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BROWNIAN EXIT DISTRIBUTION OF A BALL

by

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Summary

Let B be a ball in R^d and $X = \{X_t, t > 0\}$ be the standard Brownian motion in R^d. Define $\tau_B = \inf\{t > 0 : X_t \notin B\}$, the first exit time of X from the ball. We compute explicitly the transition density function of the killed Brownian motion $X^o = \{X_t, t < \tau_B\}$ and the joint distributic of $(\tau_B, X(\tau_B))$. A result of Wendel [5] is deduced as a simple consequence of the explicit joint density function.

Let D be a bounded domain in R^d and $X = \{X_t, t > 0\}$ the standard Brownian motion in R^d. The first exit time of X for domain D is defined to be

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

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The first exit is $X(\tau_D)$. Because of the sample path continuity of Brownian motion, $X(\tau_D)$ lies on ∂D.

In this note we compute explicitly the joint distribution of (τ_D, X_{τ_D}) when $D = B$, the ball of radius 1 centered at the origin. The method used here was indicated in [1]. Previously, Wendel [5] computed the expectations of a family of functions of (τ_B, X_{τ_B}), which determines uniquely the joint distribution; but the explicit joint distribution was not given. As we will see later, Wendel's result can be obtained from our explicit density function by a simple integration. For the sake of brevity, we only treat the ball problem. The shell problem can be treated by the same method; see [5].

Let $p_D(t,x,y)$ be the transition density function of the Brownian motion killed at time τ_D. The existence of $p_D(t,x,y)$ is proved in [4]. In fact, we have

$$p_D(t,x,y) = p(t,x,y) - E^x[p(t - \tau_D, X_{\tau_D}, y);\ t < \tau_D],$$

where $p(t,x,y) = (2\pi t)^{-d/2}\exp\{-\|y - x\|^2/2t\}$ is the transition density function of the (free) Brownian motion on R^d. When D is bounded and smooth, $p_D(t,x,y)$ can also be defined as the unique minimal solution of the heat equation with Dirichlet boundary condition:

$$(1)\begin{cases} \frac{\partial}{\partial t} p_D(t,x,y) = \tfrac{1}{2}\,\Delta_y p_D(t,x,y), & t > 0,\ x \in D,\ y \in D; \\ p_D(t,x,y) = 0 & t > 0,\ x \in D,\ y \in \partial D; \\ \lim_{t \to 0} p_D(t,x,y) = \delta_x(y), & x \in D,\ y \in D; \end{cases}$$

where δ_x is the Dirac delta function at x.

THEOREM 1. Let D be a bounded domain of C^3 boundary and $x \in D$. We have

$$(2) \qquad P^x[\tau_D \in dt,\ X_{\tau_D} \in dy] = \tfrac{1}{2}\frac{\partial p_D(t,x,y)}{\partial n_y}dt\,\sigma(dy),$$

where n_y is the inward normal direction at $y \in \partial D$ and σ is the $(d-1)$ dimensional volume measure on ∂D.

PROOF. Let f be a nonnegative continuous function on ∂D and $\alpha > 0$. Define

$$(3) \qquad u(x) = E^x[e^{-\alpha\tau_D} f(X_{\tau_D})].$$

Then u is the unique solution of the Dirichlet problem

$$\begin{cases} (\frac{\Delta}{2} - \alpha)u = 0, & \text{on } D; \\ u = f & \text{on } \partial D; \end{cases}$$

see [2]. One the other hand,

$$(4) \qquad G_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_D(t,x,y)dt$$

is Green's function of the first kind for the operator $\frac{\Delta}{2} - \alpha$ on D. Thus by Green's representation formula, we have

$$(5) \qquad u(x) = \tfrac{1}{2}\int_{\partial D} \frac{\partial G_\alpha(x,y)}{\partial n_y} f(y)\sigma(dy).$$

From (3), (4) and (5) it follows that

$$(6)\quad E^x[e^{-\alpha\tau_D}f(X_{\tau_D})] = \int_0^\infty e^{-\alpha t}\int_{\partial D}\frac{1}{2}\frac{\partial p_D(t,x,y)}{\partial n_y}f(y)\sigma(dy)dt.$$

It can be verified that under our assumptions on the domain, the integral converges absolutely and the exchange of derivation and integration needed in deriving (6) is legitimate. Formula (2) now follows at once from (6) by inverting the Laplace transform.

