$

and Propositions 7 and 12(i), we have

$$\epsilon \log u_f^\epsilon(x) \geq -2\sqrt{I_1^+(qa^{-1}; x, z_i) + h} - 2h$$

for all $\epsilon \leq \epsilon_4(z_i, \Delta_i, f, h)$. It follows that

$$\epsilon \log u_f^\epsilon(x) \geq -2\sqrt{I_1^+(qa^{-1}; x, S_f) + 2h} - 2h$$

for all $\epsilon \leq \epsilon_5(f, h)$. This proves the theorem. □

3. Bounds of the Solution on the Boundary

The results concerning the estimates of u_f^ϵ used in the last section will be restated and proved in this section.

PROPOSITION 3. *Suppose that Condition (C) is satisfied and that* $q(x_0) > 0$. *Then there exist positive constants* c_1 *and* ϵ_0 *such that*

$$\sup_{x \in \bar{D}} u_f^\epsilon(x) \leq c_1 \|f\|_\infty \epsilon$$

for all $\epsilon \leq \epsilon_0$.

Proof. Let $T > 0$, whose value will be determined later. From the probabilistic representation (1.2), we have (by the Markov property),

$$u_f^\epsilon(x) = \sum_{n=0}^{\infty} E_x^\epsilon\left[\int_{nT}^{(n+1)T} e_q(s) f(X_s)\, \phi(dx)\right]$$

$$\leq \|f\|_\infty \sum_{n=0}^{\infty} E_x^\epsilon[e_q(nT)(\phi(nT+T) - \phi(nT))] \tag{3.1}$$

$$\leq \|f\|_\infty \sum_{n=0}^{\infty} E_x^\epsilon[e_q(nT)] \sup_{z \in \bar{D}} E_z^\epsilon[\phi(T)].$$

Using the Markov property, it is easy to show by induction that

$$\sup_{z \in \bar{D}} E_z^\epsilon[e_q(nT)] \leq \left\{\sup_{z \in \bar{D}} E_z^\epsilon[e_q(T)]\right\}^n. \tag{3.2}$$

We will prove the following two inequalities: There exist positive T, ϵ_0, c_3, and $0 < c_2 < 1$ such that, for any $\epsilon \leq \epsilon_0$,

$$\sup_{z \in \bar{D}} E_z^\epsilon[e_q(T)] \leq c_2 < 1 \tag{3.3}$$

and

$$\sup_{z \in \bar{D}} E_z^\epsilon[\phi(T)] \leq c_3 \epsilon. \tag{3.4}$$

If (3.3) and (3.4) hold, then we obtain from (3.1) and (3.2) that

$$u_f^\epsilon(x) \leq \frac{c_2 c_3}{1 - c_2} \|f\|_\infty \epsilon$$

for all $x \in \bar{D}$ and $\epsilon \leq \epsilon_0$. This will prove the proposition.

To prove (3.3) and (3.4), we first recall a result about diffusion with small parameter. Let X satisfy the stochastic differential equation (1.1). Let ϕ^x be the solution of $\dot{\phi}_t^x = V(\phi_t)$, $\phi_0^x = x$. Then we have, for each fixed $T > 0$,

$$\sup_{x \in \bar{D}} E_x^\epsilon[|X-\phi|_T^2] \leq c_4(T)\epsilon^2. \tag{3.5}$$

[For unrestricted diffusion (1.4) (without boundary condition) this can be verified in the usual way using Doob's martingale inequality and Gromwall's inequality. For diffusion (1.1) with boundary condition, one can reduce it to the case without boundary condition by the method of Anderson and Orey [1].] Let us establish (3.3). Since $q(x_0) > 0$, there exist positive numbers λ, μ such that $q(x) \geq \mu$ on $B_\lambda(x_0)$. Now Condition (C) implies the following fact: There is a $T_0 > 0$ such that $\phi_t^x \in B_\lambda(x_0)$ for all $t \geq T_0$ and all $x \in \bar{D}$. Let us now choose $T = T_0 + 1$. The choice of T_0 and the fact that $q(x_0) > 0$ implies the existence of a small $\delta > 0$ such that, for all $x \in \bar{D}$,

$$|\phi - \phi^x|_T \leq \delta \quad \text{implies} \quad \int_0^T q(\phi_s)\, ds \geq \frac{\mu}{2}.$$

Now we have

$$\begin{aligned} E_x^\epsilon[e_q(T)] &\leq E_x^\epsilon[e_q(T); |X-\phi|_T \leq \delta] + P_x^\epsilon[|X-\phi|_T \geq \delta] \\ &\leq e^{-\mu/2} + c_4(T)\epsilon^2/\delta^2. \end{aligned}$$

[We have used (3.5) and Chebyshev's inequality in the second step.] Thus (3.3) holds if ϵ is small.

To establish (3.4), we choose a function $g \in C^2(\bar{D})$ such that $\partial g/\partial\gamma = 1$ on ∂D. Using Itô's formula on $g(X_t)$ with X_t determined by (1.1), we have

$$\begin{aligned} \phi(T) &= \epsilon \int_0^T (\nabla g)^T \sigma(X_s)\, dB_s + \epsilon^2 \int_0^T Lg(X_s)\, ds \\ &\quad + \int_0^T \langle V, \nabla g\rangle(X_s)\, ds - g(X_T) + g(x). \end{aligned} \tag{3.6}$$

For the last three terms in the above equation we have

$$\begin{aligned} &\left| \int_0^T \langle V, \nabla g\rangle(X_s)\, ds - g(X_T) + g(x) \right| \\ &\leq \left| \int_0^T \langle V, \nabla g\rangle(\phi_s^x)\, ds - g(\phi_T^x) + g(x) \right| + c_5(T+1)|X - \phi^x|_T \\ &= c_5(T+1)|X-\phi^x|_T. \end{aligned} \tag{3.7}$$

[The first term after the inequality sign vanishes because of Condition (C1).] Taking the expectation in (3.6) and using (3.5) and (3.7), we obtain (3.4) immediately. The proof of Proposition 3 is completed. □

PROPOSITION 4. *Suppose that $V \equiv 0$. Then, uniformly on $\bar{D}$,*

$$\lim_{\epsilon \to 0} u_f^\epsilon(x) = 0.$$

Proof. We prove the following two inequalities:

(i) For each positive δ, there exists $\lambda=\lambda(\delta)>0$ such that

$$\sup_{x\in\bar{D}} E^{\epsilon}[\phi(\lambda\epsilon^{-2})]\leq\delta \tag{3.8}$$

for all $\epsilon\geq 0$.

(ii) For each fixed $\lambda>0$, there exists $\epsilon_0(\lambda)>0$ such that

$$\sup_{x\in\bar{D}} E_x^{\epsilon}[e_q(\lambda\epsilon^{-2})]\leq\frac{1}{2}, \tag{3.9}$$

for all $\epsilon\leq\epsilon_0(\lambda)$.

