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Brownian Motion and Riemannian Geometry

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Recent years have seen intensive activities of using probability theory as a powerful tool or at least as an intuitive language in the study of problems from other fields of mathematics. Taking advantage of their knowledge in stochastic analysis, probabilists venture into fields previously unfamiliar to them. Many interesitng new problems and results emerge as a result of this combination of probability theory with fields such as differential geometry, partial differential equations, and mathematical physics. In my opinion, the advantage of probabilistic methods lies in its intuitive language, its explicitness in expressing certain analytical quantities and its flexibility in handling these quantities.

This paper discuss a few problems related to Brownian motion on Riemannian manifold. Our goal is to show how Brownian motion can be used in geometric problems. Since the paper intends to be expository, the proofs, if any, will be brief and references are given. The choice of material reflects my own interests in the subject and does not claim to be representative.

§1. Brownian motion on manifold and stochastic parallel transport

Let (M,g) be a Riemannian manifold of dimension m. Brownian motion $X = \{X_s, s \geq 0\}$ on M is an M-valued (minimal) diffusion process whose infinitesmal generator is $\Delta/2$, the Lapalce-Beltrami operator. As such the transition density function of X with respect to the standard Riemannian volume is the minimal fundamental solution of $\partial/\partial t - \Delta/2$. Since

$$\Delta = g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} - g^{ij}\Gamma^k_{ij}\frac{\partial}{\partial x^k}$$

locally Brownian motion $X_t = (X_t^1, \ldots, X_t^m)$ can be generated as the solution of the stochastic differential equation

$$dX_s^i = \sigma_j^i(x_s)dB_s^j - \frac{1}{2}g^{jk}(X_s)\Gamma^i_{jk}(X_s)ds \tag{1.1}$$

where $\sigma = (\sigma_j^i)$ is a smooth square root of the matrix $g^{-1} = (g^{ij})$. Alternatively, X is a Brownian motion if

$$X_t^i + \frac{1}{2}\int_0^t g^{jk}(X_s)\Gamma^i_{jk}(X_s)\,ds = \text{ martingale; and } d\langle X^i, X^j\rangle_t = g^{ij}(X_t)\,dt. \tag{1.2}$$

One can also define Brownian motion as the projection (on M) of the horizontal Brownian motion on the bundle of orthonormal frames. The advantage of this point of view is that the

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Brownian motion in the frame bundle can be formulated globally (see Ikeda-Watanabe[12], §V.4.).

Geometric quantities often appear in probabilistic methods as expectations of functionals on sample paths of Brownian motion. With sample paths we have much more flexibility. We have several explicit formulas such as Feynmann-Kac formula, Martin-Girsanov transform, solutions of partial differental equations, etc., which often enable us to carry out necessary computations in a more explicit way. Also Brownian motion often provides new intuition and interpretation to a geometric problem.

In general, we can divide problems which can be treated by Brownian motion method into two classes: 1) local problems, such as short-time asymptotic behavior of heat kernel; and 2) global problems, such as the existence of bounded harmonic functions. The latter type problem is in general more difficult and results can only be proved under some global geometric assumptions about the manifold.

On of the most important concept of stochastic analysis on manifold is that of stochastic parallel transport (or displacement, or translation). Suppose $\pi : E \to M$ is a real or hermitian vector bundle on M equipped with a connection ∇ compatible with its metric. Let $X = \{X_s, s \geq 0\}$ be a diffusion process on M (not necessarily Brownian motion). The stochastic parallel transport τ_t is a (random) linear map from E_{X_t} to E_{X_0} uniquely defined by the following property: If f is a global section of E, then the E_{X_0}-valued process $\tau_t f(X_t)$ satisfies the stochastic differential equation

$$d\left[\tau_t f(X_t)\right] = \tau_t \nabla_i f(X_t) \circ dX_t^i, \qquad \tau_0 = I \tag{1.3}$$

where $\circ$ denote the Stratonovich integral and $\nabla_i = \nabla_{\partial/\partial x^i}$. If $\{e_1, \dots, e_n\}$ is a local frame for E, and $\nabla_i e_j = \gamma_{ij}^k e_k$, then $\tau_t^{-1} e_i(X_0) = a_i^j(t) e_j(X_t)$ with $a_i^j(t)$ determined by the equation

$$da_i^j(t) = -\gamma_{kl}^j(X_t) a_i^l(t) \circ dX_t^k.$$

This is the local formula for the stochastic parallel transport.

