Brownian Excursions From Extremes

*Pei Hsu**

Courant Institute of Mathematical Sciences, New York.

*P. March***

McGill University, Montréal.

Let $B = (B_t, \mathcal{F}_t, P; t \geq 0)$ be standard Brownian motion starting at zero and define its extreme processes as

$$M_t = \max_{0 \leq s \leq t} B_s \qquad \text{and} \qquad m_t = \min_{o \leq s \leq t} B_s.$$

The point of this note is to observe a mapping property of Brownian motion and use it to derive some results about excursions of B from its extremes which are related to the work of Groeneboom [4], Bass[1] and Pitman[9] and of Imhof[7]. It must be pointed out that these results are consequences of general excursion theory as expounded by Getoor [2],[3] and Jacobs[8], for example. However this mapping property is new and its application to excursions is direct.

Let $r_t = M_t - m_t$ be the range process and for each $\epsilon > 0$ define the increasing processes

$$a(t, \epsilon) = \int_\epsilon^t 4 r_s^{-2} ds$$

and

$$\tau(t, \epsilon) = \inf\{s : a(s, \epsilon) > t\}.$$

Let

$$X_t = \frac{2B_t - M_t - m_t}{M_t - m_t} \tag{1}$$

and define $X_t^\epsilon = X_{\tau(t,\epsilon)}$.

Proposition 1. *The process $X^\epsilon = (X_t^\epsilon, \mathcal{F}_{\tau(t,\epsilon)}, P; t \geq 0)$ is a reflecting Brownian motion on $[-1, 1]$. Its local times at ± 1 are*

$$\phi_t^{\epsilon,+} = \int_\epsilon^{\tau(t,\epsilon)} 4 r_s^{-1} dM_s$$

and

$$\phi_t^{\epsilon,-} = \int_\epsilon^{\tau(t,\epsilon)} 4 r_s^{-1} d(-m_s) \qquad \textit{respectively}$$

Proof. We may write equation (1) as $X_t = F(B_t, M_t, m_t)$ where

$$F(x, y, z) = (2x - y - z)/(y - z).$$

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Since F is smooth on $\{y \neq z\}$, we may apply Itô's formula there to obtain

$$dX_s = \frac{2dB_s}{M_s - m_s} + \frac{2(M_s - B_s)d(-m_s)}{(M_s - m_s)^2} - \frac{2(B_s - m_s)dM_s}{(M_s - m_s)^2}. \tag{2}$$

Because each $\tau(t,\epsilon)$ is an $\mathcal{F}_t$-stopping time we may write (2) in the integrated form

$$X_{\tau(t,\epsilon)} = X_\epsilon + W_t^\epsilon + \frac{1}{2}\phi_t^{\epsilon,-} - \frac{1}{2}\phi_t^{\epsilon,+} \tag{3}$$

where

$$\begin{cases} W_t^\epsilon = \int_\epsilon^{\tau(t,\epsilon)} 2r_s^{-1}dB_s \\ \phi_t^{\epsilon,-} = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-1}d(-m_s) \\ \phi_t^{\epsilon,+} = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-1}dM_s \end{cases} \tag{4}$$

To finish the proof we check that W^ϵ is a standard Brownian motion and that (3) is its Skorohod equation (Tanaka[11]). Clearly W^ϵ is an $\mathcal{F}_{\tau(t,\epsilon)}$-martingale and

$$[W^\epsilon]_t = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-2}ds = a(\tau(t,\epsilon),\epsilon) = t.$$

By Lévy's criterion, W^ϵ is a Brownain motion, independent of X_ϵ. Now $\phi_t^{\epsilon,\pm}$ are continuous, incereasing $\mathcal{F}_{\tau(t,\epsilon)}$-adapted processes which increase only when B attains a new extremum, that is only when $X^\epsilon = \pm 1$. $\diamond$

As the next propostion shows, we may write the extreme processes in terms of the local times $\phi^{\epsilon,\pm}$. Set $\phi_t^\epsilon = \phi_t^{\epsilon,+} + \phi_t^{\epsilon,-}$.

