The Annals of Probability
1991, Vol. 19, No. 2, 486–508

SOME POTENTIAL THEORY FOR REFLECTING BROWNIAN MOTION IN HÖLDER AND LIPSCHITZ DOMAINS

BY RICHARD F. BASS[1] AND PEI HSU[2]

University of Washington and Northwestern University

Bounds are found on the transition densities and Green functions for Brownian motion with normal reflection in Hölder and Lipschitz domains. For Lipschitz domains, reflecting Brownian motion and boundary local time are constructed, a Harnack inequality valid up to the boundary is proved, a probabilistic solution to the Neumann problem is given and the Kuramochi boundary is identified.

1. Introduction. Quite a bit is known about Brownian motion with normal reflection in C^2 domains (see [16]), and even C^1 domains (see [13]). On the other hand, Fukushima [8] has given a construction of reflecting Brownian motion in arbitrary domains in $\mathbb{R}^d$; much less is known here, although see [11] for some recent work related to this subject.

This paper is concerned with some intermediate cases where the domains have some regularity but not a great deal. In Section 2, we consider Hölder domains and obtain an upper bound on the transition densities for reflecting Brownian motion (abbreviated as RBM) by means of results of [5] and [6]. We then turn to Lipschitz domains in $\mathbb{R}^d$, $d \geq 3$, in Section 3. We consider upper and lower bounds on the transition densities and on the Green functions. We establish a Harnack inequality valid up to the boundary for harmonic functions having zero normal derivative on the boundary and also the Hölder continuity of such harmonic functions.

Using the estimates in Section 3 and the theory of Dirichlet forms, we prove in Section 4 the existence of RBM on a bounded Lipschitz domain. We also construct the boundary local time corresponding to surface measure on the boundary. In Section 5, we discuss the Neumann boundary value problem for Lipschitz domains and give a representation of the solution in terms of RBM and its boundary local time. Finally, in Section 6 we consider the ideal boundary for the Neumann problem and RBM. In particular we look at the Kuramochi compactification for bounded Hölder and Lipschitz domains.

The letter c, with or without subscripts, will denote constants whose value is unimportant and which may change from line to line. The open ball of radius r with center x will be denoted $B(x; r)$. Other notation will be introduced as needed.

Received June 1989; revised October 1989.
[1]Research partially supported by NSF Grant DMS-87-01073.
[2]Research partially supported by NSF Grant DMS-86-01977.
AMS 1980 *subject classifications.* Primary 60J65; secondary 60J35, 60J50.
Key words and phrases. Reflecting Brownian motion, Lipschitz domain, Hölder domain, Kuramochi boundary, transition density, Green function, Neumann problem.

2. Hölder domains. The main result of this section is an upper bound on the transition densities for RBM in a Hölder domain. Our sole contribution to this result is the observation that [5] supplies the Sobolev inequality needed in [6].

Let D be a bounded C^γ domain, $\gamma \in (0,1)$. By this we mean the following: There exist a finite number of balls $B(x_i; r_i)$, $i = 1, \dots, N_D$, whose union contains D, and for each $i = 1, \dots, N_D$, there exists a C^γ function $F_i\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that for some coordinate system, $B(x_i; r_i) \cap D$ is equal to the intersection of $B(x_i; r_i)$ with the region above the graph of F_i.

THEOREM 2.1. *Suppose* $0 < \varepsilon < 4\gamma/(d-1-\gamma)$. *Then*

$$\|u\|_{2+\varepsilon} \le c_1(\varepsilon)\left(\|u\|_2 + \sum_{i=1}^{d} \|\partial u/\partial x_i\|_2\right). \tag{2.1}$$

(*Here* $\|\cdot\|_p$ *denotes the* L^p *norm on* D.)

PROOF. Let $\alpha_1 = \gamma^{-1}$. Define

$$I = \left\{(x_1,\dots,x_d)\colon 0 < x_d < b_1,\ 0 < \left(\sum_{i=1}^{d-1} x_i^2\right)^{1/2} < b_2 x_d^{\alpha_1}\right\},$$

where b_1, b_2 are small numbers to be chosen later. If f is a C^γ function on $\mathbb{R}^{d-1}$ with $f(0) = 0$ and if $x \in \mathbb{R}^{d-1}$, then

$$|f(x)| = |f(x) - f(0)| \le c|x|^\gamma.$$

Hence if b_2 is small enough, $I \subseteq \{(x,y)\colon x \in \mathbb{R}^{d-1},\ y > f(x)\}$.

Therefore taking b_1 and b_2 small enough and m large enough, we see that there exist a finite number of orthogonal transformations of $\mathbb{R}^d$, say $\omega_1, \dots, \omega_m$, such that at least one of the translates $x + \omega_i I$, $i = 1, \dots, m$, is contained in D whenever $x \in D$.

Writing $I(x)$ for $x + \omega_i I$ for some ω_i such that $x + \omega_i I \subseteq D$, Lemma I.1 of [5] says that

$$|u(x)| \le c\int_{I(x)}\left[|u(y)| + \sum_{i=1}^{d}\left|\frac{\partial u(y)}{\partial y_i}\right|\right]|x-y|^{-\alpha_1(d-1)}\,dy, \tag{2.2}$$

while Lemma II.1 of [5] says that

$$\int_I |x|^{-\beta}\,dx < \infty \quad \text{whenever } \beta < \alpha_1(d-1)+1. \tag{2.3}$$

We let $r = 1$, $q = 2$, $\alpha_2 = \alpha_1(d-1)+1$ and $\beta = 2+\varepsilon$. Note $rq = 2 < \alpha_2$ and $\beta < q\alpha_2/(\alpha_2 - rq)$. Hypothesis (2.1) of [5], page 115, is satisfied by the proof of Lemma I.1 of [5]. Then Theorem I.2 of [5] gives our result. □

COROLLARY 2.2. *Let $\nu = 2 + 4/\varepsilon$. Then*

$$\|u\|_2^{2+4/\nu} \le c_2\left[\|u\|_2^2 + \|\nabla u\|_2^2\right]\|u\|_1^{4/\nu}.$$

PROOF. Write

$$\int u^2(x)\,dx = \int |u(x)|^{(2+\varepsilon)/(1+\varepsilon)}|u(x)|^{\varepsilon/(1+\varepsilon)}\,dx$$

$$\le \left[\int |u(x)|^{2+\varepsilon}\,dx\right]^{1/(1+\varepsilon)}\left[\int |u(x)|\,dx\right]^{\varepsilon/(1+\varepsilon)},$$

and then apply (2.1). □

THEOREM 2.3. *If $p(t,x,y)$ is the transition density for reflecting Brownian motion in D and $\nu > (d-1+\gamma)/\gamma$, then for $t \le 1$,*

$$p(t,x,y) \le c_3(\nu)t^{-\nu/2}\exp\left(-|x-y|^2/c_4 t\right). \tag{2.4}$$

PROOF. We know that $p(t,x,y)$ exists and is continuous in x and y (see Remark 4.1). It is well known ([9]) that reflecting Brownian motion is associated with the Dirichlet form

$$\mathscr{E}(f,f) = \tfrac{1}{2}\int_D |\nabla f|^2(x)\,dx, \qquad D(\mathscr{E}) = \{f \in L^2(D) : \mathscr{E}(f,f) < \infty\}.$$

Fix $x_0, y_0 \in D$, $t_0 \le 1$. Set $\alpha = (y_0 - x_0)/4t_0$ and $\psi(x) = \alpha \cdot x$. Note that

$$e^{-2\psi}|\nabla e^{\psi}|^2 = |\nabla\psi|^2 = |\alpha|^2, \qquad e^{2\psi}|\nabla e^{-\psi}|^2 = |\nabla\psi|^2 = |\alpha|^2.$$

Now by Theorem 3.25 of [6],

$$p(t,x,y) \le ct^{-\nu/2}\exp\left(-|\psi(y) - \psi(x)| + 2t|\alpha|^2\right), \quad \text{a.e. } y,\ t \le 1. \tag{2.5}$$

By the smoothness of p, we may drop the a.e. Taking $t = t_0$, $x = x_0$ and $y = y_0$ in (2.5) completes the proof. □

For large t we have:

THEOREM 2.4. *There exist $T > 0$ and $c_5 > 0$ such that for all $(x,y) \in D \times D$ and $t \ge T$,*

$$\left|p(t,x,y) - \frac{1}{|D|}\right| \le e^{-c_5 t}.$$

In other words, the transition density function approaches the stationary distribution uniformly and exponentially.