§2. Solution of heat equation on sections of a vector bundle

If we are given a global section f of the vector bundle E, then $\tau_t f(X_t)$ is a E_{X_0}-valued process. Thus $u(t,x) = E_x\left[\tau_t f(X_t)\right]$ is a global section of E. We claim that it satisfies the heat equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta^H u; \qquad u(0,\cdot) = f. \tag{2.1}$$

where Δ^H is the horizontal laplacian on E, whose local form is given below. For the proof of this fact, we need only to use (1.2) and (1.3), from which we obtain

$$d[\tau_t f(X_t)] = d[\text{martingale}] + \frac{1}{2}\tau_t\left[g^{jk}(X_t)\nabla_j\nabla_k f(X_t) - g^{jk}(X_t)\Gamma_{jk}^i(X_t)\nabla_i f(X_t)\right] dt.$$

It follows that

$$u(t,x) = f(x) + \frac{1}{2}E_x\left[\int_0^t \tau_s \Delta^H f(X_s) ds\right] \tag{2.2}$$

with

$$\Delta^H f = g^{jk}\nabla_j\nabla_k f - g^{jk}\Gamma_{jk}^i \nabla_i f,$$

or invariantly

$$\Delta^H f = \sum_{i=1}^{n} \nabla^2 f(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis of E. From (1.3) it is also easy to show that

$$E_x \{\tau_s E_{X_s} [\tau_t f(X_t)]\} = E_x [\tau_{s+t} f(X_{s+t})].$$

Now (2.1) follows from (2.2) in the usual way.

Oftentimes what is geometrically more interesting is a heat equation slightly different from (2.1):

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \qquad u(0,\cdot) = f \tag{2.3}$$

where Δ is a second order elliptic operator such that the difference $S = \Delta - \Delta^H$ is a linear transform *on each fibre*. In such case, we can use the Feynmann-Kac formula to express the solution of (2.3). Namely, define $M_t : E_{X_0} \to E_{X_0}$ by the following ordinary differential equation along path:

$$dM_t = \frac{1}{2} M_t \tau_t S(X_t)\, \tau_t^{-1} dt.$$

The solution of (2.3) is simply

$$u(t,x) = E_x [M_t \tau_t f(X_t)]. \tag{2.4}$$

An important example is the case when E is the bundle of differential forms on M. In this case $\Delta = -(d^* d + dd^*)$ is the Kodaira-DeRham laplacian and the difference S can be explicitly expressed by the curvature tensor of the metric (Weitzenböck's formula). Formula (2.4) then gives the solution of the heat equation on differential forms. We will use this formula to prove the Gauss-Bonnet-Chern formula in §6.

§3. Large deviation properties of Brownian motion and Brownian bridge

A sequence of probability measures $\{Q^t, t > 0\}$ on a metric space Ω is said to obey the large deviation principle with rate function $I : \Omega \to [0, \infty]$ if for any closed $F \subset \Omega$,

$$\limsup_{t\to 0} t \log Q^t(F) \le - \inf_{\omega\in F} I(\omega)$$

and for any open $G \subset \Omega$,

$$\liminf_{t\to 0} t \log Q^t(G) \ge - \inf_{\omega\in G} I(\omega).$$

If B is the ordinary m-dimensonal Brownian motion in R^m, then the law of $\{B_{st}, 0 \le s \le 1\}$ obeys the large deviation principle with the rate function

$$I(\omega) = \frac{1}{2}\int_0^1 |\dot\omega(s)|^2 ds \tag{3.1}$$

if $|\dot\omega(s)| \in L^2[0,1]$ and $I(\omega) = \infty$ otherwise. For Brownian motion $\{X_s, s \ge 0\}$ on M, let Q_x^t be the law of $X^t = \{X_s^t = X_{st}, 0 \le s \le 1\}$ in the path space $\Omega_x = \{\omega : [0,1] \to M, \omega(0) = x\}$. Then the probabilities $\{Q_x^t, t > 0\}$ obeys the large deviation principle with the same rate function (3.1) but $|\dot\omega(s)|$ is understood to be the norm in the Riemanian metric. The major step of the proof of this fact is to justify the use of the contraction principle in the large deviation theory to the solution of Itô's stochastic differential equation. This can be accomplished by approximating the solution by Euler polygonal method; see Varadhan[18]; see also Azencott[1].