Proposition 2. *For $t \geq \epsilon$,*

$$M_t = M_\epsilon + r(\epsilon)\int_0^{a(t,\epsilon)} \exp\{\phi_s^\epsilon/4\}\, d\phi_s^{\epsilon,+} \tag{i}$$

$$m_t = m_\epsilon + r(\epsilon)\int_0^{a(t,\epsilon)} \exp\{\phi_s^\epsilon/4\}\, d\phi_s^{\epsilon,-} \tag{ii}$$

where $a(t,\epsilon) = \inf\{s : \tau(s,\epsilon) > t\}$ and

$$\tau(t,\epsilon) = \epsilon + \frac{1}{4}r(\epsilon)^2\int_0^t \exp\{\phi_s^\epsilon/2\}\, ds.$$

Proof. By (4) we have

$$\phi_t^\epsilon = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-1}dr_s = 4\log\frac{r(\tau(t,\epsilon))}{r(\epsilon)},$$

hence

$$r(\tau(t,\epsilon)) = r(\epsilon)\exp\{\phi^{\epsilon}_t/4\} \tag{5}$$

Since $a(\tau(t,\epsilon),\epsilon) = t$ it follows that $d\tau(t,\epsilon) = r(\tau(t,\epsilon))^2 dt/4$. Thus by (5),

$$\begin{aligned}\tau(t,\epsilon) &= \tau(0,\epsilon) + \frac{1}{4}\int_0^t r(\tau(t,\epsilon))^2\,ds \\ &= \epsilon + \frac{1}{4}r(\epsilon)^2\int_0^t \exp\{\phi^{\epsilon}_s/2\}\,ds.\end{aligned}$$

It follows that τ and hence a are defined solely in terms of M_ϵ, m_ϵ and $\phi^{\epsilon,\pm}$.

Next, by (4)

$$\phi^{\epsilon,+}_t = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-1}\,dM_s = \int_0^t 4r(\tau(s,\epsilon))^{-1}\,dM_{\tau(s,\epsilon)}, \tag{6}$$

and so

$$M_{\tau(t,\epsilon)} = M_\epsilon + \frac{1}{4}r_\epsilon\int_0^t \exp\{\phi^\epsilon_s/4\}\,d\phi^{\epsilon,+}_s, \tag{7a}$$

Similarly, we have

$$m_{\tau(t,\epsilon)} = m_\epsilon + \frac{1}{4}r_\epsilon\int_0^t \exp\{\phi^\epsilon_s/4\}\,d\phi^{\epsilon,-}_s, \tag{7b}$$

and the proposition follows from a time change in (7a) and (7b).◇

These propositions allow us to compare excursions of B from its extremes with excursions of reflecting Brownian motion in $[-1,1]$. To be precise, let

$$f(t,\epsilon) = \inf\{s : \phi^\epsilon_s > t\} \tag{8}$$

be the inverse of boundary local time and let q^ϵ be the point process of excursions of X^ϵ. That is, let

$$D_{q^\epsilon} = \{s : f(s,\epsilon) > f(s-,\epsilon)\}$$

and for each $s \in D_{q^\epsilon}$ let

$$q^\epsilon_s(u) = X^\epsilon(f(s-,\epsilon) + u \wedge l^\epsilon_s), \qquad u \geq 0 \tag{9}$$

where $l^\epsilon_s = f(s,\epsilon) - f(s-,\epsilon)$ is the duration of the excursion. Similarly, consider the point process p of excursions of B from its extremes. Let

$$\mu(t) = \inf\{s : r(s) > t\}, \tag{10}$$

let the domain of p be $D = \{t : \mu(t) > \mu(t-)\}$ and for each $t \in D$ let

$$p_t(u) = B(\mu(t-) + u \wedge \lambda(t)), \qquad u \geq 0 \tag{11}$$

where $\lambda(t) = \mu(t) - \mu(t-)$. Proposition 4 provides a formula for p in terms of q^ϵ. To ensure the formula is well defined we need the

Lemma 3. *Let $D^\epsilon = \{t : \mu(t) > \mu(t-)$ and $\mu(t) > \epsilon\}$. Then*
(i) $f(t,\epsilon) = a(\mu(r(\epsilon)e^{t/4}),\epsilon)$
(ii) $D_{q^\cdot} = \{s(u,\epsilon) : u \in D^\epsilon\}$ *where* $s(t,\epsilon) = 4\log(tr(\epsilon)^{-1})$.