Suppose these two inequalities hold. Taking $T=\lambda\epsilon^{-2}$ in (3.1) and (3.2), we have immediately that $u_f^{\epsilon}(x)\leq\|f\|_{\infty}\delta$ for $\epsilon\leq\epsilon_0(\lambda)$, which proves the proposition.

Let us prove (3.8). In (3.6), we may choose g such that $\|g\|_{\infty}\leq\delta/3$. Since $V\equiv 0$, the term involving V disappears. We obtain (3.8) by taking $\lambda=\delta(3\|Lg\|_{\infty})^{-1}$ and $T=\lambda\epsilon^{-2}$.

To prove (3.9), we note from (1.1) that the law of $\{X_t, 0\leq t\leq 1\}$ under probability P_x^{ϵ} is identical to that of $\{X_{\epsilon^2 t}, 0\leq t\leq 1\}$ under probability P_x^1. Therefore we can write

$$\begin{aligned} E_x^{\epsilon}[e_q(\lambda\epsilon^{-2})] &= E_x^{\epsilon}\left[\exp\left\{-\int_0^{\lambda\epsilon^{-2}} q(X_s)\,ds\right\}\right] \\ &= E_x^{1}\left[\exp\left\{-\frac{1}{\epsilon^2}\int_0^{\lambda} q(X_s)\,ds\right\}\right] \\ &\stackrel{\text{def}}{=} \chi_{\epsilon}(x). \end{aligned} \tag{3.10}$$

Now each $\chi_{\epsilon}(x)$ is continuous on $\bar{D}$. Since we assume that $q\not\equiv 0$ on $\bar{D}$, we have $\chi_{\epsilon}(x)\downarrow 0$ as $\epsilon\downarrow 0$ for fixed $x\in\bar{D}$. Therefore $\chi_{\epsilon}\downarrow 0$ uniformly on $\bar{D}$ (Dini's theorem). (3.9) follows immediately. □

We now turn to the lower bound for the solution u_f^{ϵ}. First, we need a simple lemma. Recall the definition of f_{ϵ} in the last section. We have the following.

LEMMA 5. *There is a constant $c_1>0$ such that, for any two points x, y on the boundary ∂D satisfying $|y-x|\leq\epsilon$,*

$$|f_{\epsilon}(x)-f_{\epsilon}(y)|\leq c_1\|f\|_{\infty}\frac{|y-x|}{\epsilon}.$$

Proof. From the definition of f_{ϵ}, we have

$$\begin{aligned} |f_{\epsilon}(x)-f_{\epsilon}(y)| \leq{}& \frac{\|f\|_{\infty}}{\sigma[\Delta(x,\epsilon)]}\{\sigma[\Delta(x,\epsilon)\backslash\Delta(y,\epsilon)]+\sigma[\Delta(y,\epsilon)\backslash\Delta(x,\epsilon)]\} \\ &+\|f\|_{\infty}\left|\frac{1}{\sigma[\Delta(x,\epsilon)]}-\frac{1}{\sigma[\Delta(y,\epsilon)]}\right|\sigma[\Delta(y,\epsilon)]. \end{aligned}$$

By simple geometric considerations, we see that the expression inside the braces in the above inequality is bounded by $c_2|y-x|\epsilon^{n-1}$. The assertion of the lemma follows easily. □

Recall the definition of Q_f from the previous section. We have the following lower bound for u_f^ϵ.

PROPOSITION 6. *Suppose that vector field V is uniformly bounded. Fix $\alpha>1$ and $x\in Q_f$. Then, for each $h>0$, there exists $\epsilon_0(x,f,h)>0$ such that*

$$\inf_{z\in\Delta(x,\epsilon^\alpha)} u_f^\epsilon(z)\geq e^{-h\epsilon^{-2}}$$

for all $\epsilon\leq\epsilon_0(x,f,h)$.

Proof. Set $\beta=\|q\|_\infty$ for simplicity. Define the process $Y=Y^{x,\epsilon}$ by the stochastic differential equation

$$dY_t=\epsilon\sigma(Y_t)dB_t+\epsilon^2 b(Y_t)dt-\gamma(Y_t)\psi(dt),\quad Y_0=z. \tag{3.11}$$

Set $\phi_f(t)=\int_0^t f(X_s)\,\phi(ds)$ and $\psi_f(t)=\int_0^t f(Y_s)\,\psi(ds)$. We have from the probabilistic representation (1.2) that, for any $T>0$,

$$\begin{aligned} u_f^\epsilon(z)&\geq E_z^\epsilon\left[\int_0^T e^{-\beta t}\phi_f(dt)\right]\geq e^{-\beta T}E_z^\epsilon[\phi_f(T)]\\ &\geq e^{-\beta T}E_z^\epsilon[M_T\psi_f(T)]. \end{aligned}\tag{3.12}$$

In the last step we have used the well-known theorem of Girsanov on absolute continuity of diffusion measures. The exponential martingale $\{M_t,t\geq 0\}$ is

$$M_t=\exp\left\{\frac{1}{\epsilon}N_t-\frac{1}{2\epsilon^2}\langle N\rangle_t\right\},$$

with

$$N_t=\int_0^t V^T\sigma(Y_s)^{-1}dB_s.$$

By Schwarz's inequality, we have

$$\begin{aligned} E_z^\epsilon[\psi_f(T)]^2&\leq E_z^\epsilon[M_T\psi_f(T)]E_z^\epsilon[M_T^{-1}\psi_f(T)]\\ &\leq E_z^\epsilon[M_T\psi_f(T)]E_z^\epsilon[M_t^{-2}]^{1/2}E_z^\epsilon[\psi_f(T)^2]^{1/2}. \end{aligned}$$

This is equivalent to

$$E_z^\epsilon[M_t\psi_f(T)]\geq\frac{E_z^\epsilon[\psi_f(T)]^2}{E_z^\epsilon[M_T^{-2}]^{1/2}E_z^\epsilon[\psi_f(T)^2]^{1/2}}. \tag{3.13}$$

From (3.12) and (3.13) we need (a) an upper bound for $E_z^\epsilon(M_T^{-2})$; (b) an upper bound for $E_z^\epsilon[\psi_f(T)^2]$; and (c) a lower bound for $E_z^\epsilon[\psi_f(T)]$.