We now discuss Brownian bridge. The Browian bridge ${}^tX^{x,y}$ from x to y with lifetime t is obtained from the Browian motion starting from x by conditioning to reach y at time t. As such it has the transition function

$$\frac{p(s_2 - s_1, z_1, z_2)\, p(t - s_2, z_2, y)}{p(t - s_1, z_1, y)}$$

and the absolute distribution

$$\frac{p(s, x, z)p(t - s, z, y)}{p(t, x, y)}$$

It is convenient to make the time chamge $s \mapsto st$ and consider the process $X_s^{x,y;\,t} = {}^tX_{st}^{x,y}, 0 \le s \le 1$. Another equivalent formulation of the Bronwnian bridge is the following: If $Q_{x,y}^t$ denotes the law of $X^{x,y;\,t}$ on $\Omega_{x,y} = \{\omega : [0,1] \to M, \omega(0) = x, \omega(1) = y\}$, then

$$\left.\frac{dQ_{x,y}^t}{dQ_x^t}\right|_{\mathcal{F}_s} = \frac{p(t(1-s), \omega(s), y)}{p(t, x, y)}, \qquad 0 \le s < 1 \tag{3.2}$$

where $\{\mathcal{F}_s, 0 \le s \le 1\}$ is the standard filtration of σ-fields on Ω_x. This formula often help us reduce problems about Brownian bridge to that of Brownian motion and that of the heat kernel.

Let us assume that M is complete. Brownian bridge measures $\{Q_{x,y}^t, t > 0\}$ obeys the large deviation principle with the rate function

$$J_{x,y}(\omega) = I(\omega) - \frac{1}{2}\rho(x,y)^2,$$

where $\rho(x, y)$ is the Riemannian distance. This result can be derived from the large deviation of $\{Q_x^t, t > 0\}$ by using (3.2) and some local and global estimates of the heat kernel. Care must be taken since the drift of Brownian bridge is singular. One may consider, for example, the Brownian bridge from x to y up to time $1 - \epsilon$ and then pass to the limit as $\epsilon \to 0$; see Hsu[8].

Let us mention some other interesting properties of Brownian bridge discussed in Hsu[7]. We can prove easily that the measures $\{Q_{x,y}^t, t > 0\}$ is sequentially compact as $t \to 0$. From the large deviation property, any limiting measure μ must be supported by the set $\{\omega : J_{x,y}(\omega) = 0\}$, i.e., the set $\Gamma_{x,y}$ of minimizing geodesics joining x and y. The question arises whether $Q_{x,y}^t$ converges weakly as $t \to 0$. It can be shown that this is the case for *any* pair of points (x, y) if M and its metric are analytic. This is also the case if $\Gamma_{x,y}$ is a smooth manifold of dimension k and each geodesic joining x, y has exactly multiplicity k. In the latter case the limiting measure can be descirbed explicitly. For example if $\Gamma_{x,y}$ consists of l nondegenerate geodesics $\gamma_1, \ldots, \gamma_l$, then

$$\mu(\{\gamma_i\}) = \frac{\det[d\exp_x(\dot\gamma_i(0))]^{-1/2}}{\sum_{j=1}^l \det[d\exp_x(\dot\gamma_j(0))]^{-1/2}}.$$

§4. Existence and nonexistence of bounded harmonic functions

Brownian motion can be used to study harmonic functions on Riemannian manifolds. Of course the most important fact about harmonic functions in this respect is that f is harmonic on M if and only if $f(X_t)$ is a local martingale.