PROOF. Let $\{-\lambda_n, \phi_n\}$, $n = 0, 1, \ldots$ be the normalized eigenpairs of the generator $\frac{1}{2}\Delta$ of P_t. We have the eigenexpansion

$$p(t,x,y) = \frac{1}{|D|} + \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y). \tag{2.6}$$

Hence

$$
\begin{aligned}
p(t,x,x) - \frac{1}{|D|} &= \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)^2 \\
&\le e^{-\lambda_1 t/2} \sum_{n=1}^{\infty} e^{-\lambda_n t/2}\phi_n(x)^2 \qquad (2.7)\\
&\le e^{-\lambda_1 t/2}\left[p(t/2,x,x) - \frac{1}{|D|}\right].
\end{aligned}
$$

Note that by (2.6), $p(t,x,x)$ is decreasing in t. Thus by Theorem 2.3, there exists $c > 0$ such that for $t \ge 1$,

$$p(t,x,x) - \frac{1}{|D|} \le ce^{-\lambda_1 t/2}.$$

Now

$$p(t,x,y) - \frac{1}{|D|} = \int_D \left[p(t/2,x,z) - \frac{1}{|D|}\right]\left[p(t/2,z,y) - \frac{1}{|D|}\right] dz.$$

Using the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
&\left|p(t,x,y) - \frac{1}{|D|}\right| \qquad (2.8)\\
&\quad\le \sqrt{p(t/2,x,x) - \frac{1}{|D|}}\sqrt{p(t/2,y,y) - \frac{1}{|D|}} \le c_1 e^{-\lambda_1 t/2}.
\end{aligned}
$$

The theorem is proved. □

The eigenfunction expansion (2.6) provides some additional useful information. For example, using Cauchy–Schwarz as in (2.8),

$$p(t,x,y) \le \sup_z p(t,z,z). \qquad (2.9)$$

Taking the Laplacian in either variable of $p(t,x,y)$,

$$
\begin{aligned}
|\Delta p(t,x,y)| = |\partial p(t,x,y)/\partial t| &= \left|\sum_{n=1}^{\infty} \lambda_n e^{-\lambda_n t}\phi_n(x)\phi_n(y)\right| \\
&\le \left(\sum_{n=1}^{\infty} \lambda_n e^{-\lambda_n t}\phi_n(x)^2\right)^{1/2}\left(\sum_{n=1}^{\infty} \lambda_n e^{-\lambda_n t}\phi_n(y)^2\right)^{1/2},
\end{aligned}
$$

using Cauchy–Schwarz. But since $\lambda_n e^{-\lambda_n t/2}$ is uniformly bounded in n,

$$|\Delta p(t,x,y)| \le c \sup_z p(t/2,z,z) \le c_6 t^{-\nu/2}, \qquad t \le 1. \qquad (2.10)$$

Finally note that if $T \ge 0$ and $f(\cdot) = -\frac{1}{2}\Delta p(T,\cdot,y)$, then Dynkin's formula (or the eigenfunction expansion) tells us that

$$\int_0^T \int_D p(t,\cdot,z) f(z)\, dz\, dt = p(T,\cdot,y). \qquad (2.11)$$

Let $G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y)\,dt$ be the λ-resolvent kernel. If K is a compact subset of D with smooth boundary, we use $p^K(t,x,y)$ to denote the transition density of the Brownian motion which is reflected at ∂D and absorbed on ∂K. The corresponding resolvent kernel is denoted by $G_0^K(x,y)$.

COROLLARY 2.5. *For any* $\nu > (d-1+\gamma)/\gamma$, *we have* $G_\lambda(x,y) \le c_6(\nu,\lambda)|x-y|^{-\nu+2}$ *and* $G_0^K(x,y) \le c_6|x-y|^{-\nu+2}$.

PROOF. Given t_0, the estimate in Theorem 2.3 is valid for $t \le t_0$ by adjusting c_3 and c_4. So using Theorem 2.3 for t small and Theorem 2.4 for t large, the case of $G_\lambda(x,y)$ follows by integration. As for $G_0^K(x,y)$, we have

$$G_0^K(x,y) = \int_0^1 p^K(t,x,y)\,dt + \int_1^\infty p^K(t,x,y)\,dt.$$

Since $p^K \le p$, the bound for the first term is given by Theorem 2.3 as before. For the second term, we have the inequality $p^K(t,x,y) \le e^{-ct}$ for all x,y in D and $t \ge 1$. The proof of this inequality is similar to the proof of Theorem 2.4. Using this inequality, we see that the second term is bounded by a constant. The corollary is proved. □

Since the kernel $|x-y|^{-\nu+2}$ is not locally integrable for ν large, it is useful to have:

COROLLARY 2.6. $G_0^K 1_A(x) \le c_7|A|^{2/\nu}$.

(Here $|A|$ denotes the Lebesgue measure of A.)

PROOF. Using Theorem 2.3 and (2.9), $p^K(t,x,y) \le ct^{-\nu/2}$ for t small, while as in the proof of Corollary 2.5, $p^K(t,x,y) \le c\exp(-c't)$ for t large. Hence $p^K(t,x,y) \le ct^{-\nu/2}$ for all t.

Since $\int_D p^K(s,x,y)\,dy \le 1$, we write

$$G_0^K 1_A(x) \le T + \int_T^\infty \int_D p^K(s,x,y) 1_A(y)\,dy\,ds$$

$$\le T + c|A|T^{1-\nu/2}.$$

Setting $T = |A|^{2/\nu}$ completes the proof. □

An important consequence of Theorem 2.3 is that it allows us to obtain a tightness estimate for Brownian motion. In the next theorem we assume that D is smooth (in order that RBM be well-defined), but c_8 and c_9 depend only on the constants used in the definition of D as a Hölder domain and not on any additional smoothness assumptions on D.

THEOREM 2.7. *If $t < 1$, $x \in D$, then*

$$P^x\Big[\sup_{s \le t}|X_s - x| \ge \lambda\Big] \le c_8 t^{(d-\nu)/2}\exp(-\lambda^2/c_9 t). \tag{2.12}$$

PROOF. Integrating (2.4), we get

$$P^x[|X_t - x| \ge \lambda] \le ct^{(d-\nu)/2}\exp(-\lambda^2/c't), \qquad t \le 1, \tag{2.13}$$

if $\lambda^2 \ge 4t$. Since $\nu > d$, (2.13) holds for $\lambda^2 < 4t$ as well.

Now let $\tau_\lambda = \inf\{t > 0: |X_t - X_0| \ge \lambda\}$. By the strong Markov property, we have

$$\begin{aligned}
P^x&\Big[\sup_{0 \le s \le t}|X_s - x| \ge \lambda\Big] \\
&\le P^x[|X_t - x| \ge \lambda/2] + P^x\big[\tau_\lambda < t, |X_t - X_{\tau_\lambda}| \ge \lambda/2\big] \\
&\le P^x[|X_t - x| \ge \lambda/2] + \int_0^t E^x\big[P^{X_s}[|X_{t-s} - X_0| \ge \lambda/2]; \tau_\lambda \in ds\big] \\
&\le P^x[|X_t - x| \ge \lambda/2] + \sup_{y \in D,\, 0 \le u \le t} P^y[|X_u - y| \ge \lambda/2] \\
&\le 2 \sup_{y \in D,\, 0 \le u \le t} P^y[|X_u - y| \ge \lambda/2].
\end{aligned}$$

Using this and (2.13), we obtain (2.12). □

3. Lipschitz domains. In this section we suppose that $d \ge 3$. For this section we also suppose that we have a function $F: \mathbb{R}^{d-1} \to \mathbb{R}$ that is uniformly bounded and Lipschitz with constant γ,

$$|F(x_1) - F(x_2)| \le \gamma|x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}^{d-1},$$

and that our domain D is given by

$$D = \{(x,y): x \in \mathbb{R}^{d-1}, F(x) < y < \infty\}.$$

Without loss of generality we may assume $\gamma \ge 1$. We assume F is smooth, but our estimates will only depend on γ and not on any further smoothness of F.

We introduce the notation

$$D_\varepsilon = \{x \in D: \operatorname{dist}(x, \partial D) < \varepsilon\}, \tag{3.1}$$

$$A(x;r) = B(x;r) \cap D, \tag{3.2}$$

and if X_t is RBM in D,

$$\tau_r = \inf\{t: |X_t - X_0| > r\}, \tag{3.3}$$

$$T_B = \inf\{t: X_t \in B\} \quad \text{for Borel sets } B. \tag{3.4}$$

Note that if a is a constant, aX_{t/a^2} is again a Brownian motion in the interior of aD, and it is easy to check that aX_{t/a^2} is RBM in aD. Also aD is the region above a Lipschitz function with the same Lipschitz constant γ as F. We refer to this property as scaling.

Let $H = \mathbb{R}^{d-1} \times [0, \infty)$. We start with the Sobolev inequality (see [15], page 124):

$$\|u\|_{2d/(d-2)} \le c\|\nabla u\|_2. \tag{3.5}$$

By considering functions that are symmetric about ∂H, (3.5) gives

$$\int_H |u(x)|^{2d/(d-2)}\,dx \le c\left[\int_H |\nabla u(x)|^2\,dx\right]^{d/(d-2)}. \tag{3.6}$$

Define $\varphi(x, y) = (x, y + F(x))$, let v be a function on D and let $u\colon H \to R$ be defined by $u = v \circ \varphi$.