(a) Since a^{-1} and V are uniformly bounded, we have

$$\langle N\rangle_T=\int_0^T V^Ta^{-1}V(Y_s)\,ds\leq c_1T.$$

Hence

$$
\begin{aligned}
E_z^\epsilon[M_T^{-2}] &= E_z^\epsilon\left[\exp\left\{-\frac{2}{\epsilon}N_T - \frac{2}{\epsilon^2}\langle N\rangle_T + \frac{3}{\epsilon^2}\langle N\rangle_T\right\}\right] \\
&\le E_z^\epsilon\left[\exp\left\{-\frac{2}{\epsilon}N_T - \frac{2}{\epsilon^2}\langle N\rangle_T\right\}\right]\exp\left\{\frac{3c_1T}{\epsilon^2}\right\} \qquad (3.14)\\
&= \exp\left\{\frac{3c_1T}{\epsilon^2}\right\}.
\end{aligned}
$$

(b) To obtain an upper bound for $E_z^\epsilon[\psi_f(T)^2]$, we note first that $\psi_f(T)$ has the same law under the probability P_z^ϵ as $\psi_f(\epsilon^2 T)$ has under the probability E_z^1. Let us use $p_Y(t,z,y)$ to denote the transition density function of Y under the probability E_z^1. We have

$$
E_z^1[\psi_f(t)] = \int_0^t ds\int_{\partial D} p_Y(s,z,y)f(y)\,\sigma(dy). \qquad (3.15)
$$

Using the method of parametrix, we can show that $p_Y(s,z,y)$ has the following exponential bounds (see [9], where the heat kernel with the oblique derivative boundary condition is treated in great detail): There is a $t_0>0$ such that, for $s\le t_0$,

$$
p_Y(s,z,y) \ge c_2 s^{-n/2}\exp\{-c_2^{-1}|y-z|^2/s\} - c_3 s^{-(n-1)/2}\exp\{c_3^{-1}|y-z|^2/s\},
$$

$$
p_Y(s,z,y) \le c_3 s^{-n/2}\exp\{-c_3^{-1}|y-z|^2/s\}.
$$

The upper bound of $p_Y(s,z,y)$ and (3.14) imply by simple integration that there is a constant c_4 such that, for all $t\le t_0$,

$$
\sup_{z\in\bar D} E_z^1[\psi_f(t)] \le c_4\|f\|_\infty\sqrt{t}.
$$

Using the Markov property, we have

$$
\begin{aligned}
E_z^1[\psi_f(t)^2] &= 2E_z^1\left[\int_0^t \{\psi_f(t)-\psi_f(s)\}\,\psi_f(ds)\right] \\
&= 2E_z^1\left[\int_0^t E_{X_s}^1\{\psi_f(t-s)\}\,\psi_f(ds)\right] \\
&\le c_5\|f\|_\infty^2 t.
\end{aligned}
$$

Hence we have, for $\epsilon^2 T\le t_0$,

$$
E_z^\epsilon[\psi_f(T)^2] = E_z^1[\psi_f(\epsilon^2 T)] \le c_5\|f\|_\infty^2\epsilon^2 T. \qquad (3.16)
$$

(c) To obtain a lower bound of $E_z^\epsilon[\psi_f(T)] = E_z^1[\psi_f(\epsilon^2 T)]$ with $z\in\partial D$, we integrate the lower bound of the heat kernel $p_Y(s,z,y)$ to get

$$
\int_0^t p_Y(s,z,y)\,ds \ge c_6|y-z|^{-d+2}H_0\left(\frac{c_6 t}{|y-z|^2}\right) - c_7|y-z|^{-d+3}H_1\left(\frac{c_7 t}{|y-z|^2}\right),
$$

where

$$H_i(u)=\int_0^u s^{-(n-i)/2}e^{-1/s}\,ds, \quad i=0,1.$$

For fixed $u_0>0$ there exists $\lambda=\lambda(u_0)$ such that, for all $u\ge u_0$,

$$\begin{array}{lll} H_0(u)\ge\lambda\sqrt{u}, & H_l(u)\le\lambda^{-1}u, & \text{if } n=1; \\ H_0(u)\ge\lambda\log u, & H_1(u)\le\lambda^{-1}\sqrt{u}, & \text{if } n=2; \\ H_0(u)\ge\lambda, & H_1(u)\le\lambda^{-1}\log u, & \text{if } n=3; \\ H_0(u)\ge\lambda, & H_1(u)\le\lambda^{-1}, & \text{if } n\ge 4. \end{array}$$

Take $u_0=T\min(c_6,c_7)$. Using the above estimates, we can verify that in all dimensions the following assertion holds: There exists $\epsilon_1(T)>0$ such that, for all $\epsilon\le\epsilon_1(T)$ and all pair of points z,y satisfying $|y-z|\le\epsilon$,

$$\int_0^{\epsilon^2T} p_Y(s,z,y)\,ds\ge c_8(T)\epsilon^{-d+2}. \tag{3.17}$$

Let $t=\epsilon^2T$ in (3.15) and use (3.17). By the definition of f_ϵ, we have immediately that

$$E_z^\epsilon[\psi_f(T)]=E_z[\psi_f(\epsilon^2T)]\ge c_9(T)\epsilon f_\epsilon(z). \tag{3.18}$$

Collecting (3.13), (3.14), (3.16) and (3.18), we have proved that

$$E_z^\epsilon[M_T\psi_f(T)]\ge c_{10}(T)\|f\|_\infty^{-1}f_\epsilon(z)^2\exp\{-3c_1T\epsilon^{-2}\}. \tag{3.19}$$

Now let $x\in Q_f$. Then $f_\epsilon(x)\to l_f(x)>0$. Since $\alpha>1$, Lemma 5 implies that there exists $\epsilon_2(x,f)>0$ such that

$$\inf_{z\in\Delta(x,\epsilon^\alpha)} f_\epsilon(z)\ge\frac{1}{2}l_f(x) \tag{3.20}$$

for all $\epsilon\le\epsilon_2(x,f)$. Combining this inequality with (3.12) and (3.19), we see that

$$\inf_{z\in\Delta(x,\epsilon^\alpha)} u_f^\epsilon(z)\ge c_{11}(T)l_f(x)^2\|f\|_\infty^{-1}\exp\{-3c_1T\epsilon^{-2}\}$$

for $\epsilon\le\epsilon_3(x,f,T)$. To obtain the proposition it is enough to take $T=h/6c_1$ in the above inequality. □

PROPOSITION 7. *Suppose that $V\equiv 0$ and that f is continuous on ∂D. Then there exist $\delta=\delta(x,f)$ and $\epsilon_0(x,f)>0$ such that*

$$\inf_{z\in\Delta(x,\epsilon^\alpha)} u_f^\epsilon(z)\ge c_1\epsilon f(x)$$

for all $\epsilon\le\epsilon_0(x,f)$.

Proof. We have in this case

$$u_f^\epsilon(z)\ge e^{-\beta T}E_z^\epsilon[\psi_f(T)].$$

Choose $\delta > 0$ so that $f(z) > f(x)/2$ for $z \in \Delta(x, \delta)$. We have from (3.18) that

$$E_z^\epsilon[\psi_f(T)] \geq \frac{c_9(T)}{2} \epsilon f(x)$$

for all $\epsilon \leq \min\{\delta, \epsilon_1(T)\}$ and $z \in B_\delta(x)$. The proposition follows by taking $T = 1$. □

4. Large Deviation Bounds for Feynman–Kac Functional $E_x^\epsilon[e_q(T_A)]$

A result of Anderson and Orey [1] (see also [2]) states that $I_T^+(a^{-1}, \phi)$ introduced in Section 1 is the large deviation rate function (in the path space Ω_T) of the diffusion (1.1) with oblique reflection. (Refer to their work for the exact statement of this result.) We will need a modified version of the large deviation lower bound. See Lemma 9 below.