Let M be a Cartan-hadarmard manifold, namely, a simply connected manifold of nonpositive sectional curvature. M is isomorphic to the tangent plane at any point $x_0 \in M$ by its exponential map. Thus the Brownian motion on M pulled back to the tangent plane at $x_0 \in M$ can be decomposed into a radial process r_t on with values in $[0, \infty)$ and an angular process θ_t with values in S^{m-1}. We prove the existence of bounded nonconstant harmonic functons on M by proving that the angular process converges to a limit: $\lim_{t\to 0}\theta_t = \Theta$. Let us assume either (a) there are $\alpha > 1, 0 \le \beta < 1$ with $\alpha(1-\beta) > 2$ such that $\mathrm{Sect}(x) \le \alpha(\alpha-1)/\rho(x)^2$ and $\mathrm{Ric}(x) \ge -\rho(x)^{2\beta}$; or (b) there are constants $U > 0, L > 0$ with $U^2/L > 1$ such that $\mathrm{Sect}(x) \le U^2$ and $\mathrm{Ric}(x) \ge -L^2\rho(x)^2(\rho(x) = \rho(x, x_0))$. Under one of these two assumptions, we claim that the limiting angle of Brownian motion exists. The idea is to consider the successive stopping times

$$\tau_1 = \inf\{t > 0 : \rho(X_t, X_0) = 1\}, \qquad s_1 = \tau_1$$

$$\tau_n = \tau_1 \circ \sigma_{s_{n-1}}, \qquad s_n = \tau_n + s_{n-1}$$

where σ is the shifting operator. Let $\Delta\theta_n = \sup_{s_{n-1}\le t\le s_n} d(\theta_{s_{n-1}}, \theta_t)$ (d is the distance on S^{m-1}). It is enough to show that the series $\sum_n \Delta\theta_n$ converges. Let us discuss case (a). By the upper bound of the sectional curvature anc a comparison theorem in differential geometry, we have $\Delta\theta_n \le c/r_{s_n}^{\alpha}$. Thus we want to show that r_{s_n} goes to infinity sufficiently fast. Since the sectional curvature is nonpositive, Brownian motion wanders to infinity as least as fast as ordinary Brownian motion in R^m. This yields the estimate $r_t \ge t^{1/2-\epsilon}$ for arbitrarily fixed small ϵ and all large t. Thus we have $r_{s_n} \ge s_n^{1/2-\epsilon}$ for large n. The problem now reduces to estimating $s_n = \tau_1 + \cdots + \tau_n$. Now the lower bound of the Ricci curvature has exactly the opposite effect of the upper bound, i.e., the Brownian motion will not drift too fast from where it starts. If the Ricci curvature on a geodesic ball centered at x and with radius 1 is bounded from below by $-L^2$, then we can prove the estimate

$$P_x\left[\tau_1 \le c_1 L^{-1}\right] \le c_2 L^{-1/2} e^{-c_3 L}. \tag{4.1}$$

(c_1, c_2, c_3 are universal constants). Since at nth step, we have $\rho(x, X_t) \le n$, the above estimate and our assumption on the lower bound of the Ricci curvature yield

$$P_x\left[\tau_n \le c_4 n^{-\beta}\right] \le c_5 n^{-\beta/2} e^{-c_6 n^{\beta}}.$$

It follows from Borel-Cantelli lemma that $\tau_n \ge c\, n^{-\beta}$ for large n and hence s_n grows at least as fast as $n^{1-\beta}$ and $\Delta\theta_n \le c_4 n^{-\alpha(1-\beta)(1/2-\epsilon)}$. This prove our assertion.

The existence of the limiting angle enables us to prove there are many bounded nonconstant harmonic functions on M. Indeed, let f be any continuous function on S^{m-1} then the function $u_f(x) = E_x[f(\Theta)]$ is harmonic. A more refined argument shows u_f is not constant as long as f is not. We refer to Hsu-March[9] for details.

It is not known whether in general the lower bound on Ricci curvature is necessary for the existence of nonconstant bounded harmonic functions. The lower bound is not needed if $m = 2$ or if M is radially symmetric; see Kendall[14] and March[16] for probabilistic proof of these two facts. We also mention that in the case when sectional curvatures are bounded from below and from above by constants, Kifer[15] succeeded in giving a probabilistic proof that the Martin boundary of M can be identified with the sphere at infinity.

A complete Riemannian manifold is said to be stochastically complete if the lifetime ζ of Brownian motion on M is infinite a.s. Analytically this means that $\int_M p(t, x, y)dy = 1$ for all $t > 0$ and all $x \in M$. A by-product of the above proof is the result that if the Ricci curvature of a compact Riemannian manifold has the lower bound $\mathrm{Ric}(x) \ge -L^2\rho(x)^2$ for some $L > 0$, then M is stochastically complete. Compare with Elworthy[7], §IX. 6.