Since F is Lipschitz, the Jacobians of both φ and φ^{-1} are bounded. Then

$$\begin{aligned}\int_D |v(x)|^{2d/(d-2)}\,dx &\le c\int_H |u(x)|^{2d/(d-2)}\,dx \le c\left[\int_H |\nabla u(x)|^2\,dx\right]^{d/(d-2)} \\ &\le c\left[\int_D |\nabla v(x)|^2\,dx\right]^{d/(d-2)}.\end{aligned} \tag{3.7}$$

We now have:

THEOREM 3.1. *If $p(t, x, y)$ is the transition density for reflecting Brownian motion in D, then for all $t > 0$,*

$$p(t, x, y) \le c_1 t^{-d/2} \exp\bigl(-|x - y|^2/c_2 t\bigr). \tag{3.8}$$

PROOF. As in the proof of Corollary 2.2, Hölder's inequality and (3.7) yield the Nash inequality

$$\|v\|_2^{2+4/d} \le c\|\nabla u\|_2^2 \|u\|_1^{4/d}. \tag{3.9}$$

We then proceed just as in the proof of Theorem 2.3, the only difference being that we take ψ smooth with $|\nabla\psi|$ bounded by $|\alpha|$ and $\psi(x) = \alpha \cdot x$ for $|x| \le 2(|x_0| + |y_0|)$. □

Just as with Theorem 2.4, we get:

THEOREM 3.2. $P^x[\sup_{s<t} |X_s - x| \ge \lambda] \le c_3 \exp(-\lambda^2/c_4 t), \qquad t > 0,\ x \in D.$

Let $G(x, y)$ be the Green function for D. Integrating (3.8) gives:

COROLLARY 3.3.

$$G(x, y) \le c_5 |x - y|^{-d+2}, \qquad x, y \in D. \tag{3.10}$$

We now work on a lower bound for $p(t, x, y)$. Recall that if W_t is Brownian motion in $\mathbb{R}^d$ and $\psi(s)$, $s \in [0, t]$, is a continuous curve with $\psi(0) = x$, then we have the support theorem ([17], pages 168–169):

$$P^x\Big[\sup_{s\le t}|W_s-\psi(s)|<\varepsilon\Big]>0. \tag{3.11}$$

Moreover, if $\psi(s)$ is a Lipschitz curve, one can give a lower bound for the probability in terms of ε, t and the Lipschitz constant of ψ.

THEOREM 3.4. *For all* t,

$$p(t,x,y)\ge c_6 t^{-d/2}\exp\big(-|x-y|^2/c_7 t\big),\qquad x,y\in D. \tag{3.12}$$

PROOF. Suppose $x,y\in D$ with $|x-y|\le 1$. We can find a point $z\in D$ and a constant c_8 independent of x and y such that $\operatorname{dist}(z,\partial D)>2c_8$ but $|x-z|, |y-z|\le 2$. Using Theorem 3.2, we can find $\lambda>0$ such that

$$P^x\Big[\sup_{s\le 1}|X_s-x|\ge\lambda\Big]\le\tfrac14. \tag{3.13}$$

By the fact that F is Lipschitz,

$$|D_\varepsilon\cap B(x;\lambda)|\le c\varepsilon,$$

and so using (3.10),

$$\begin{aligned}P[X_s\in D_\varepsilon\cap B(x;\lambda)\text{ for all } s\le 1]&\le E^x\Big[\int_0^1 1_{D_\varepsilon\cap B(x;\lambda)}(X_s)\,ds\Big]\\&\le\int_{D_\varepsilon\cap B(x;\lambda)}G(x,w)\,dw\le\tfrac14\end{aligned} \tag{3.14}$$

if ε, dependent only on γ, is sufficiently small. So from (3.13) and (3.14), we have

$$P^x[X_s\in A(x;\lambda)-D_\varepsilon\text{ for some } s\le 1]\ge\tfrac12.$$

By the strong Markov property and the support theorem (3.11), there exists $\delta>0$, depending only on γ (via c_8, λ and ε) such that

$$P^x[X_2\in B(z;c_8)]\ge\delta. \tag{3.15}$$

Let p^0 be the transition density for Brownian motion killed on leaving $B(z,2c_8)$. It is well known that $p^0(1,\cdot,\cdot)$ is bounded below on $B(z;c_8)\times B(z;c_8)$. Since $p\ge p^0$ on $B(z;2c_8)\times B(z;2c_8)$, there exists $\delta'>0$ such that

$$p(3,x,w)\ge\delta',\qquad w\in B(z,c_8).$$

The same estimate holds when x is replaced by y. Using the symmetry of p,

$$p(6,x,y)\ge\int_{B(z,c_8)}p(3,x,w)p(3,w,y)\,dw\ge(\delta')^2|B(z,c_8)|\ge c. \tag{3.16}$$

Using scaling (i.e., the fact that aX_{t/a^2} is RBM on aD), we see that there are constants c and c' such that

$$p(t,x,y)\ge c\quad\text{whenever } |x-y|\le c'\sqrt{t}.$$

We now apply the argument of [7], Section 3, to obtain our result. □

Integrating (3.12), we get:

COROLLARY 3.5. $G(x,y) \geq c_9|x-y|^{-d+2}, \qquad x,y \in D.$

We now turn to some properties of functions which are harmonic in a part of D and have zero normal derivative on ∂D. First we need:

PROPOSITION 3.6. *Let $x \in D$. Given η, there exists δ depending on η but not x, such that if $C \subseteq A(x;1)$ and $|C| > \eta$, then*

$$P^x[T_C < \tau_{3\gamma}] > \delta. \tag{3.17}$$

Recall that $|C|$ is the Lebesgue measure of C, that T_A and $\tau_{3\gamma}$ are defined by (3.4) and (3.3) and that $\gamma \geq 1$.

PROOF. As in the proof of Theorem 3.4,

$$|D_\varepsilon \cap B(x;1)| \leq c\varepsilon,$$

where D_ε is defined by (3.1). So if ε is taken small enough, $C' = C - D_\varepsilon$ will be a positive distance from ∂D and $|C'| > \eta/2$. We can find a large integer N, depending only on ε, η and γ, such that we can cover $B(x;2) - D_\varepsilon$ by at most N balls of radius $\varepsilon/4$ with centers in $B(x;2) - D_\varepsilon$. For at least one of these balls, say $B(y;\varepsilon/4)$,

$$|C' \cap B(y;\varepsilon/4)| > \eta/2N.$$

We then take $C'' = C' \cap B(y;\varepsilon/4)$, and we show there exists δ such that

$$P^x[T_{C''} < \tau_{3\gamma}] > \delta.$$

Arguing as in the derivation (3.15), by Theorem 3.2 we can find t_0 such that

$$P^x[\tau_1 < t_0] \leq \tfrac{1}{4}.$$

As in the proof of (3.14), we can take $\varepsilon' \in (0, \varepsilon/4)$ sufficiently small so that

$$P^x[X_s \in D_{\varepsilon'} \text{ for all } s \leq t_0] \leq \tfrac{1}{4}.$$

So

$$P^x[X_s \in A(x;1) - D_{\varepsilon'} \text{ for some } s < \tau_1] \geq \tfrac{1}{2}.$$

By the strong Markov property, the support theorem (3.11) and geometrical considerations, there exists δ' such that

$$P^x[X_s \in B(y;\varepsilon/4) \text{ for some } s < \tau_{3\gamma}] \geq \delta'. \tag{3.18}$$

And if $z \in B(y;\varepsilon/4)$,

$$P^z[T_{C''} < \tau_{\varepsilon/2}] \geq \int_{C''} p^0(1,z,w)\,dw \geq c|C''|, \tag{3.19}$$

where p^0 is the transition density for Brownian motion killed on exiting $B(y;\varepsilon/2)$ (cf. proof of Theorem 3.4). The strong Markov property, (3.18) and (3.19) give our result with $\delta = c\eta\delta'/2N$. □

Define

$$\operatorname*{Osc}_C f = \sup_C f - \inf_C f. \tag{3.20}$$

PROPOSITION 3.7. *There exists $\rho \in (0,1)$, depending only on γ, such that if $x \in D$, $r > 0$, h is harmonic in $A(x;r)$ and continuous on $\overline{B(x;r)} \cap D$ and h has zero normal derivative on $B(x;r) \cap \partial D$, then*

$$\operatorname*{Osc}_{A(x;r/3\gamma)} h \le \rho \operatorname*{Osc}_{A(x;r)} h. \tag{3.21}$$

PROOF. By considering $ah + b$ for suitable a and b, we may assume $\sup_{A(x;r)} h = 1$, $\inf_{A(x;r)} h = 0$. Moreover, by considering $1 - h$ if necessary, we may assume

$$|\{x \in A(x;r/3\gamma)\colon h(x) \ge \tfrac{1}{2}\}| \ge \tfrac{1}{2}|A(x;r/3\gamma)|.$$

Let $C = \{x \in A(x;r/3\gamma)\colon h(x) \ge \frac{1}{2}\}$. Then by Proposition 3.6, scaling and the fact that ∂D is smooth,

$$h(y) = E^y\big[h\big(X_{\tau_r \wedge T_C}\big)\big] \ge \tfrac{1}{2}P^y[T_C < \tau_r] \ge \delta > 0, \qquad y \in A(x;r/3\gamma).$$

Since $h \le 1$ in $A(x;r/3\gamma)$ by the maximum principle,

$$\operatorname*{Osc}_{A(x;r/3\gamma)} h \le 1 - \delta = (1-\delta) \operatorname*{Osc}_{A(x;r)} h.$$

Now take $\rho = 1 - \delta$. □

COROLLARY 3.8. *Suppose h is as in Proposition 3.7. Then there exist c_{10} and α, depending only on γ, such that*

$$|h(x) - h(y)| \le c_{10} \sup_{A(x;r)} |h| |x-y|^\alpha, \qquad x,y \in A(x;r/3\gamma),$$

i.e., h is Hölder continuous.