We now specialize the situation by taking D to be the unit ball B centered at the origin. To apply Theorem 1, we need to compute the transition density function $p_B(t,x,y)$ explicitly. In the following, we will assume that $d > 3$. The same method is applicable to the cases $d = 1$ and 2, but the final formulas look slightly different. We use $J_\nu(r)$ to denote the Bessel function of order ν and use $C_m^\nu(t)$ to denote the Gegenbauer polynomial. The latter is defined via its generating function:

$$(7)\qquad (1 - 2\alpha t + \alpha^2)^{-\nu} = \sum_{n=0}^\infty C_n^\nu(t)\alpha^n.$$

We set $q = (d - 2)/2$.

THEOREM 2. <u>The transition density function</u> $p_B(t,x,y)$ <u>for the killed Brownian motion in the unit ball</u> $B = \{x \in R^d : \|x\| < 1\}$ <u>is equal to</u>

$$\frac{2}{q\omega_{d-1}}(\|x\|\|y\|)^{-q}$$

$$\sum_{\substack{m>0\\n>0}}(q + m)\frac{J_{m+q}(\mu_{n,m}\|x\|)}{J'_{m+q}(\mu_{n,m})}\frac{J_{m+q}(\mu_{n,m}\|y\|)}{J'_{m+q}(\mu_{n,m})}C_m^q(\cos\theta)e^{-\mu_{n,m}^2 t/2},$$

where ω_{d-1} = the $(d-1)$-dimensional volume of ∂B, θ is the angle $x0y$ and $\{\mu_{n,m}, n > 0\}$ are nonnegative zeros of J_{m+q} in the ascending order.

PROOF. This is a standard exercise in the separation variable technique of mathematical physics. We only indicate the main steps. Fix $x \in B$ and choose a spherical coordinate system $y = (r, \theta_1, \ldots, \theta_{d-2}, \phi)$ so that $x = (\|x\|, 0, \ldots, 0)$. The volume element is $dy = r^{d-1}\sin^{d-1}\theta_1 \cdots \sin\theta_{d-2}\, dr\, d\theta_1 \cdots d\theta_{d-2}\, d\phi$. We regard x as fixed. By symmetry, $p_B(t,x,y)$ is a function of $(t,r,\theta) = (t, \|y\|, \measuredangle x0y)$. (Note that $\theta = \theta_1$). It follows from (1) that $Q(t,r,\theta) = p_B(t,x,y)$ is the solution of

$$(9)\begin{cases} \dfrac{\partial Q}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 Q}{\partial r^2} + \dfrac{d-1}{2r}\dfrac{\partial Q}{\partial r} + \dfrac{1}{2r^2\sin^{d-2}\theta}\dfrac{\partial}{\partial\theta}\left(\sin^{d-2}\theta\dfrac{\partial Q}{\partial\theta}\right); \\ Q(t,1,\theta) = 0; \\ \lim_{t\to 0} Q(t,r,\theta) = \delta_{(\|x\|,0)}(r,\theta). \end{cases}$$

This equation can be solved by the standard separation variable technique. Let $Q = T(t)R(r)\Theta(\theta)$, we have

$$\frac{dT}{dt} + \frac{\mu^2}{2}T = 0;$$

$$r^2\frac{d^2R}{dr^2} + (d-1)r\frac{dR}{dr} + [\mu^2 r^2 - m(m+d-2)]R = 0;$$

$$\frac{d}{d\theta}\left(\sin^{d-2}\theta\,\frac{d\Theta}{d\theta}\right) + m(m+d-2)\sin^{d-2}\theta\cdot\Theta = 0.$$

The solutions which are meaningful to our problem are

$$T = e^{-\mu_{n,m}^2 t/2},$$

$$R = r^{-q} J_{m+q}(\mu_{n,m} r);$$

$$\Theta = C_m^q(\cos\,\theta);$$

see [3], p. 971 p. 1031. The completeness of the system

(10) $$r^{-q} J_{m+q}(\mu_{n,m} r) C_m^q(\cos\,\theta), \quad n > 0,\ m > 0$$

follows because we can recover the Poisson kernel from this system, a fact which will be proved at the end of this note. Now we seek a representation of the form

(11) $$Q(t,r,\theta) = \sum_{n>0,m>0} B_{n,m} r^{-q} J_{m+q}(\mu_{n,m} r) C_m^q(\cos\,\theta) e^{-\mu_{n,m}^2 t/2}.$$

The $B_{n,m}$'s can be determined by multiplying (11) with $r^{-q} J_{m+q}(\mu_{n,m} r) C_m^q(\cos\,\theta)$ and integrating over B. Using the last condition in (9), and with the help of [3], we find that

$$B_{n,m} = A_{n,m} \|x\|^{-q} J_{m+q}(\mu_{n,m} \|x\|),$$

where

(12) $$A_{n,m} = C_m^q(1) \left(\int_0^1 r J_{m+q}^2(\mu_{n,m} r)\, dr\right)^{-1} \left(\int_{\partial B} C_m^q(\cos\,\theta)^2 \sigma(dy)\right)^{-1} = \frac{2(q+m)}{q\omega_{d-1} [J'_{m+q}(\mu_{n,m})]^2}.$$

Substituting this in (11), we obtain (8).