Our first result in this section is the following.

PROPOSITION 8. *Suppose that Condition (C) is satisfied and that* $q(x_0) > 0$. *Then, for each closed subset F of the boundary* ∂D *and each* $h > 0$, *there exists* $\epsilon_0(F, h) > 0$ *such that*

$$\epsilon^2 \log E_x^\epsilon\left[\exp\left\{-\int_0^{T_F} q(X_s)\, ds\right\}\right] \leq -I^+(a^{-1}; x, F) + h$$

for all $x \in \bar{D}$ *and all* $\epsilon \leq \epsilon_0(F, h)$.

Proof. Recall that x_0 is the unique equilibrium point of the deterministic dynamical system $\dot{\phi}_s = V(\phi_s)$. We use the well-known method of Freidlin and Wentzell [4] to handle the equilibrium point. Let h be an arbitrary positive number. Choose a positive r such that:

(1) the closure of $B_r(x_0)$ is contained in D;
(2) $I^+(x_0, y) \leq I^+(z, y) + h$ for any $z \in B_r(x_0)$ and $y \in \bar{D}$; and
(3) q is positive on $\overline{B_r(x_0)}$.

Let $K_1 = \partial B_r(x_0)$ and $K_2 = \partial B_{r/2}(x_0)$. Define the following sequence of stopping times:

$$\begin{aligned}
&\sigma_0 = 0,\\
&\tau_1 = T_{K_1}\\
&\sigma_1 = \tau_1 + T_{K_2} \circ \theta_{\tau_1},\\
&\quad\vdots\\
&\tau_n = \sigma_{n-1} + T_{K_1} \circ \theta_{\sigma_{n-1}},\\
&\sigma_n = \tau_n + T_{K_2} \circ \theta_{\tau_n}.
\end{aligned}$$

No matter where the process starts, we have either $T_F < T_{K_1} = \tau_1$ or $T_F \in (\tau_n, \sigma_n)$ for some $n \geq 1$. Therefore, using the Markov property we can write, for any $x \in \bar{D}$,

$$E_x^\epsilon[e_q(T_F)]=E_x^\epsilon[e_q(T_F);T_F<T_{K_1}]$$

$$(4.1)\qquad +\sum_{n=1}^{\infty} E_x^\epsilon[e_q(\tau_n)E_{X_{\tau_n}}^\epsilon\{e_q(T_F);T_F<T_{K_2}\},\tau_n<T_F]$$

$$\le P_x^\epsilon[T_F\le T_{K_1}]+\sup_{z\in K_1} P_z^\epsilon[T_F\le T_{K_2}]\sum_{n=1}^{\infty} E_x^\epsilon[e_q(\tau_n)].$$

We will prove the following two inequalities. There exist $\epsilon_0(F,h)>0$ such that

$$(4.2)\qquad \epsilon^2\log P_x^\epsilon[T_F\le T_{K_1}]\le -I^+(x,F)+h$$

and

$$(4.3)\qquad \sum_{n=1}^{\infty} E_x^\epsilon[e_q(\tau_n)]\le c_1$$

for all $\epsilon\le\epsilon_0(F,h)$.

Proof of Inequality (4.2). Fix $T>0$. We have

$$(4.4)\qquad P_x^\epsilon[T_F\le T_{K_1}]\le P_x^\epsilon[T_F\le T]+P_x^\epsilon[T_{K_1}>T].$$

By Proposition 4 of [1] and the additivity of $I_T^+(\phi)$ in T, the argument of [4, Lemma 2.2(a), pp. 110–111] shows that there exist constants c_2 and T_0 such that if ϕ: $[0,T]\to\bar D\setminus B_r(x_0)$, then $I_T^+(\phi)\ge c_2(T-T_0)$. This implies, as in [4, Lemma 2.2(b)], that

$$(4.5)\qquad P_x^\epsilon[T_{K_1}>T]\le \exp\{-c_3(T-T_0)/\epsilon^2\}.$$

On the other hand, since the set $\{\phi\in\Omega_T:\phi(0)=x,\ T_F(\phi)\le T\}$ is at a positive distance from the set $\{\phi\in\Omega_T:\phi(0)=x,\ I_T^+(\phi)\le I_T^+(x,F)-h/2\}$, the large deviation upper bound implies that there exists $\epsilon_1(F,T,h)>0$ such that

$$(4.6)\qquad \epsilon^2\log P_x^\epsilon[T_F\le T]\le -I_T^+(x,F)+h\le -I^+(x,F)+h$$

for all $x\in\bar D$ and $\epsilon\le\epsilon_1(F,T,h)$. (The uniformity in x is not explicitly stated in [1] for the large deviation upper bound, but it can be verified easily in the present case.) It follows from (4.4)–(4.6) that

$$\epsilon^2\log P_x^\epsilon[T_F\le T_{K_1}]\le -\min\{I^+(x,F),c_3(T-T_0)\}+2h.$$

Taking $T\ge T_0+\max_{x\in\bar D} I^+(x,F)/c_3$, we obtain (4.2).

Proof of Inequality (4.3). Since

$$\sigma_n=\sum_{k=0}^{n-1}(T_{K_1}+T_{K_2}\circ\theta_{T_{K_1}})\circ\theta_{\sigma_k},$$

we have (by the Markov property) that

$$E_x^\epsilon[e_q(\sigma_n)]\le\left\{\sup_{z\in K_2} E_z^\epsilon[e_q(T_{K_1}+T_{K_2}\circ\theta_{T_{K_1}})]\right\}^n\le\left\{\sup_{z\in K_1} E_z^\epsilon[e_q(T_{K_2})]\right\}^n.$$

To show (4.3), it is sufficient to prove that there exists $0<c_4<1$ such that

$$\sup_{z\in K_1} E_z^\epsilon[e_q(T_{K_2})]\le c_4 \quad \text{for } \epsilon\le\epsilon_2(h), \tag{4.7}$$

because then the left-hand side of (4.3) is bounded by $c_4(1-c_4)^{-1}$.