We can now prove a Harnack inequality valid up to the boundary of D for harmonic functions with zero normal derivative.

THEOREM 3.9. *There exists c_{11}, depending only on γ, such that if $z \in D$, $r > 0$, h is nonnegative and harmonic in $A(z;6r)$ and h has zero normal derivative on $B(z;6r) \cap \partial D$, then*

$$c_{11}^{-1} \le h(x)/h(y) \le c_{11}, \qquad x,y \in A(z;r/3\gamma). \tag{3.22}$$

PROOF (cf. [4]). Fix $y \in A(z;r/3\gamma)$ and assume $h(y) = 1$. Recalling that ∂D is smooth, we may assume h is bounded in $A(z;5r)$; we need to show that we can bound h in $A(z;r/3\gamma)$ by a constant depending only on γ.

First we obtain an estimate on hitting small balls. Suppose $x \in A(y;3r)$. By Proposition 3.6 and scaling,

$$P^y\big[T_{A(x;r/3\gamma)} < \tau_{4r}\big] \ge \delta,$$

for some δ depending only on γ. If $w \in D$, with $|w - x| = r/3\gamma$, then

$$P^w\left[T_{A(x;r/9\gamma^2)} < \tau_{2r}\right] \geq \delta.$$

So by the strong Markov property,

$$P^y\left[T_{A(x;r/9\gamma^2)} < \tau_{4r}\right] \geq \delta^2.$$

Continuing by induction,

$$P^y\left[T_{A(x;r/(3\gamma)^k)} < \tau_{4r}\right] \geq \delta^k. \tag{3.23}$$

We use (3.23) to get a bound on h over small balls. Since

$$1 = h(y) \geq E^y\left[h\left(X_{T_{A(x;r/(3\gamma)^k)}}\right); T_{A(x;r/(3\gamma)^k)} < \tau_{4r}\right] \geq \delta^k \inf_{A(x;r/(3\gamma)^k)} h,$$

then

$$\inf_{A(x;r/(3\gamma)^k)} h \leq \delta^{-k}, \qquad x \in A(y;3r). \tag{3.24}$$

Next we want to compare the oscillation of h on small balls. Recall from Proposition 3.7 that there exists $\rho < 1$ such that

$$\rho \operatorname*{Osc}_{A(x;r/(3\gamma)^k)} h \geq \operatorname*{Osc}_{A(x;r/(3\gamma)^{k+1})} h, \qquad x \in D.$$

Taking m large enough so that $\rho^{-m} \geq \delta^{-2}/(\delta^{-1} - 1)$ and letting $M = (3\gamma)^m$,

$$\operatorname*{Osc}_{A(x;Mr/(3\gamma)^k)} h \geq \rho^{-m} \operatorname*{Osc}_{A(x;r/(3\gamma)^k)} h \geq \frac{\delta^{-2}}{\delta^{-1} - 1} \operatorname*{Osc}_{A(x;r/(3\gamma)^k)} h. \tag{3.25}$$

We now proceed to bound h on $A(y;2r)$. Take K so that $M(3\gamma)^{-K} < \frac{1}{2}$. Suppose there exists $x_0 \in A(y;2r)$ such that $h(x_0) \geq \delta^{-K-1}$. We use induction to construct a sequence $x_0, x_1, \dots$. Suppose we have $x_n \in A(x_{n-1}; Mr/(3\gamma)^{K+n-1})$ with $h(x_n) \geq \delta^{-K-n-1}$. Since $|x_j - x_{j-1}| < Mr/(3\gamma)^{K+j-1}$, $1 \leq j \leq n$, and $|x_0 - y| < 2r$, then

$$|x_n - y| < 3r. \tag{3.26}$$

By the induction hypothesis, $h(x_n) \geq \delta^{-K-n-1}$. By (3.24),

$$\inf_{A(x_n;r/(3\gamma)^{K+n})} h \leq \delta^{-K-n}.$$

So

$$\operatorname*{Osc}_{A(x_n;r/(3\gamma)^{K+n})} h \geq \delta^{-K-n}(\delta^{-1} - 1).$$

By (3.25),

$$\operatorname*{Osc}_{A(x_n;Mr/(3\gamma)^{K+n})} h \geq \delta^{-K-n-2}.$$

Since $h \geq 0$, this implies there exists a point $x_{n+1} \in A(x_n; Mr/(3\gamma)^{K+n})$ with $h(x_{n+1}) \geq \delta^{-K-n-2}$.

But then we have a sequence $x_0, x_1, \dots$, lying in $A(z;4r)$ by (3.26), with $h(x_n) \to \infty$. This contradicts the boundedness of h on $A(z;5r)$. Therefore h is bounded on $A(z;r)$ by δ^{-K-1}. □

REMARK 3.10. We learned from Jerison and Kenig an alternative way to prove Theorem 3.9. Recall the map φ defined following (3.6). One can show that the function $h \circ \varphi$ satisfies an elliptic equation in divergence form with bounded measurable coefficients in the open upper half space H. By symmetric reflection one can extend the equation and the function to the lower half space so that the resulting function satisfies an elliptic equation of the same type on the whole space. The condition of zero normal derivative on the boundary is used here to ensure that no boundary terms appear. After this reflection, Theorem 3.9 can be obtained from Moser's Harnack inequality.

With some extra work, Theorems 3.1 and 3.4 can also be obtained along the same lines. After applying the map, we transpose the problem of constructing the heat kernel with Neumann boundary condition on D to the same problem for Δ_H, the Laplace–Beltrami operator with respect to the transformed metric; this operator is in divergence form with respect to the Riemannian volume measure. We now extend Δ_H to the whole space by reflection σ: $(x, y) \mapsto (x, -y)$. The resulting operator is an elliptic operator with bounded measurable coefficients on the whole space which is in divergence form with respect to the Riemannian volume measure. For these operators, the argument in [7] still applies and we obtain a heat kernel $q(t, x, y)$ (defined on the whole space) for Δ_H with respect to the Riemannian volume measure. The desired heat kernel on H with Neumann boundary condition is then $p(t, x, y) = q(t, x, y) + q(t, x, \sigma y)$. The upper bound and the lower bound in [7] then give the bounds we have proved in this section.

REMARK 3.11. In the remaining sections of this paper, we will consider bounded domains. Bounded Lipschitz domains satisfy the cone condition ([1], page 66), and hence we have ([1], pages 95–112) the Sobolev inequality

$$\|u\|_{d/(d-2)} \leq c(\|u\|_2 + \|\nabla u\|_2).$$

By following the proofs of Section 2 closely, we get the estimates of Theorem 2.3 and Corollary 2.5 with $\nu = d$. With only minor modifications to the proof, the analogue of Theorem 2.7 still holds. By the definition of bounded Lipschitz domain, if r is sufficiently small, then $A(x; 6r)$ equals the intersection of $B(x; 6r)$ with a domain of the type defined in the beginning of this section. Hence Corollary 3.8 and Theorem 3.9 remain valid provided we add "for $r < r_0$ for some $r_0 > 0$" to their statements. With a little more work, one can show using a localization procedure that the lower bounds also hold provided $t \leq 1$ and x and y are sufficiently close; this is a bit lengthy, and since we do not need this in what follows, we omit the proof.

4. Reflecting Brownian motion on Lipschitz domains. The existence of RBM on a smooth Euclidean domain (say, C^2) is well known. The goal of this section is to prove the existence of RBM and boundary local time on a bounded Lipschitz domain. A proof of this seems not to have appeared in print before. Fukushima's method ([8]), for example, leads only to RBM on a certain compactification of the domain D, not necessarily on $\overline{D}$ itself.

In this section, we assume that D is a bounded Lipschitz domain. Let $p(t,x,y)$ denote the transition density whose various estimates we have discussed in the last section. Let $G_\lambda(x,y)$ be the λ-resolvent as before. These are the density kernel and the resolvent kernel of the $L^2(D)$-semigroup P_t associated with the Dirichlet form,

$$\mathscr{E}(u,u) = \frac{1}{2}\int_D |\nabla u(x)|^2\,dx, \qquad D(\mathscr{E}) = H^1(D).$$

Here $H^1(D)$ is the set of functions whose $H^1(D)$ norm is finite, where

$$\|u\|^2_{H^1(D)} = (u,u) + \mathscr{E}(u,u), \qquad (u,u) = \int_D u^2(x)\,dx.$$

(See [9] for details about Dirichlet forms.) Furthermore, $p(t,x,y)$ is smooth on $(0,\infty)\times D\times D$ and $G_\lambda(x,y)$ is smooth on $D\times D$ off the diagonal. We will prove that in the case of a Lipschitz domain, these functions can be extended continuously to $\overline{D}$, the Euclidean closure of D.