Combining Theorem 1 and Theorem 2 we have

THEOREM 3. The joint density function of (τ_B, X_{τ_B}) with respect to $dt\sigma(dy)$ is

$$(13)\quad -\frac{1}{q\omega_{d-1}}\|x\|^{-q}$$

$$\sum_{n>0,m>0}(q+m)\mu_{n,m}e^{-\mu_{n,m}^2 t/2}\frac{J_{m+q}(\mu_{n,m}\|x\|)}{J'_{m+q}(\mu_{n,m})}C_m^q(\cos\theta),$$

The next theorem was first proved in [5].

THEOREM 4. Let θ_t be the angle xOX_t. We have

$$E^x[e^{-\alpha\tau_B}C_m^q(\cos\theta_{\tau_B})] = C_m^q(1)\|x\|^{-q}\frac{I_{m+q}(\sqrt{2\alpha}\|x\|)}{I_{m+q}(\sqrt{2\alpha})},$$

where $I_{m+q}(r)$ is the Bessel function of imaginary argument

PROOF. We have

$$E^x[e^{-\alpha\tau_B}C_m^q(\cos\theta_{\tau_B})]$$

$$= \int_0^\infty\int_{\partial D} e^{-\alpha t}C_m^q(\cos\theta)P^x[\tau_B\ \ dt, X_{\tau_B}\ \ dy].$$

Using (12) and (13), we get

$$(14)\quad E^x[e^{-\alpha\tau_B}C_m^q(\cos\theta_{\tau_B})]$$

$$= -2C_m^q(1)\|x\|^{-q}\sum_{n>0}\frac{\mu_{n,m}}{\mu_{n,m}^2+2\alpha}\frac{J_{m+q}(\mu_{n,m}\|x\|)}{J'_{m+q}(\mu_{n,m})}.$$

But the last sum is just the partial fractional expansion of the meromorphic function $I_{m+q}(\sqrt{2\alpha}\|x\|)/I_{m+q}(\sqrt{2\alpha})$; namely,

$$(15) \quad \frac{I_{m+q}(\sqrt{2\alpha}\|x\|)}{I_{m+q}(\sqrt{2\alpha})} = -2 \sum_{n>0} \frac{\mu_{n,m}}{\mu_{n,m}^2 + 2\alpha} \frac{J_{m+q}(\mu_{n,m}\|x\|)}{J'_{m+q}(\mu_{n,m})}.$$

The theorem is proved.

Finally, we prove that the Poisson kernel $P(x,y)$ for B can be recovered from the explicit density formula (13). As we pointed out early, this implies that the system (10) is complete on B. Integrating (13) from 0 to ∞, we obtain

$$P(x,y) = P^x[X_{\tau_B} \in dy]/\sigma(dy)$$

$$= -\frac{2}{q\omega_{d-1}}\|x\|^{-q} \sum_{n>1,m>0} \frac{q+m}{\mu_{n,m}} \frac{J_{m+q}(\mu_{m,n}\|x\|)}{J'_{m+q}(\mu_{n,m})} C_m^q(\cos\theta).$$

In (15), letting $\alpha = 0$, we have

$$-2 \sum_{n>1} \frac{1}{\mu_{m,n}} \frac{J_{m+q}(\mu_{m,n}\|x\|)}{J'_{m+q}(\mu_{m,n})} = \|x\|^{m+q}.$$

Put this in the expression from $P(x,y)$. By (7), we have

$$P(x,y) = \frac{1}{q\omega_{d-1}} \sum_{m>0} (q+m)\|x\|^m C_m^q(\cos\theta)$$

$$= \frac{1}{\omega_{d-1}}(1 - 2\|x\|\cos\theta + \|x\|^2)^{-q}$$

$$+ \frac{1}{q\omega_{d-1}} \frac{d}{d\|x\|}(1 - 2\|x\|\cos\theta + \|x\|^2)^{-q}$$

$$= \frac{1}{\omega_{d-1}} \frac{1 - \|x\|^2}{(1 - 2\|x\|\cos\theta + \|x\|^2)^{d/2}}.$$

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