Define

$$\tau^{z,0}=\inf\{t>0: \phi_t^z\in K_2\}.$$

Let $s_0=\inf_{z\in K_1}\tau^{z,0}$. It is clear that $s_0>0$. Let $\delta\in(0,s_0)$. Since the distance between the set

$$\{\phi\in\Omega_{\tau_0}: \phi(0)=z,\ T_{K_2}(\phi)\le\tau_0-\delta\}$$

and the set

$$\{\epsilon\in\Omega_{\tau_0}: \phi(0)=z,\ I_{\tau_0}^+(\phi)=0\}=\{\phi\in\Omega_{\tau_0}: \phi(0)=z,\ \dot\phi_s=V(\phi_s)\}$$

is positive, the large deviation upper bound shows that there is a positive $c_5(\delta)>0$ such that

$$\epsilon^2\log P_z^\epsilon[T_{K_2}\le\tau^0-\delta]\le -c_5(\delta). \tag{4.8}$$

[We have written τ^0 instead of $\tau^{z,0}$ for simplicity.] By the choice of r, function $q(z)$ is strictly positive on the closure of $B_r(x_0)$. Thus it is possible to choose δ so small that there exists a c_6 with the property that

$$|\phi-\phi^z|_{\tau^0-\delta}\le\delta \quad\text{implies}\quad \int_0^{\tau^0-\delta} q(\phi_s)\,ds\ge c_6>0 \tag{4.9}$$

for all $z\in K_1$. We now can write

$$\begin{aligned}
E_z^\epsilon[e_q(T_{K_2})]&=E_z^\epsilon[e_q(T_{K_2}); T_{K_2}>\tau^0-\delta]+E_z^\epsilon[e_q(T_{K_2}); T_{K_2}\le\tau^0-\delta]\\
&\le E_z^\epsilon[e_q(\tau^0-\delta)]+P_z^\epsilon[T_{K_2}\le\tau^0-\delta]\\
&\le E_z^\epsilon[e_q(\tau^0-\delta); |X-\phi^z|_{\tau^0-\delta}\le\delta]+P_z^\epsilon[|X-\phi^z|_{\tau^0-\delta}>\delta]\\
&\quad +P_z^\epsilon[T_{K_2}\le\tau^0-\delta].
\end{aligned}$$

The first term after the last inequality sign is bounded by e^{-c_6} by (4.9). As $\epsilon\to0$, the second term and the third term go to zero by (4.8) and (3.5). Thus (4.7) is proved.

Finally, we show that (4.2) and (4.3) imply our proposition. Indeed, replacing K_1 by K_2 in (4.2), we have

$$\epsilon^2\log P_z^\epsilon[T_F\le T_{K_2}]\le -I^+(z,F)+h\le -I^+(x_0,F)+2h$$

for all $z\in K_1$ and $\epsilon\le\epsilon_1(F,h)$. Now (4.1), (4.2), (4.3), and the above inequality together give immediately that

$$\epsilon^2\log E_x^\epsilon[e_q(T_F)]\le -\min\{I^+(x,F), I^+(x_0,F)\}+3h$$

for all $\epsilon\le\epsilon_3(F,h)$ and $x\in\bar D$. Since x_0 is the unique equilibrium point, we have $I^+(x_0,F)\ge I^+(x,F)$ for all points $x\in\bar D$. The proposition follows. □

Our next goal is Proposition 10. We first prove a modified version of the large deviation lower bound. In the large deviation theory one usually computes a large deviation lower bound for the probability of a fixed open set in the path space. In our situation we need the open set to shrink as $\epsilon \downarrow 0$. If the size of the open set does not decrease too fast, we expect the same lower bound should hold. The following proposition makes this idea explicit.

LEMMA 9. *Fix $\alpha < 2$. Let $T > 0$ and $\phi \in \Omega_T$ such that $\phi(0) = x$ and $I_T^+(\phi) < \infty$. Then there exists an $\epsilon_0 = \epsilon_0(T, \phi, h) > 0$ such that*

$$\epsilon^2 \log P_x^\epsilon[X_{[0,T]} \in O_{\epsilon^\alpha}(\phi)] \geq -I_T^+(\phi) - h$$

for all $\epsilon \leq \epsilon_0(T, \phi, h)$.

REMARK. The above result claims that the size of an open set in the path space can shrink as fast as ϵ^α $(\alpha < 2)$ without affecting the exponential large deviation lower bound. Because of Lemma 5, it is important for our application that the radius can go to zero faster than ϵ. The lower bound presented in [5, p. 332] implies that the exponential lower bound does not change if the radius does not go to zero faster than $c_0\sqrt{T}\epsilon$ for some constant c_0. This result is not sufficient for our application. Also, here we consider diffusion process with oblique reflection rather than unrestricted diffusion (1.4). We will handle the oblique reflection by the method of Anderson and Orey [1].

Proof of Lemma 9. First of all we notice that the problem can be localized, so we may assume that D is the half-space $D = \{x = (x^1, x^2, \ldots, x^n) : x^1 > 0\}$ and γ is the unit outward normal vector field $\gamma(x) = (-1, 0, \ldots, 0)$. Introduce the map $\Gamma : C(R^n) \to C(R^n_+)$ on the paths space as follows. For $\phi = (\phi^1, \ldots, \phi^n)$, the image $\Gamma(\phi) = \psi$ is defined by $\psi^i = \phi^i$ for $i = 2, \ldots, n$ and $\psi_t^1 = \phi_t^1 - \min_{0 \leq s \leq t}(\phi_s^1 \wedge 0)$ (the Skorohod equation). The following three facts hold.

(α) If $Y = Y^{x,\epsilon}$ is the solution of the stochastic differential equation

$$dY_t = \epsilon\sigma(\Gamma(Y)_t)dB_t + \epsilon^2 b(\Gamma(Y)_t)dt + V(\Gamma(Y)_t)dt, \quad Y_0 = x, \tag{4.10}$$

then the process $X = \Gamma(Y)$ is the solution of (1.1) with desired oblique reflection.

(β) The large deviation rate function for the above diffusion process is

$$I_T^\Gamma(\phi) = \frac{1}{2}\int_0^T |\dot{\phi}_s - b(\Gamma(\phi))_s)|^2_{a(\Gamma(\phi)_s)^{-1}}\, ds.$$

It is related to the rate function of the oblique reflecting diffusion by

$$I_T^+(\phi) = \inf\{I_T^+(\psi) : \psi \in \Omega_T(R^n),\ \Gamma(\psi) = \phi\}.$$

(γ) The map Γ is Lipschitz in the path space

$$|\Gamma(\phi_1) - \Gamma(\phi_2)|_T \leq 2|\phi_1 - \phi_2|_T$$

for any two paths ϕ_1, ϕ_2 in $C(R^n)$.

Using (γ), we can write

$$P_x^\epsilon[X_{[0,T]}\in O_{\epsilon^\alpha}(\phi)]=P_x^\epsilon[Y_{[0,T]}\in\Gamma^{-1}(O_{\epsilon^\alpha}(\phi))] \geq P_x^\epsilon[Y_{[0,T]}\in O_{\epsilon^\alpha/2}(\Gamma^{-1}(\phi))]. \tag{4.11}$$

This reduces the lemma to the following large deviation lower bound for the diffusion (4.10) (with nonanticipating coefficients):

$$\epsilon^2\log P_x^\epsilon[Y_{[0,T]}\in O_{\epsilon^\alpha}(\psi)]\geq -I_T^\Gamma(\psi)-h. \tag{4.12}$$

Indeed, choose $\psi\in\Gamma^{-1}(\phi)$ such that

$$I_T^\Gamma(\psi)\leq \inf_{\tilde\psi\in\Gamma^{-1}(\phi)} I_T^\Gamma(\tilde\psi)+h=I_T^+(\phi)+h.$$

Since $O_{\epsilon^\alpha}(\psi)\subset O_{\epsilon^\alpha}(\Gamma^{-1}(\phi))$, we have from (4.11) and (4.12) (ignoring the unimportant factor 1/2)

$$\epsilon^2\log P_x^\epsilon[X_{[0,T]}\in O_{\epsilon^\alpha}(\phi)]\geq -I_T^+(\phi)-2h,$$

which is equivalent to what we want.