REMARK 4.1. The existence of $G_\lambda(x,y)$ and $p(t,x,y)$ can be proved for *arbitrary* bounded domains without any smoothness conditions. They can be constructed by the usual L^2-method. The smoothness of these functions in the interior can then be verified by a standard interior regularity argument (cf. [10]). It can also be proved that if K_1, K_2 are two compact subsets of D such that K_1 and K_2 are disjoint, then $G_\lambda(\cdot,\cdot)$ is bounded on $K_1\times(D-K_2)$. See [8] for details.

WARNING. We let Δ denote both twice the infinitesimal generator of the strongly continuous semigroup P_t associated with the Dirichlet form $\mathscr{E}$ and the Laplacian; it should be clear from the context which one we mean. As the generator of a semigroup, Δ is more than just the Laplace operator. Each function $u\in D(\Delta)$, the domain of Δ, satisfies a lateral (boundary) condition. If D is smooth, this condition is simply $\partial u/\partial n = 0$ on the boundary (cf. [9], pages 21–22).

LEMMA 4.2. *Suppose that $u\in C^2(D)$, both u and Δu are bounded on D, and $u\in D(\Delta)$. Then u is uniformly continuous on D and may be extended to a continuous function on $\overline{D}$.*

PROOF. Choose $x_0\in D$ and $\varepsilon_0>0$ such that $\overline{B(x_0;2\varepsilon_0)}\subset D$. Let $K(\varepsilon) = B(x_0;\varepsilon)$. Let D_n be a sequence of smooth domains increasing to D, all containing $\overline{B(x_0;2\varepsilon_0)}$, such that the Lipschitz constants γ_{D_n} are uniformly bounded in n. Let $G_\lambda^{n,K(\varepsilon)}(x,y)$ be the λ-resolvent kernel for the Brownian motion which is reflected on ∂D_n and absorbed on $\partial K(\varepsilon)$.

Now if $\varepsilon<\varepsilon_0$ and $g = G_0^{n,K(\varepsilon)}f$ for bounded functions f and g, we write

$$g(x) = \int_D G_0^{n,K(\varepsilon)}(x,y)\,f(y)\,dy. \tag{4.1}$$

For any $x_1 \in D_n$, we split the integral in (4.1) into two integrals I_1 and I_2; the first one is on the set $D_n - B(x_1; \delta)$, and the second is on the set $B(x_1; \delta) \cap D_n$. Recall that we have the upper bound $G_0^{n, K(\varepsilon)}(x, y) \le c|x - y|^{-d+2}$. For a fixed positive δ, the first one is a continuous function of x on $D_n \cap B(x_1; \delta/2)$, by Corollary 3.8. The second integral goes to zero uniformly as $\delta \to 0$ by the upper bound on $G_0^{n, K(\varepsilon)}(x, y)$ given by Corollary 2.5 (see Remark 3.11). It follows that g is uniformly continuous on $D_n - B(x_0; 2\varepsilon_0)$, with the modulus of continuity independent of n and $\varepsilon < \varepsilon_0$.

Next suppose $\varepsilon < \varepsilon_0$ and $g = G_1^{n, K(\varepsilon)} f$ for bounded functions f and g. By the resolvent equation,

$$g = G_0^{n, K(\varepsilon)}(f - g),$$

and by what we just proved, g is uniformly continuous on $D_n - B(x_0; 2\varepsilon_0)$ with a modulus independent of n and ε.

If E_n^x is the expectation operator corresponding to RBM on D_n, it is not hard to see that $E_n^x \exp(-T_{K(\varepsilon)}) \to 0$ as $\varepsilon \to 0$ uniformly in $D_n - B(x_0; 2\varepsilon_0)$ at a rate independent of n. It follows that $G_1^{n, K(\varepsilon)} f \to G_1^n f$ uniformly on $D_n - B(x_0; 2\varepsilon_0)$ at a rate independent of n, and hence that if $g = G_1^n f$ for bounded functions f and g, then g is uniformly continuous on $D_n - B(x_0; 2\varepsilon_0)$ with modulus of continuity independent of n. (Here G_λ^n is the resolvent operator for RBM on D_n.)

Finally, by [8], Lemma 2.5, $G_1^n f \to G_1 f$ uniformly as $n \to \infty$. So if $g = G_1 f$ for bounded functions f and g, then g is uniformly continuous on $D - B(x_0; 2\varepsilon_0)$.

Now let u satisfy the hypotheses of the lemma. Clearly we need only worry about u near the boundary of D. But using the hypotheses, $u = G_1 f$ for a bounded function f. It follows that u is uniformly continuous on D and therefore may be extended to be continuous on $\overline{D}$. The lemma is proved. □

LEMMA 4.3. (a) *The function $p(t, x, y)$ defined on $(0, \infty) \times D \times D$ can be extended continuously to $(0, \infty) \times \overline{D} \times \overline{D}$ so that it is a transition density function on $\overline{D}$. Moreover, $p(t, \cdot, y) \in D(\Delta)$.*

(b) *The resolvent kernel $G_\lambda(x, y)$ defined on $D \times D$ off the diagonal can be extended continuously to $\overline{D} \times \overline{D}$ off the diagonal so that it is a resolvent kernel on $\overline{D}$.*

(c) *For all $(x, y) \in \overline{D} \times \overline{D}$,*

$$G_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y)\, dy. \tag{4.2}$$

PROOF. By (2.10) and Remark 4.1, we see that $p(t, \cdot, y)$ will satisfy the hypotheses of Lemma 4.2 provided we show that $p(t, \cdot, y)$ is in $D(\Delta)$. Let P_t be the semigroup associated with $p(t, \cdot, \cdot)$. Note we have the identity

$$p(t, \cdot, y) = G_1 G_1 f(\cdot)$$

for $f = (1 - \Delta/2)^2 p(t, \cdot, y)$. But then

$$f(\cdot) = p(t, \cdot, y) - \Delta p(t, \cdot, y) + (\Delta P_{t/2})(\Delta p(t/2, \cdot, y))/4,$$

and so by (2.10), f is a bounded function on D. This implies that $G_1 f$ is in the closure of $D(\Delta)$ and hence $G_1 G_1 f \in D(\Delta)$, as required.

Therefore by Lemma 4.2, $p(t, \cdot, y)$ may be extended to be continuous on $\overline{D}$. Since $p(t,x,y)$ is symmetric in (x,y), the same argument applies to the y variable. The fact that $p(t,x,y)$ is jointly continuous in (x,y) now follows from

$$p(t,x,y) = \int_D p(t/2,x,z)p(t/2,z,y)\,dz$$

and the dominated convergence theorem. Finally, the conservativity [i.e., $\int_D p(t,x,y)\,dy = 1$] and the Chapman–Kolmogorov equation hold for the extended kernel by passing to the limit in the same formulas for the interior points.

Since we know that (4.2) holds for x, y in the interior, assertions (b) and (c) follow by passing to the limit again. Note that $p(t,x,y)$ is uniformly bounded for $t \geq 1$. □

THEOREM 4.4. *There is a unique $\overline{D}$-valued, continuous, normal strong Markov process (RBM) whose associated Dirichlet form is $\mathscr{E}$.*

PROOF. According to the theory of Dirichlet forms (Theorems 6.2.1 and 4.5.1 of [9]), we need to verify that the Dirichlet form $\mathscr{E}$ is regular on $\overline{D}$, i.e., we need to show that the set $Z = H^1(D) \cap C(\overline{D})$ is dense in both $H^1(D)$ and $C(\overline{D})$ (with their respective norms). That Z is dense in $C(\overline{D})$ is clear, since every continuous function can be approximated uniformly on compact subsets by smooth functions. Now suppose $u \in H^1(D)$. We have to demonstrate a sequence of functions in $C(\overline{D})$ tending to u in $H^1(D)$. Due to the special form of the norm $\|u\|^2_{H^1(D)} = \mathscr{E}(u,u) + (u,u)$, we see that the bounded functions in $H^1(D)$ form a dense subset. Therefore, we may assume without loss of generality that u is bounded. Define $u_\lambda = \lambda G_\lambda u$. We claim that $u_\lambda \to u$ in $H^1(D)$. We have first of all $u_\lambda \to u$ in $L^2(D)$ by the general theory of strongly continuous semigroups. On the other hand, since $H^1(D) = D(\sqrt{-\Delta}) = D(\mathscr{E})$,

$$\begin{aligned}\mathscr{E}(u_\lambda - u, u_\lambda - u) &= \left(\sqrt{-\Delta}(u_\lambda - u), \sqrt{-\Delta}(u_\lambda - u)\right)\\ &= \left(\sqrt{-\Delta}\,u, \sqrt{-\Delta}\,u\right) - 2\lambda\left(G_\lambda\sqrt{-\Delta}\,u, \sqrt{-\Delta}\,u\right)\\ &\quad + \lambda^2\left(G_\lambda\sqrt{-\Delta}\,u, G_\lambda\sqrt{-\Delta}\,u\right).\end{aligned}$$

Hence as $\lambda \to \infty$, we have

$$\begin{aligned}\mathscr{E}(u_\lambda - u, u_\lambda - u) \to &\left(\sqrt{-\Delta}\,u, \sqrt{-\Delta}\,u\right) - 2\left(\sqrt{-\Delta}\,u, \sqrt{-\Delta}\,u\right)\\ &+ \left(\sqrt{-\Delta}\,u, \sqrt{-\Delta}\,u\right) = 0.\end{aligned}$$

Thus we have proved that $u_\lambda \to u$ in $H^1(D)$. It remains to show that $u_\lambda \in C(\overline{D})$. It suffices to show that the resolvent operator G_λ maps bounded functions on D into $C(\overline{D})$. If u is a bounded function on D, then by Lemma 4.3, $P_s u$ is continuous on $\overline{D}$ for each s, where P_s denotes the semigroup operator associated with $\mathscr{E}$. That $G_\lambda u$ is continuous on $\overline{D}$ follows by integration and dominated convergence.