Proof. The lemma follows easily from the equality $\tau(f(t,\epsilon),\epsilon) = \mu(r(\epsilon)e^{t/4})$, which we now show. Let $\alpha(t)$ and $\beta(t)$ denote the left and right side of this equality, respectively. On the one hand, by (7a) and (7b), we have

$$r(\alpha(t)) = r(\epsilon)\exp\{\phi^\epsilon(f(t,\epsilon))\} = r(\epsilon)e^{t/4} \overset{\text{def.}}{=} g(t).$$

On the other hand, by definition $r(\beta(t)) = g(t)$. Thus $r(\alpha(t)) = r(\beta(t))$. Since $g(t)$ is strictly increasing, for any $\delta > 0$

$$\begin{aligned}\alpha(t+\delta) \geq& \mu(r(\alpha(t+\delta))-) = \mu(g(t+\delta)-)\\ \geq& \mu(g(t)) = \mu(r(\beta(t))) = \beta(t).\end{aligned}$$

Letting $\delta \downarrow 0$ we get $\alpha(t) \geq \beta(t)$. Since the reverse inequality is similar, the lemma is proved.◇

Proposition 4. *Let $\{p_t; t \in D\}$ be the point process of excursions of B from its extremes. For each $t \in D^\epsilon$*

$$p_t(u) = \frac{t}{2}q^\epsilon_{s(t,\epsilon)}\left(\frac{4u}{t^2}\right) + \frac{1}{2}\left(M_{\mu(t)} + m_{\mu(t)}\right)$$

Proof. First note that the statement makes sense, by Lemma 3. Let $s \in D_{q^\cdot}$ where $s = s(t,\epsilon)$ and $t \in D^\epsilon$. The durations $l^\epsilon(s)$ of q^ϵ_s and $\lambda(t)$ of p_t are related, according to Lemma 3, by

$$\begin{aligned}l^\epsilon(s) =& f(s(t,\epsilon),\epsilon) - f(s(t,\epsilon)-,\epsilon)\\ =& a(\mu(t),\epsilon) - a(\mu(t)-,\epsilon)\\ =& 4\int_{\mu(t-)}^{\mu(t)} r_u^{-2}\,du\\ =& \frac{\mu(t)-\mu(t-)}{r(\mu(t))^2} = 4\frac{\lambda(t)}{t^2}.\end{aligned} \tag{12}$$

Thus by the formulas

$$X_u = \frac{2B_u - M_u - m_u}{M_u - m_u}, \qquad X^\epsilon_v = X_{\tau(v,\epsilon)}$$

and the definition of q^ϵ and p we get

$$q^\epsilon_{s(t,\epsilon)}(u) = \frac{1}{t}\left(2p_t\left(\frac{t^2u}{4}\right) - M_{\mu(t)} - m_{\mu(t)}\right),$$

from which the proposition follows. ◇

An immediate corollary is the identification of the conditional law of excursions of B from its extremes. Indeed, let $-\infty < c < d < \infty$ and introduce the transition density of Brownian motion in $[c,d]$ with absorption at the endpoints (Port-Stone[10]):

$$p_0^{c,d}(t,x,y) = \frac{2}{d-c}\sum_{n=0}^{\infty}\sin\left(n\pi\frac{x-c}{d-c}\right)\sin\left(n\pi\frac{y-c}{d-c}\right)\exp\left\{-\frac{n^2\pi^2}{(d-c)^2}\frac{t}{2}\right\} \tag{14}$$

as well as the functions

$$(15)\qquad \begin{cases} g^{c,d}(t,y;a) = \dfrac{1}{2}\dfrac{\partial}{\partial n_a} p_0^{c,d}(t,a,y), & a = c,d \\ \theta^{c,d}(t,a,b) = \dfrac{1}{4}\dfrac{\partial^2}{\partial n_a \partial n_b} p_0^{c,d}(t,a,b), & a,b = c,d. \end{cases}$$