Let us assume that M is complete with nonnegative Ricci curvature. Let us prove that such manifold does not have nonconstant bounded harmonic functions. The following probabilistic proof is due to Debiard-Gaveau-Mazet[6]. If f is a bounded harmonic function then $f(X_t)$ is bounded martingale (note that from the above discussion the lifetime of the Browian motion is infinite). Since bounded martiangale converges, a little argument shows that its total quadratic variation of $f(X_t)$ must be integrable, i.e.,

$$E_x\left[\int_0^\infty |df|^2(X_t)dt\right] < \infty. \tag{4.2}$$

On the other hand, since τ_t preserves the metric, we have from (1.3),

$$\begin{aligned} d|df|^2(X_t) &= d\langle \tau_t df(X_t), \tau_t df(X_t)\rangle \\ &= 2\langle d\tau_t df(X_t), \tau_t df(X_t)\rangle + \frac{1}{2}\langle d\tau_t df(X_t), d\tau_t df(X_t)\rangle \\ &\geq d[\text{martingale}] + \langle \Delta^H df(X_t), df(X_t)\rangle dt. \end{aligned}$$

Now Weitzenböck's formula gives $\Delta^H(df) = \Delta(df) + \text{Ric}(df) = \text{Ric}(df)$ because for harmonic f we we have $\Delta(df) = 0$. It follows from the above that

$$|df|^2(X_t) \geq |df|^2(x) + \text{martingale} + \int_0^t \text{Ric}(df, df)ds.$$

Taking expectation and integrating with respect to the time variable, we have

$$E\left[\int_0^t |df|^2(X_t)dt\right] = |df|^2(x)\, t + \int_0^t (t-s)E_x \,\text{Ric}(df, df)(X_s)ds.$$

From (4.2) and the assumption of nonnegative Ricci curvature, the above equality can hold for all t only if $df(x) = 0$. It follows that f must be a constant.

§5. Asymptotic behavior of heat kernel

For a long time, analytic properties of the heat kernel were used to study diffusion processes. With the introduction of powerful probabilstic methods (stochastic differential equations, in particular), diffusion theory becomes a useful tool in studying the heat kernel. Works in this direction was initiated by Molchanov[17] and Azencott et al[3] contributed much toward the clarification of the work of Molchanov.

The basic asymptotic relation

$$\lim_{t\to 0} t \log p(t, x, y) = -\frac{1}{2}\rho(x, y)^2$$

was proved by Varadhan to hold for any x, y on a manifold. For x, y such that y does not belong to the cut-locus of x, one can prove, either by analytic method or by probability, the asymptotic relation

$$p(t, x, y) \sim \left(\frac{1}{2\pi t}\right)^{m/2} H(x, y)e^{-\rho(x,y)^2/2t} \tag{5.1}$$

where $H(x, y) = \det[d\exp_x(\dot\gamma_{x,y}(0))]^{-1/2}$ and $\gamma_{x,y}$ is the unique minimizing geodesic joining x, y. Let Γ_{xy} be the space of minimizing geodesics from x to y with uniform speed. Let O be a neighborhood of the middle cross section of $\Gamma_{x,y}(1/2) = \{z : z = \gamma(1/2) \text{ for some } \gamma \in \Gamma_{x,y}\}$.

Since $\Gamma_{x,y}$ intersects the cut-locus of neither x nor y, one can choose O close to $\Gamma_{x,y}(1/2)$ so that it keeps a positive distance from the cut-locus of x and that of y. Now the large deviation result in §2 implies easily that as $t \to 0$,

$$Q^t_{x,y}\left[\omega(1/2) \in O\right] = \frac{1}{p(t,x,y)} \int_O p\left(\frac{t}{2}, x, z\right) p\left(\frac{t}{2}, z, y\right) dz \to 1.$$

It follows that

$$p(t,x,y) \sim \int_O p\left(\frac{t}{2}, x, z\right) p\left(\frac{t}{2}, z, y\right) dz.$$

Since O does not intersect the cut-locus of x and y, we have from (5.1)

$$p(t,x,y) \sim \left(\frac{1}{\pi t}\right)^m e^{-\rho(x,y)^2/2t} \int_O H(x,z) H(z,y) e^{-E(z)/2t} dz, \tag{5.2}$$

where

$$E(z) = 2[\rho(x,z)^2 + \rho(z,y)^2] - \rho(x,y)^2.$$

Formula (5.2) together with Laplace's method can be used effectively to compute the asymptotic behavior of the heat kernel for distant points.