Up to now, we have proved that $\mathscr{E}$ on a bounded Lipschitz domain is a regular Dirichlet form. This implies by the general theory of Dirichlet forms that there is a (Hunt) process X associated with $\mathscr{E}$. To prove X has continuous sample paths, we need to show that $\mathscr{E}$ has the local property (see Theorem 4.5.1 of [9]): If $u, v \in H^1(D)$ and the measures $u(x)\,dx$, $v(x)\,dx$ have disjoint supports, then $\mathscr{E}(u,v) = 0$. This is easy and can be proved with the help of Lemma 7.7 in [10].

So far, we have not proved the existence of P^x for all $x \in \overline{D}$, because the theory of Dirichlet forms only asserts that P^x exists except for a set of capacity zero. For any $x \in \overline{D}$, we define P^x by

$$\text{(4.3)} \qquad \forall\, t > 0,\ A \in \mathscr{F}, \qquad P^x[A \circ \theta_t] = \int_D p(t,x,y) P^y[A]\,dy,$$

where θ_t is the shift operator and $\mathscr{F}$ is the filtration generated by X_t. This uniquely defines a probability measure P^x on $\mathscr{F}$. Note that the right-hand side of (4.3) makes sense because a set of capacity zero has Lebesgue measure zero. It is routine to verify that the family of probabilities $\{P^x\}_{x \in \overline{D}}$ thus defined is a Markov process whose associated Dirichlet form is $\mathscr{E}$.

The normality of the process, i.e., that $P^x[X_0 = x] = 1$, follows immediately from the upper bound (3.8), using Remark 3.11. □

We now turn to the boundary local time. Recall that σ is the surface measure of the boundary.

THEOREM 4.5. *There is a unique positive continuous additive functional L_t (the boundary local time) such that for any $x \in \overline{D}$ and any $\lambda > 0$,*

$$\text{(4.4)} \qquad G_\lambda \sigma(x) = E^x\left[\int_0^\infty e^{-\lambda t}\,dL_t\right].$$

PROOF. We first show that the measure σ has finite energy integral; namely, for any $u \in H^1(D) \cap C(\overline{D})$,

$$(u, \sigma) \le C\sqrt{\|u\|_{H^1(D)}}\,.$$

First, we claim $G_\lambda \sigma \in D(\sqrt{-\Delta})$. In fact, let $D_\varepsilon = \{x \in \overline{D}: \operatorname{dist}(x, \partial D) \le \varepsilon\}$ and $\sigma_\varepsilon(dx) = \varepsilon^{-1} 1_{D_\varepsilon}(x)\,dx$. The fact that $G_\lambda(x,y)$ is continuous on $\overline{D} \times \overline{D}$ off the diagonal and the estimate $G_\lambda(x,y) \le c|x-y|^{-(d-2)}$ imply that $G_\lambda \sigma_\varepsilon(x)$ is uniformly bounded in x, ε and converges uniformly to $G_\lambda \sigma(x)$ on $\overline{D}$ as $\varepsilon \to 0$. Hence the convergence also takes place in $L^2(D)$. On the other hand, as $\varepsilon \to 0$, $\varepsilon' \to 0$,

$$\begin{aligned}\left(\sqrt{-\Delta}\,G_\lambda \sigma_\varepsilon, \sqrt{-\Delta}\,G_\lambda \sigma_{\varepsilon'}\right) &= (-\Delta G_\lambda \sigma_\varepsilon, G_\lambda \sigma_{\varepsilon'}) \\ &= 2(\sigma_\varepsilon, G_\lambda \sigma_{\varepsilon'}) - 2\lambda(G_\lambda \sigma_\varepsilon, G_\lambda \sigma_{\varepsilon'}) \\ &\to 2(\sigma, G_\lambda \sigma) + 2\lambda(G_\lambda \sigma, G_\lambda \sigma).\end{aligned}$$

It follows that $\|\sqrt{-\Delta}\,G_\lambda\sigma_\varepsilon - \sqrt{-\Delta}\,G_\lambda\sigma_{\varepsilon'}\|_{L^2(D)} \to 0$ as $\varepsilon \to 0$, $\varepsilon' \to 0$. Since $\sqrt{-\Delta}$ is closed in $L^2(D)$, we have $G_\lambda\sigma \in D(\sqrt{-\Delta})$.

Now assume $u \in H^1(D) \cap C(\overline{D})$. We have

$$\begin{aligned} 2(u,\sigma) &= \big((1-\Delta)G_{1/2}u,\sigma\big) \\ &= \big(G_{1/2}u,\sigma\big) + \big(\sqrt{-\Delta}\,G_{1/2}\sqrt{-\Delta}\,u,\sigma\big) \qquad (4.5)\\ &= \big(u,G_{1/2}\sigma\big) + \lim_{\lambda\to\infty}\big(\lambda G_\lambda\sqrt{-\Delta}\,G_{1/2}\sqrt{-\Delta}\,u,\sigma\big). \end{aligned}$$

The second term on the right is equal to

$$\begin{aligned} \big(\sqrt{-\Delta}\,u,\lambda G_{1/2}\sqrt{-\Delta}\,G_\lambda\sigma\big) &= \big(\sqrt{-\Delta}\,u,\lambda G_\lambda\sqrt{-\Delta}\,G_{1/2}\sigma\big) \\ &\le \|\sqrt{-\Delta}\,u\|_2\|\sqrt{-\Delta}\,G_{1/2}\sigma\|_2. \end{aligned}$$

It follows from (4.5) that

$$2(u,\sigma) \le \|u\|_2\|G_{1/2}\sigma\|_2 + \|\sqrt{-\Delta}\,u\|_2\|\sqrt{-\Delta}\,G_{1/2}\sigma\|_2 \le \|G_{1/2}\|_{H^1(D)}\|u\|_{H^1(D)}.$$

By Theorem 5.1.1 of [9], the boundary local time L_t exists and (4.4) holds quasi everywhere, hence except for a set of measure zero. Let us show that it holds for all $x \in D$. Denote the right-hand side of (4.4) by $\psi(x)$. Since $\psi(x) = G_\lambda\sigma(x)$ almost everywhere and P_t has a density, we have for any $x \in \overline{D}$,

$$\begin{aligned} \psi(x) &= \lim_{t\to 0} E^x\left[\int_t^\infty e^{-\lambda s}\,dL_s\right] = \lim_{t\to 0} e^{-\lambda t}P_t\psi(x) \\ &= \lim_{t\to 0} e^{-\lambda t}P_tG_\lambda\sigma(x) = G_\lambda\sigma(x). \end{aligned}$$

In the last step we used the continuity of $G_\lambda\sigma$. The theorem is proved. □

REMARK 4.6. We could have constructed RBM directly from the transition density function $p(t,x,y)$, which, as we have shown, has a continuous extension to the closure $\overline{D}$. This transition density generates a nice Feller semigroup P_t which sends the space of bounded measurable functions into $C(\overline{D})$. The sample path continuity can be proved by verifying Dynkin's condition: For all $\varepsilon > 0$,

$$\lim_{t\to 0}\frac{1}{t}\sup_{x\in\overline{D}} P^x\big[X_t \in B(x;\varepsilon)^c\big] = 0,$$

which follows immediately from Theorem 3.2. The existence of boundary local time can also be proved by showing that the potential $G_\lambda\sigma$ satisfies the conditions of Theorem 4.22 in [2]. We adopted the approach of Dirichlet forms because we wanted to emphasize the fact that the Dirichlet form $\mathscr{E}$ is regular on a bounded Lipschitz domain. Furthermore, using the Dirichlet form $\mathscr{E}$ with $D(\mathscr{E}) = H^1(D)$ is an easy way to identify the process as RBM without referring to any smoothness conditions on D.

5. The Neumann boundary value problem. We consider the following Neumann boundary value problem on a bounded Lipschitz domain D:

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ \dfrac{\partial u}{\partial n} = f & \text{on } \partial D, \end{cases} \tag{5.1}$$

where $f \in B(\partial D)$ (the space of bounded measurable functions),

$$\int_{\partial D} f(x)\sigma(dx) = 0$$

and n denotes the outward pointing normal. A rigorous definition of a generalized solution is given as follows.