There exist unique probability laws $P_{c,d}^{a,b;l}$ on $C([0,\infty),[c,d])$ with absolute distribution:

$$(16)\qquad P_{c,d}^{a,b;l}\left(e(u) \in dy\right) = \frac{g^{c,d}(u,y;a)g^{c,d}(l-u,y;b)}{\theta^{c,d}(l,a,b)}dy, \qquad 0 \le u \le l$$

and transition density

$$(17)\quad P_{c,d}^{a,b;l}\left(e(v) \in dy \middle| e(u) = x\right) = p_0^{c,d}(v-u,x,dy)\frac{g^{c,d}(l-v,y;b)}{g^{c,d}(l-u,x;b)} \qquad 0 \le u < v \le l.$$

Indeed, if $X^{c,d}$ is reflecting Brownian motion in $[c,d]$ then $P_{c,d}^{a,b;l}$ is just the law of the excursion process of $X^{c,d}$ conditioned to begin at a, end at b and have duration l. This is a simple extension of the well-known case of one reflecting barrier (e.g. Ikeda-Watanabe[6]) and also can be proved by imitating the calculations of Hsu[5]. Finally let us note a scaling property of the laws $P_{c,d}^{a,b;l}$ which follows from the invariance of the family $\{p_0^{c,d}, -\infty < c < d < \infty\}$ under affine changes of variable:

(18) If $Z = \{Z(t); 0 \le t \le l\}$ has the law $P_{c,d}^{a,b;l}$ then $\{\alpha Z(\alpha^{-2}t) + \beta; 0 \le t \le \alpha^2 l\}$ has the law $P_{\alpha c+\beta,\alpha d+\beta}^{\alpha a+\beta,\alpha b+\beta;\alpha^2 l}$.

Theorem 5. *Let $t \in D$. Let $-\infty < c < d < \infty$ and let $l > 0$. Then conditional on the event $\xi = [m_{\mu(t)} = c, M_{\mu(t)} = d, p_t(0) = a, p_t(\lambda(t)) = b, \lambda(t) = l]$, the law of the excursion process $p_t(\cdot)$ is $P_{c,d}^{a,b;l}$.*

Proof. Fix some ϵ with $t \in D^\epsilon$ and let $s = s(t,\epsilon)$. By (9) and (12), we have

$$\xi = [q_s^\epsilon(0) = \mathrm{sgn}(a), q_s^\epsilon(l^\epsilon(s)) = \mathrm{sgn}(b), l^\epsilon(s) = |d-c|^2 l/4]$$

But then conditional on ξ, the process $q_s^\epsilon(\cdot)$ has law $P_{-1,1}^{e,f;m}$ with $e = \mathrm{sgn}(a), f = \mathrm{sgn}(b)$ and $m = |d-c|^2 l/4$. So by Proposition 4 and the invariance property (18) we find that conditional on ξ, $p_t(\cdot)$ has the law $P_{c,d}^{a,b;l}$. $\diamond$

It is known that if X is reflecting Brownian motion in an interval then conditional on the σ-field generated by the boundary local time of X, the various excursions of X from the boundary are mutually independent. This is evident from the construction of the excursions law characterizing the excursion point process in the one reflecting barrier case (Ikeda-Watanabe[6]). Or again, one can either imitate the argument of Hsu[5] or simply quote the results in Jacobs[8]. Let us show that this conditional independence property is shared by excursions of Brownian motion B from its extremes, conditional on $\sigma\{M_s, m_s; s \ge 0\}$.