To prove (4.12), we follow the usual proof of the large deviation lower bound ([5], [4]) and pay special attention to the size of the open set involved. Let $U_t=Y_t-\psi_t$. Then, from (4.10),

$$dU_t=\epsilon\sigma(\Gamma(U+\psi)_t)dB_t+\epsilon^2 b(\Gamma(U+\psi)_t dt+V(\Gamma(U+\psi)_t)dt-\dot\psi_t\,dt.$$

Consider also the process

$$dW_t=\sigma(\Gamma(\epsilon W+\psi)_t)dB_t. \tag{4.13}$$

We have by Girsanov's theorem that

$$P_x^\epsilon[Y_{[0,T]}\in O_{\epsilon^\alpha}(\psi)]=P_x^\epsilon[|U|_T\leq\epsilon^\alpha]=P_x^\epsilon[\Lambda_T;|W|_T\leq\epsilon^{\alpha-1}], \tag{4.14}$$

where $\{\Lambda_t,\ t\geq 0\}$ is the exponential martingale

$$\Lambda_t=\exp\left\{\frac{1}{\epsilon}\Phi_t-\frac{1}{2\epsilon^2}\langle\Phi\rangle_t\right\},$$

with

$$\Phi_t=\int_0^t[\epsilon^2 b(\Gamma(\epsilon W+\psi)_s)+V(\Gamma(\epsilon W+\psi)_s)-\dot\psi_s]^T\sigma^{-1}(\Gamma(\epsilon W+\psi)_s)\,dB_s.$$

The quadratic variation of Φ is given by

$$\langle\Phi\rangle_T=\int_0^T|\epsilon^2 b(\Gamma(\epsilon W+\psi)_s)+V(\Gamma(\epsilon W+\psi)_s)-\dot\psi_s|^2_{a(\Gamma(\epsilon W+\psi)_s)^{-1}}\,ds.$$

By the Lipschitz property of the map Γ,

$$|W|_T\leq\epsilon^{\alpha-1}\quad\text{implies}\quad\langle\Phi\rangle_T\leq I_T^\Gamma(\psi)+\epsilon^\alpha c_1(\psi).$$

Here $c_1(\psi)$ can be taken as $c_0\int_0^T|\dot\psi|^2\,ds+c_0T$ for some constant c_0. It follows from (4.14) that

(4.15)
$$P_x^\epsilon[Y_{[0,T]}\in O_{\epsilon^\alpha}(\psi)]\geq \exp\left\{-\frac{I_T^\Gamma(\psi)}{\epsilon^2}-\frac{c_1(\psi)}{\epsilon^{2-\alpha}}\right\}E_x^\epsilon\left[\exp\left\{\frac{\Phi_T}{\epsilon}\right\};|V|_T\leq\epsilon^{\alpha-1}\right].$$

For the last expectation we will show that

$$(4.16)\qquad E_x^\epsilon\left[\exp\left\{\frac{\Phi_T}{\epsilon}\right\};|W|_T\leq\epsilon^{\alpha-1}\right]\geq c_3\epsilon^{\alpha-1}\exp\{-\epsilon^{-3/2}c_2(\psi)\}.$$

(4.12) is a consequence of (4.15) and 4.16), and the proof will be completed.

We now establish (4.16). First we notice that there is a constant c_4 such that

$$\langle\Phi\rangle_T\leq c_4\int_0^T|\dot\psi_s|^2\,ds+c_4T\overset{\text{def}}{=}c_5(\psi).$$

Using Chebyschev's inequality, we have

$$(4.17)\qquad P_x^\epsilon[|\Phi_T|\geq\epsilon^{-1/2}c_5(\psi)^{1/2}]\leq\frac{E_x^\epsilon[\langle\Phi\rangle_T]}{c_5(\psi)}\epsilon\leq\epsilon.$$

Next, define a continuous martingale W_0 by

$$W_t^0=\int_0^t\sigma(\Gamma(\psi)_s)\,dB_s.$$

Subtract this from equation (4.13) and use the martingale moment inequality [8, p. 110] twice. We obtain

$$\begin{aligned}E_x^\epsilon[|W-W^0|_T^{2p}]&\leq c_6(p)\epsilon^{2p}E_x^\epsilon\left[\left(\int_0^T|W_s|^2\,ds\right)^p\right]\\&\leq c_6(p)\epsilon^{2p}T^pE_x^\epsilon[|W|_T^{2p}]\\&\leq c_7(p)\epsilon^{2p}T^{2p}.\end{aligned}$$

Hence we have

$$P_x^\epsilon\left[|W-W^0|_T\geq\frac{\epsilon^{\alpha-1}}{2}\right]\leq c_8(p)\epsilon^{2p(2-\alpha)}T^{2p}.$$

On the other hand, since W^0 is a continuous martingale with $\langle W^0\rangle_t\leq$ const. t, we have

$$P_x^\epsilon\left[|W^0|_T\leq\frac{\epsilon^{\alpha-1}}{2}\right]\geq c_9(T)\epsilon^{\alpha-1}.$$

Since $\alpha<2$, it follows (by choosing p sufficiently large) that

$$\begin{aligned}P_x^\epsilon[|W|_T\leq\epsilon^{\alpha-1}]&\geq P_x^\epsilon\left[|W^0|_T\leq\frac{\epsilon^{\alpha-1}}{2}\right]-P_x^\epsilon\left[|W-W^0|_T\geq\frac{\epsilon^{\alpha-1}}{2}\right]\\(4.18)\qquad&\geq c_9(T)\epsilon^{\alpha-1}+c_8(p)\epsilon^{2p(2-\alpha)}T^{2p}\\&\geq c_{10}(T)\epsilon^{\alpha-1}.\end{aligned}$$

From (4.17) and (4.18) we see that there is a set of probability greater than $c_{10}(T)\epsilon^{\alpha-1}/2$ on which $\Phi_T \geq -\epsilon^{-1/2} c_5^{1/2}(\psi)$ and $|W|_T \leq \epsilon^{\alpha-1}$. (4.16) follows immediately. The proof of the lemma is completed. □

We now prove the following.