DEFINITION 5.1. A function $u \in C(\overline{D})$ is said to be a generalized solution to the Neumann boundary value problem (5.1) if for any $\phi \in C^2(\overline{D})$, we have

$$\int_D u(x)\,\Delta\phi(x)\,dx + \int_{\partial D} f(x)\phi(x)\sigma(dx) = \int_{\partial D} u(x)\frac{\partial \phi}{\partial n}(x)\sigma(dx). \tag{5.2}$$

REMARK 5.2. It is not difficult to verify that our definition is equivalent to the definition of generalized solutions used in [12]. Hence, by [12], Section 4, a generalized solution has the property that the normal derivative equals f a.e. on the boundary. The existence and uniqueness of a generalized solution is also given there.

Note that for smooth domains, the classical solution to the Neumann problem (5.1) satisfies (5.2) by virtue of Green's second identity.

In this section, we want to derive a representation for the solution to the Neumann problem in terms of RBM. Such a probabilistic representation for smooth domains was discussed in Brosamler ([3]).

THEOREM 5.3. *Let D be a bounded Lipschitz domain and let $f \in B(\partial D)$ with $\int_{\partial D} f(x)\sigma(dx) = 0$. Then there is a unique generalized solution u to the Neumann boundary value problem* (5.1) *satisfying the condition $\int_D u(x)\,dx = 0$. Furthermore, we have for each $x \in D$,*

$$u(x) = \lim_{t\to\infty} \tfrac{1}{2} E^x\left[\int_0^t f(X_s)\,dL_s\right], \tag{5.3}$$

where X is reflecting Brownian motion on D and L_t is boundary local time for X.

In order to prove the above theorem, we need Lemma 5.4.

LEMMA 5.4. *Let $\phi \in C^2(\overline{D})$. Then*

$$\begin{aligned} &\frac{1}{2}\int_0^t ds \int_D p(s,x,\cdot)\,\Delta\phi(x)\,dx \\ &\quad = P_t\phi - \phi + \frac{1}{2}\int_{\partial D} p(t,x,\cdot)\frac{\partial\phi}{\partial n}(x)\sigma(dx). \end{aligned} \tag{5.4}$$

PROOF. Let $D_k \subset D$ be a sequence of smooth domains exhausting D such that for any $\phi \in C^2(\overline{D})$ and $\psi \in C(\overline{D})$, we have

$$\int_{\partial D_k} \psi(x) \frac{\partial \phi}{\partial n_{D_k}} \sigma_{D_k}(dx) \to \int_D \psi(x) \frac{\partial \phi}{\partial n} \sigma(dx).$$

Let $p^k(t,x,y)$ be the transition density function for RBM on D_k. The proof of Lemma 4.2 and the uniform boundedness of $p^k(t,x,y)$ show in fact that $p^k(t,x,\cdot)$ (for fixed t,x) is a sequence of uniformly continuous functions on D. Hence a subsequence of $p^k(t,x,\cdot)$ converges, say, to $q(t,x,\cdot)$. But then $q(t,x,y)$ must be the transition density function for the RBM on D, i.e., $q = p$. Indeed, (4.2) holds if we replace G_λ by G_λ^k and p by p^k. Since $G_\lambda^k \to G_\lambda$ by [8], Lemma 2.5, passing to the limit we have $G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} q(t,x,y)\,dt$. By the uniqueness of Laplace transforms we must have $q = p$. Hence we have $p^k(t,x,y) \to p(t,x,y)$ (even without taking subsequences). Since the D_k are assumed to be smooth, (5.4) holds with D replaced by D_k and p by p^k [Green's second identity and the relation $\partial p(t,x,y)/\partial t = \Delta p(t,x,y)/2$]. We obtain (5.4) for the domain D by passing to the limit. The lemma is proved. □

We now turn to the proof of Theorem 5.3.

PROOF OF THEOREM 5.3. We know existence and uniqueness by [12]. So we must show that the function defined in (5.3) is indeed a solution to the problem. We have for $x \in D$,

$$\begin{aligned} E^x\left[\int_0^t f(X_s)\,dL_s\right] &= \int_0^t ds \int_{\partial D} p(s,x,y) f(y) \sigma(dy) \\ &= \int_0^t ds \int_{\partial D} \left[p(s,x,y) - \frac{1}{|D|}\right] f(y) \sigma(dy). \end{aligned} \tag{5.5}$$

[We leave the verification of the first equality for general $f \in B(\partial D)$ to the reader.] Hence we have

$$\left| E^x\left[\int_{t_1}^{t_2} f(X_s)\,dL_s\right]\right| \le \int_{t_1}^{t_2} ds \int_{\partial D} \left| p(s,x,y) - \frac{1}{|D|}\right| |f(y)| \sigma(dy).$$

It follows from Theorem 2.4 that the convergence in (5.3) is uniform over $\overline{D}$ and in fact

$$u(x) = \frac{1}{2}\int_0^\infty ds \int_{\partial D} \left[p(s,x,y) - \frac{1}{|D|}\right] f(y) \sigma(dy). \tag{5.6}$$

Using the boundedness of f, the continuity of $p(t,x,y)$ on $\overline{D}$ and the upper bound (3.8) for $p(t,x,y)$, we can verify easily that $u \in C(\overline{D})$. It is also clear that $\int_D u(x)\,dx = 0$. To show that u satisfies (5.2), let $u_t(x)$ denote the expression after the limit sign in (5.3). We have

$$(u_t, \Delta\phi) = \frac{1}{2}\left(\int_0^t \left[\int_{\partial D} p(s,x,y)\, f(y)\sigma(dy)\right] ds, \Delta\phi\right)$$
$$= \frac{1}{2}\int_{\partial D} f(y)\left[\int_0^t P_s\, \Delta\phi\, ds\right]\sigma(dy).$$

Using the preceding lemma we have immediately

$$(5.7)\quad \begin{aligned}(u_t, \Delta\phi) &= \int_{\partial D} f(P_t\phi - \phi)\\ &\quad + \frac{1}{2}\int_{\partial D}\frac{\partial\phi}{\partial n}(x)\sigma(dx)\int_0^t ds\int_{\partial D}\left[p(s,x,y) - \frac{1}{|D|}\right] f(y)\sigma(dy).\end{aligned}$$

Here we have used the assumption that $\int_{\partial D} f(y)\sigma(dy) = 0$. We have $u_t \to u$ and $P_t\phi \to \bar{\phi} = |D|^{-1}\int_D \phi(x)\, dx$, both uniformly on D. Letting $t \to \infty$ and using (5.6), we obtain (5.2). □

6. Kuramochi boundary. Analytically and probabilistically, the Kuramochi boundary of a bounded domain is the ideal boundary for the Neumann problem and RBM just as the Martin boundary is the ideal boundary for the Dirichlet problem and absorbing Brownian motion. In this section, following Ohtsuka ([14]), we recall the definition of the Kuramochi boundary of a bounded domain. Then we state the result that RBM with continuous sample paths exists on the Kuramochi compactification. The proof of this result is similar to [8] where RBM is constructed on a slightly different compactification. The main result of this section is that if D is a bounded Lipschitz domain, then the Kuramochi boundary coincides with the usual Euclidean boundary. We will also prove that in the case of Lipschitz domains, every boundary point is minimal. For Hölder domains we prove that two Euclidean boundary points cannot collapse into one Kuramochi boundary point.

We start by recalling the definition of the Kuramochi boundary, which is rather similar to the definition of Martin boundary. The reader is referred to [14] for further details. Let D be a bounded Euclidean domain (without any smoothness conditions). Take a compact subset $K \subset D$ with smooth boundary. For example, one may take K to be a small ball. Let $G_\lambda^K(x,y)$ be the λ-resolvent kernel of the Dirichlet form $\mathcal{E}^K$ with domain $D(\mathcal{E}^K) = H^1(D) \cap H_0^1(\mathbb{R}^d - K)$. Here $H_0^1(\mathbb{R}^d - K)$ means functions in $H^1(\mathbb{R}^d - K)$ that vanish on ∂K. In other words, $G_\lambda^K(x,y)$ is the resolvent kernel of the Brownian motion which is reflected at ∂D and is absorbed at ∂K. We will denote G_0^K simply by G^K. The function $G^K(x,y)$ is smooth on $(D-K)\times(D-K)$ off the diagonal and is harmonic in x on $D - K - \{y\}$. A sequence $x_n \in D - K$ with no accumulation point in D is called fundamental if the limit $G^K(x_n, y)$ exists as $n \to \infty$ for every $y \in D - K$. Two fundamental sequences are said to be equivalent if they give rise to the same limiting harmonic function. The set of equivalence classes of fundamental sequences is denoted by Δ. For $x^* \in \Delta$, we define $G^K(x, y) = \lim_{n\to\infty} G^K(x_n, y)$, where $x^* = \{x_n\}$ is a fundamental sequence of x^*. Δ is called the Kuramochi boundary of D. Let $D^* = D \cup \Delta$. As in the Martin boundary case, D^* can be made into a compact metric space by introducing the metric

$$\rho(x,y) = \int_{D-K} \frac{|G^K(x,z) - G^K(y,z)|}{1 + |G^K(x,z) - G^K(y,z)|}\, dz.$$

It can be shown (by taking a large compact set K covering two compact sets K_1 and K_2) that the compactification thus defined is independent of the choice of K. D^* equipped with the metric ρ defined above is called the Kuramochi compactification of D. We refer to $\Delta = \partial D^*$ as the Kuramochi boundary or the ideal boundary of D.