Lemma 6. *Let $\mathcal{B}_\epsilon = \sigma\{\phi_s^{\epsilon,+}, \phi_s^{\epsilon,-}; s \ge 0\}$ and $\mathcal{B} = \sigma\{M_s, m_s; s \ge 0\}$. Then $\mathcal{B}_\epsilon \subset \mathcal{B}$ and $\lim_{\epsilon\to 0} \mathcal{B}_\epsilon = \mathcal{B}$.*

Proof. Since Proposition 2 exhibits M and m as explicit functions of $\phi^{\epsilon,\pm}$, we have the inclusions

$$\sigma\{M_s - M_\epsilon, m_s - m_\epsilon; s \geq \epsilon\} \subset \sigma\{M_\epsilon, m_\epsilon, \phi_s^{\epsilon,\pm}; s \geq 0\} \subset \sigma\{M_s, m_s; s \geq \epsilon\}$$

and the lemma follows from this.◇

Theorem 7. *Conditional on* $\mathcal{B} = \sigma\{M_s, m_s; s \geq 0\}$, *the excursions* $[p_t(\cdot); t \in D]$ *are mutually independent.*

Proof. For $n \geq 1$ consider functionals $F : C([0,\infty), R)^n \to R$ of the form

$$F(\omega_1, \omega_2, \ldots, \omega_n) = \prod_{j=1}^{n} f_j(\omega_j(s_{j,1}), \ldots, \omega_j(s_{j,m(j)}))$$

for bounded continuous functions f_j. Let $t_1, \ldots, t_n \in D$. Using Proposition 1, for all sufficiently small ϵ,

$$E\left[F(p_{t_1}, \ldots, p_{t_n})\middle|\mathcal{B}_\epsilon\right] = \prod_{j=1}^{n} E\left[f_j(p_{t_j}(s_{j,1}), \ldots, p_{t_j}(s_{j,m(j)}))\middle|\mathcal{B}_\epsilon\right]$$

by the conditional independence property of q^ϵ. Thus by the martingale convergence theorems and Lemma 6; taking the limit as $\epsilon \downarrow 0$ yields

$$E\left[F(p_{t_1}, \ldots, p_{t_n})\middle|\mathcal{B}\right] = \prod_{j=1}^{n} E\left[f_j(p_{t_j}(s_{j,1}), \ldots, p_{t_j}(s_{j,m(j)}))\middle|\mathcal{B}\right]. \qquad \diamond$$

We close by remarking that Theorem 5 and 7 show that Brownian motion consists of conditionally independent Brownian excursions properly interpolated between endpoints of flat stretches of the extreme process M and m.

References

[1] Bass, R.F., Markov Processes and Convex Minorants, Sem. Prob. XVIII, Lecture Notes in Math. No. 1059, 24–41.

[2] Getoor,R.K., Splitting Times and Shift Functionals, Z. fur Wahr. verw. Gebiete, 47(1979), 69–81.

[3] Getoor, R.K., Excursions of Markov Processes, Ann. Prob. 7(1979), 244–266.

[4] Groeneboom, P., The Concave Majorant of Brownian Motion, Ann. Prob. 6(1983), 1016–1027.

[5] Hsu, P., On Excursions of Reflecting Brownian Motion, Trans. Amer. Math. Soc., vol.296, No.1(1986), p.239–263.

[6] Ikeda, N. and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes*, North Holland and Kodansha, Amsterdam, Oxford, New York, 1981.

[7] Imhof, J-P., On the Range of Brownian Motion and its Inverse Process, Ann. Prob. 13(1985), 1011–1017.

[8] Jacobs, P.A., Excursions of a Markov Process Induced by Continuous Additive Functionals, Z. fur Wahr. verw. Gebiete, 44 (1978), 325–336.

[9] Pitman, J.W., Remarks on the Convex Minorant of Brownian Motion, Seminar on Stochastic Processes, Birkhäuser, 1982.

[10] Port, S.C. and Stone, C.J., *Brownian Motion and Classical Potential Theory*, Academic Press, New York, 1978.

[11] Tanaka, H., Stochastic Differential Equations with Reflecting Boundary Conditions in Convex Domains, Hiroshima Math. J. 9(1979), 163-179.