PROPOSITION 10. *Fix $\alpha < 2$. For each $z \in \partial D$ and each $h > 0$ there exists $\epsilon_0(z, h) > 0$ such that, for all $\epsilon \leq \epsilon_0(z, h)$,*

$$\epsilon^2 \log E_x^\epsilon[e_q(T_{\Delta(z,\epsilon^\alpha)})] \geq -I^+(x, z) - h.$$

Proof. Let $\beta = \|q\|_\infty$ and $T_\epsilon = T_{\Delta(z,\epsilon^\alpha)}$ as before. We have, for any $T \geq 0$,

$$E[e_q(T_\epsilon)] \geq e^{-\beta T} P_x^\epsilon[T_\epsilon \leq T].$$

The conclusion of the proposition is a consequence of the above inequality and the following assertion: For any $h > 0$, there are $T > 0$ and $\epsilon_0(z, h) > 0$ such that

$$\epsilon^2 \log P_x^\epsilon[T_\epsilon \leq T] \geq -I^+(x, z) - h \tag{4.19}$$

for all $\epsilon \leq \epsilon_0(z, h)$. Let us therefore prove (4.19).

Consider the following process $\tilde{X}$. It satisfies the same equation (1.1) as process X before time T_ϵ, the first hitting time of the set $\Delta(z, \epsilon^\alpha)$. After this time it satisfies the unrestricted equation (1.4).

Fix $h > 0$. By the definition of $I^+(x, z)$, there exist $T_0 < T_1 < T$ and a path $\phi \in \Omega_T(R^n)$ such that:

(a) $\phi(0) = x$ and $\phi(T_1) = z$;
(b) $\phi(t) \in \bar{D} - \{z\}$ for $t < T_1$, and $\phi(t) = z + n(z)(t - T_1)$ for $T_0 \leq t \leq T$;
(c) $I_{T_0}^+(\phi) \leq I^+(x, z) + h$ and $I_{T-T_0}(\phi(T_0 + \cdot)) \leq h$.

(See [10, p. 27].) From (a) and (b), if $\tilde{X}_{[0,T]} \in O_{\epsilon^\alpha/3}(\phi)$ for sufficiently small ϵ then $T_\epsilon(\tilde{X}) \leq T$. On the other hand, (b) implies that z is not on the path $\phi_{[0,T_0]}$. Thus, for sufficiently small ϵ, if $\tilde{X}_{[0,T_0]} \in O_{\epsilon^\alpha/3}(\phi)$ then $\tilde{X}_{[0,T_0]}$ does not intersect the neighborhood $\Delta(z, \epsilon^\alpha/3)$. We have therefore $T_\epsilon \geq T_0$ for such paths. (The reader is advised to draw a picture to convince himself of the above argument.) It follows that

$$\begin{aligned} & P_x^\epsilon[T_\epsilon \leq T] \\ & \quad = P_x^\epsilon[T_\epsilon(\tilde{X}) \leq T] \\ & \quad \geq P_x^\epsilon[\tilde{X}_{[0,T]} \in O_{\epsilon^\alpha/3}(\phi)] \\ & \quad = P_x^\epsilon[\tilde{X}_{[0,T_0]} \in O_{\epsilon^\alpha/3}(\phi); T_\epsilon \geq T_0; \tilde{X}_{[0,T-T_0]} \circ \theta_{T_0} \in O_{\epsilon^\alpha/3}(\phi(T_0 + \cdot))] \\ & \quad \geq P_x^\epsilon[X_{[0,T_0]} \in O_{\epsilon^\alpha/3}(\phi)] \inf_{y \in B_{\epsilon^\alpha/3}(\phi_{T_0})} P_y^\epsilon[\tilde{X}_{[0,T-T_0]} \in O_{\epsilon^\alpha/3}(\phi(T_0 + \cdot))]. \end{aligned} \tag{4.20}$$

Here we have used the Markov property in the fourth step. In this step we also removed the first tilde because, as we argued before, $\tilde{X}_{[0,T_0]} \in O_{\epsilon^\alpha/3}(\phi)$ implies $T_\epsilon \geq T_0$ and hence $\tilde{X}_s = X_s$ for $s \leq T_0$.

In the last expression of (4.20), the lower bound for the first factor is given by Lemma 9. As for the second factor, we notice that $\tilde{X}_0 \in B_{\epsilon^\alpha/3}(\phi_{T_0})$ and $\tilde{X}_{[0,T-T_0]} \in O_{\epsilon^\alpha/3}(\phi(T_0+\cdot))$ imply that $\tilde{X}_{[0,T-T_0]}$ does not intersect the part of the boundary $\partial D \backslash \Delta(z, \epsilon^\alpha)$. Thus, in evaluating the probability we may ignore the boundary condition and assume that the $\tilde{X}$ satisfies the unrestricted equation (1.4). Thus the usual lower bound for the large deviation of paths (modified for shrinking open sets; see the proof of Lemma 9) gives

$$\begin{aligned}\epsilon^2 \inf_{y \in B_{\epsilon^\alpha/3}(\phi_{T_0})} \log P_y^\epsilon[\tilde{X}_{[0,T-T_0]} \in O_{\epsilon^\alpha}(\phi(T_0+\cdot))] &\geq -I_{T-T_0}(\phi(T_0+\cdot)) - h \\ &\geq -2h.\end{aligned}$$

It follows that

$$\epsilon^2 \log P_x^\epsilon[T_\epsilon \leq T] \geq -I_{T_0}^+(\phi) - h - 2h \geq -I^+(x,z) - 3h, \tag{4.21}$$

which is just (4.19). The proposition is proved. □

Finally, we discuss the case $V \equiv 0$.

PROPOSITION 11. *Suppose $V \equiv 0$ and q is strictly positive on $\bar{D}$.*

(i) *Let F be a closed subset of the boundary ∂D. Then for any $h > 0$ there is $t_0(F,h) > 0$ such that, for all $t \leq t_0(F,h)$,*

$$t \log P_x^1\left[\int_0^{T_F} q(X_s)\,ds \leq t\right] \leq -I_1^+(qa^{-1}; x, F) + h.$$

(ii) *Let G be an open subset of ∂D. Then for any $h > 0$ there is a $t_1(G,h) > 0$ such that, for all $t \leq t_1(G,h)$,*

$$t \log P_x^1\left[\int_0^{T_G} q(X_s)\,ds \leq t\right] \geq -I_1^+(qa^{-1}; x, G) - h.$$

Proof. Let

$$a(t) = \int_0^t q(X_s)\,ds.$$

Consider the time-changed process $Y_t = X_{\tau(t)}$, where $\tau(t) = \inf\{s \geq 0 : a(s) \geq t\}$ is the inverse process of $a(t)$. It is clear that Y is a diffusion process associated with the operator $q^{-1}\epsilon^2 L$. Hence the large deviation rate function for Y is just $I_T^+(qa^{-1}; \phi)$. On the other hand, we have

$$\int_0^{T_F} q(X_s)\,ds = T_F(Y).$$

($T_F(\phi)$ denotes the first hitting time of set F for path ϕ.) Under the assumption $V \equiv 0$, the law of $\{X_s, s \geq 0\}$ under the probability P_x^1 is identical with the law of $\{X_{st^{-1}}, s \geq 0\}$ under the probability $P_x^{\sqrt{t}}$. Hence we have

$$P_x^1\left[\int_0^{T_F} q(X_s)\,ds \leq t\right] = P_x^{\sqrt{t}}\left[\int_0^{T_F} q(X_s)\,ds \leq 1\right] = P_x^{\sqrt{t}}[T_F(Y) \leq 1].$$