The probabilistic implication of the above definition is Theorem 6.1.

THEOREM 6.1. *There exists a unique D^*-valued continuous strong Markov process (RBM) whose associated Dirichlet form is $\mathscr{E}$ with $D(\mathscr{E}) = H^1(D)$.*

In order to save space we will not prove Theorem 6.1 here. See the remarks at the beginning of this section. Note that in general the RBM may not be normal on D, i.e., there might be branching points. However, if D is Lipschitz, all points are nonbranching (see Section 4).

LEMMA 6.2. *Suppose D is a bounded Lipschitz domain. If ν is a (signed) measure on ∂D with $G^K\nu \equiv 0$, then $\nu = 0$.*

PROOF. Let $\phi \in C(\overline{D})$. We have

$$(G^K\phi, \nu) = (\phi, G^K\nu) = 0$$

and

$$\left(P_t^K G^K\phi, \nu\right) = \left(P_t^K\phi, G^K\nu\right) = 0.$$

(P_t^K is the semigroup corresponding to G_λ^K.) Since P_t^K is a semigroup, we have

$$G^K\phi - P_t^K G^K\phi = \int_0^t P_s^K\phi\, ds.$$

It follows that

$$\int_0^t \left(P_s^K\phi, \nu\right) ds = 0,$$

or equivalently, $(P_t^K\phi, \nu) = 0$ for almost all $t > 0$. If D is a bounded Lipschitz domain, we have $P_t^K\phi(x) \to \phi(x)$ as $t \to 0$ for all $x \in \overline{D} - K$. It follows that $(\phi, \nu) = 0$, and therefore $\nu = 0$. The lemma is proved. □

THEOREM 6.3. *Let D be a bounded Lipschitz domain. Then its Kuramochi compactification is equivalent to its usual Euclidean compactification.*

PROOF. The key to our proof is part (b) of the proof of Lemma 4.2, in which we have shown that for each fixed $y \in D - K$, the function $G^K(\cdot, y)$ can be extended continuously to $\overline{D} - K - \{y\}$. Let $x^* = \{x_n\}$ be a fundamental sequence in D. By choosing a subsequence, we may assume that $x_n \to x^\partial \in \partial D$. It follows that $G^K(x_n, y) \to G^K(x^\partial, y)$. Therefore $G^K(x^*, y) = G^K(x^\partial, y)$. If x^∂ and z^∂ are two distinct boundary points, it follows from Lemma 6.2 with $\nu = \delta_{x^\partial} - \delta_{z^\partial}$, where δ_y denotes point mass at y, that $G^K(x^\partial, \cdot\,) \neq G^K(z^\partial, \cdot\,)$. Hence x^∂ depends only on x^* and is independent of a particular fundamental sequence representing x^*. Therefore we can define a map σ: $\Delta \to \partial D$ by setting $\sigma(x^*) = x^\partial$. On the other hand, if x^∂ is a point on the boundary, then by the continuity of G^K on the closure $\overline{D}$, any sequence $\{x_n\}$ of points in D which converges to x^∂ is fundamental and thus defines a unique ideal boundary point $x^* = \{x_n\}$. This shows that the map σ is one-to-one from Δ onto ∂D. Define i: $D^* \to \overline{D}$ by $i|_D =$ identity and $i|_\Delta = \sigma$. The map i provides the desired homeomorphism between D^* and $\overline{D}$. This completes the proof of our theorem. □

Let $x^* \in \Delta$ be an ideal boundary point. x^* is said to be minimal if whenever u is positive and harmonic in D and $u(\cdot) \leq G^K(x^*, \cdot\,)$, then $u = cG^K(x^*, \cdot\,)$ for some constant $c \leq 1$. The concept of minimality is independent of the choice of K. We have:

THEOREM 6.4. *If D is a bounded Lipschitz domain, then every boundary point is minimal.*

PROOF. Suppose u is positive and harmonic in D and $u \leq G^K(x^*, \cdot\,)$. By [14], there exists a finite positive measure μ_1 on Δ such that $u = G^K\mu_1$. By Theorem 6.3, we see that μ_1 is a measure on ∂D. Similarly, since $G^K(x^*, \cdot\,) - u \leq G^K(x^*, \cdot\,)$, there exists a finite positive measure μ_2 such that $G^K(x^*, \cdot\,) - u = G^K\mu_2$. Hence $G^K(x^*, \cdot\,) = G^K(\mu_1 + \mu_2)(\cdot)$. By Lemma 6.2 applied to $\nu = \delta_{x^*} - (\mu_1 + \mu_2)$, we have $\mu_1 + \mu_2 = \delta_{x^*}$ or $\mu_1 = c\delta_{x^*}$ for some $c \leq 1$. Hence $u = cG^K(x^*, \cdot\,)$. □

Finally, we prove a property of bounded Hölder domains.

THEOREM 6.5. *Let D be a bounded Hölder domain and let $D^* = D \cup \Delta$ be its Kuramochi compactification. Then two distinct Euclidean boundary points do not collapse into one ideal boundary point.*

PROOF. We will sketch the proof and leave the details to the reader.

(a) The statements of Lemmas 4.2 and 4.3 hold if we replace $\overline{D}$ throughout these two lemmas by D^*. To see this, we note that we only need to verify that in the present case, the integral I_2 in Lemma 4.2 still goes to zero uniformly on D as $\delta \to 0$. But we may pass to the limit in Corollary 2.6 by taking smooth domains increasing to D in a suitable way. Once we have the analog of Lemma 4.2, the analog of Lemma 4.3 follows word by word. In particular, $p(t, x, y)$ can be extended continuously to $D^* \times D^*$.

(b) Let $x^* \in \Delta$. To prove our theorem, we have to show that each fundamental sequence $\{x_n\}$ for x^* converges in the Euclidean metric to a unique boundary point $x^\partial \in \partial D$ depending only on x^*. Let us suppose $x_n \to x^\partial$. We have by the continuity of $p(t, x, y)$ on $D^* \times D^*$ and Theorem 2.3,

$$\begin{aligned} P^{x^*}\left[|X_t - x^\partial| > \lambda\right] &= \int_{B(x^\partial;\,\lambda)^c \cap D} p(t, x^*, y)\, dy \\ &= \lim_{n \to 0} \int_{B(x_n;\,\lambda)^c \cap D} p(t, x_n, y)\, dy \\ &\leq c_3 |D| t^{-\nu/2} e^{-\lambda^2/c_4 t}. \end{aligned}$$

Letting $t \to 0$, we find $\lim_{t \to 0} P^{x^*}[|X_t - x^\partial| \geq \lambda] = 0$ for any $\lambda > 0$. It follows by the sample path continuity that $\lim_{t \to 0} X_t = x^\partial$, P^{x^*} a.s. Thus x^∂ is uniquely determined by x^*. The theorem is proved. □

REFERENCES

[1] Adams, R. A. (1975). *Sobolev Spaces*. Academic, New York.

[2] Blumenthal, R. M. and Getoor, R. K. (1968). *Markov Processes and Potential Theory*. Academic, New York.

[3] Brosamler, G. A. (1976). A probabilistic solution of the Neumann problem. *Math. Scand.* **38** 137–147.

[4] Caffarelli, L. (1986). Métodos de continuação em equações eliticas não-lineares. *Inst. Mat. Pura e Apl.* Rio de Janeiro.

[5] Campanato, S. (1962). Il teorema di immersione di Sobolev per una classe di aperti non dotati della proprietá di cono. *Ricerche Mat.* **11** 103–122.

[6] Carlen, E. A., Kusuoka, S. and Stroock, D. W. (1987). Upper bounds for symmetric Markov transition functions. *Ann. Inst. H. Poincaré. Probab. Statist.* **23** 245–287.

[7] Fabes, E. B. and Stroock, D. W. (1986). A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. *Arch. Rational Mech. Anal.* **96** 327–338.

[8] Fukushima, M. (1967). A construction of reflecting barrier Brownian motions for bounded domains. *Osaka J. Math.* **4** 183–215.

[9] Fukushima, M. (1980). *Dirichlet Forms and Markov Processes*. North-Holland, Amsterdam.

[10] Gilbarg, D. and Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*. Springer, New York.

[11] Guanglu, G., Minping, Q. and Silverstein, M. L. (1988). Normal derivative for bounded domains with general boundary. *Trans. Amer. Math. Soc.* **308** 785–809.

[12] Jerison, D. S. and Kenig, C. E. (1982). Boundary value problems in Lipschitz domains. In *Studies in Partial Differential Equations* (W. Littman, ed.). Math. Assoc. Amer., Washington.

[13] Lions, P. L. and Sznitman, A. S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** 511–537.

[14] Ohtsuka, M. (1964). An elementary introduction of Kuramochi boundary. *J. Sci. Hiroshima* **28** 271–299.

[15] Stein, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press.

[16] Stroock, D. W. and Varadhan, S. R. S. (1971). Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.* **24** 147–225.

[17] Stroock, D. W. and Varadhan, S. R. S. (1979). *Multidimensional Diffusion Processes*. Springer, New York.

Department of Mathematics
University of Washington
Seattle, Washington 98195

Department of Mathematics
Northwestern University
Evanston, Illinois 60208