Now the large deviation upper bound immediately gives

$$t \log P_x^1\left[\int_0^{T_F} q(X_s)\, ds \le t\right] \le -I_1^+(qa^{-1}; x, F) + h.$$

This proves part (i). The proof of part (ii) is essentially the same as the proof of (4.19) in Proposition 10. Let us briefly repeat the argument. Let $z \in G$ be such that $I_1^+(qa^{-1}; x, z) \le I_1^+(qa^{-1}; x, G) + h$. By the definition of $I_1^+(qa^{-1}; x, z)$, there are $T_0 < T_1 < 1$ and a path $\phi \in \Omega_1(R^n)$ with the following properties:

(a) $\phi(0) = x$ and $\phi(T_1) = z$;
(b) $\phi(t) \in \bar{D} - \{z\}$ for $t < T_1$, and $\phi(t) = z + n(z)(t - T_1)$ for $T_0 \le t \le 1$;
(c) $I_{T_0}^+(qa^{-1}; \phi) \le I_1^+(qa^{-1}; x, z) + h$, and $I_{T-T_0}(qa^{-1}; \phi(T_0 + \cdot)) \le h$.

Choose ϵ_0 so small that $\tilde{Y}_{[0,T]} \in O_{\epsilon_0/3}(\phi)$ implies $T_G(Y) \le 1$ and that $\tilde{Y}_{[0,T_0]} \in O_{\epsilon_0/3}(\phi)$ implies $T_G(Y) \ge T_0$, where the process $\tilde{Y}$ is defined similarly to $\tilde{X}$ in the proof of Proposition 10, but with T_ϵ replaced by $T_{\Delta(z,\epsilon_0)}$. The same argument used there leads to the following inequality:

$$P_x^{\sqrt{t}}[T_G(Y) \le 1]$$
$$\ge P_x^{\sqrt{t}}[Y_{[0,T_0]} \in O_{\epsilon_0/3}(\phi)] \inf_{y \in B_{\epsilon_0/3}(\phi_{T_0})} P_y^{\sqrt{t}}[\tilde{Y}_{[0,T-T_0]} \in O_{\epsilon_0/3}(\phi(T_0 + \cdot))].$$

We therefore have, as in (4.21) of Proposition 10,

$$t \log P_x^{\sqrt{t}}[T_G(Y) \le 1] \ge -I_1^+(qa^{-1}; x, z) - 3h \ge I_1^+(qa^{-1}; x, G) - 4h.$$

The proposition is proved. □

PROPOSITION 12. *Suppose $V \equiv 0$ and q is strictly positive on $\bar{D}$.*

(i) *Let F be a closed subset of ∂D. Then for any $h > 0$ there exists $\epsilon_0(F, h) > 0$ such that, for all $\epsilon \le \epsilon_0(F, h)$,*

$$\epsilon \log E_x^\epsilon\left[\exp\left\{-\int_0^{T_F} q(X_s)\, ds\right\}\right] \le -2\sqrt{I_1^+(qa^{-1}; x, F)} + h.$$

(ii) *Let G be an open subset of ∂D. Then for any $h > 0$ there exists $\epsilon_1(G, h) > 0$ such that*

$$\epsilon \log E_x^\epsilon\left[\exp\left\{-\int_0^{T_G} q(X_s)\, ds\right\}\right] \ge -2\sqrt{I_1^+(qa^{-1}; x, G)} - h. \tag{4.22}$$

Proof. Let $J(F) = I_1^+(qa^{-1}; x, F)$ for simplicity. By Proposition 11, for any h there is a $t_0 > 0$ such that, for all $t \le t_0$,

$$t \log P_x^1[T_F(Y) \le t] \le -J(F) + h,$$
$$t \log P_x^1[T_G(Y) \le t] \ge -J(G) - h.$$

Integrating by parts, we have

$$
\begin{aligned}
E_x^\epsilon[e_q(T_F)] &= E_x^1\left[\exp\left\{-\frac{1}{\epsilon^2}T_F(Y)\right\}\right] \\
&= \int_0^\infty \exp\left\{-\frac{t}{\epsilon^2}\right\} dP_x^1[T_F(Y)\le t] \\
&\le \exp\left\{-\frac{t_0}{\epsilon^2}\right\} + \frac{1}{\epsilon^2}\int_0^{t_0}\exp\left\{-\frac{t}{\epsilon^2}\right\}P_x^1[T_F(Y)\le t]\,dt \\
&\le \exp\left\{-\frac{t_0}{\epsilon^2}\right\} + \frac{1}{\epsilon^2}\int_0^{t_0}\exp\left\{-\left(\frac{t}{\epsilon^2}+\frac{J(F)-h}{t}\right)\right\}dt \\
&\le \exp\left\{-\frac{t_0}{\epsilon^2}\right\} + \frac{t_0}{\epsilon^2}\exp\left\{-\frac{2\sqrt{J(F)-h}}{\epsilon}\right\}.
\end{aligned}
$$

This gives immediately

$$\epsilon\log P_x^\epsilon[e_q(T_F)] \le -2\sqrt{J(F)-h}+h$$

for $\epsilon \le \epsilon_0(F, h)$. Part (i) is proved. Using Proposition 11(ii), we have

$$
\begin{aligned}
E_x[e_q(T_G)] &= E_x^1\left[\exp\left\{-\frac{1}{\epsilon^2}T_G(Y)\right\}\right] \\
&\ge \int_0^{t_0}\exp\left\{-\frac{t}{\epsilon^2}\right\} dP_x^1[T_G(Y)\le t] \\
&\ge \frac{1}{\epsilon^2}\int_0^{t_0}\exp\left\{-\frac{t}{\epsilon^2}\right\}P_x^1[T_G(Y)\le t]\,dt \\
&\ge \frac{1}{\epsilon^2}\int_0^{\infty}\exp\left\{-\left(\frac{t}{\epsilon^2}+\frac{J(G)+h}{t}\right)\right\}dt - \frac{1}{\epsilon^2}\int_{t_0}^\infty \exp\left\{-\frac{t}{\epsilon^2}\right\}dt \\
&= \frac{2}{\epsilon}\sqrt{J(G)+h}\,K_1(2\sqrt{J(G)+h}/\epsilon) - \exp\left\{-\frac{t_0}{\epsilon^2}\right\}.
\end{aligned}
$$

Here $K_1(z)$ is the Bessel function of imaginary argument [6, formula 3.324.1, p. 307]. By the asymptotic formula for $K_1(z)$ [6, formula (16), p. 936], we have

$$\epsilon\log E_x^\epsilon[e_q(T_G)] \ge -2\sqrt{J(G)+h}-h$$

for $\epsilon \le \epsilon_1(G, h)$. This proves part (ii), and the proof of the proposition is completed. □

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