If M and its metric are analytic, then $E(z)$ is also analytic. In this case, we can show using (5.2) that for any $x, y \in M$, an asymptotic relation of the form

$$p(t,x,y) \sim \text{const.} t^{-\alpha} \left(\log \frac{1}{t}\right)^\beta e^{-\rho(x,y)^2/2t}$$

holds, where α is a rational number between $m/2$ and $m - 1/2$ and β is a nonnegative integer.

Let us consider now a smooth strictly convex body D in R^m. Let $p(t,x,y)$ be the heat kernel on the exterior of D with the Neumann boundary condition. Let x, y be two close points in ∂D. The precise asymptotic behavior of $p(t,x,y)$ in this ase was conjectured long ago in mathematical physics (see Buslaev[5]). This conjecture was proved recently in full generality in Hsu[10] by probabilistic method. We have

$$p(t,x,y) \sim cH(x,y)[N(0)N(\rho)]^{1/6} t^{-(m/2+1/6)} \exp\left\{-\frac{\rho^2}{2t} - \frac{\mu_1 \rho^{1/3}}{t^{1/3}} \int_0^\rho N(s)^{2/3} ds\right\} \tag{5.3}$$

where $N(s)$ is the normal curvature of the unique geodesic in ∂D joining x, y and c, μ_1 are universal constants ($\rho = \rho(x,y)$). The proof is to follow the original idea of Molchanov[17] for the proof of (5.1) and reduce the problem to the computation of the asymtotics of

$$E\left[\exp\left\{-\lambda \int_0^1 l(s)|\hat{W}|(s)\, ds\right\}\right]$$

where l is a continuous, positive function and $\hat{W}$ is the standard Brownian bridge in R^1. It is interesting to see what is the proper form of (5.3) when $N(s)$ vanishes at one to both endpoints.

§6. Probabilistic proof of the index theorem

There are several probabilistic proofs of the Atiyah-Singer index theorem (Bismut[4], Azencott[2], Ikeda-Watanabe[13], Hsu[11]). In this section, we briefly describe the main idea of the proof in Hsu[11]. Since probabilistically there is not much difference between the general case of Dirac operator acting the bundle of spinors and the simpler case of the Gauss-Bonnet-Chern formula, we will describe the latter to exemplify the general idea.

Assume M is a closed compact manifold and E^p is the bundle of p-forms on M and $E = \sum_p \oplus E^p$. The Kodaira–De Rham laplacian acts on the sections of E. The Euler characteristic $\chi(M)$ is given via the De Rham theory and the Hodge theory by

$$\chi(M) = \sum_{p=0}^{m} (-1)^p \dim\mathrm{Ker}\left(\Delta\Big|_{H^p}\right)$$

where H^p is the space of harmonic p-forms on M. Now let $e^{t\Delta/2}(x,y) : E_y \to E_x$ be the heat kernel for Δ on E. The eigenexpansion of the heat kernel gives the formula

$$\chi(M) = \int_M \rho\left(e^{t\Delta/2}(x,x)\right) dx$$

where ρ is called supertrace and is defined by

$$\rho(S) = \sum_{p=0}^{m} (-1)^p \operatorname{tr}(S|_{E^p}).$$

The probability theory comes in at this point to show that the limit

$$e(x) = \lim_{t\to 0} \rho\left(e^{t\Delta/2}(x,x)\right) \tag{6.1}$$

exists and identify the limit. Once this is done, we have

$$\chi(M) = \int_M e(x)dx$$

which is the Gauss-Bonnet-Chern formula for the Euler characteristic.

From the solution of the heat equation on differential forms (2.4) and the definition of Brownian bridge, we obtain after proper scaling

$$e^{t\Delta/2}(x,x) = E\left[M_1^t \tau_1^t\right] p(t,x,x)$$

where $\{\tau_s^t, 0 \le s \le 1\}$ is the stochastic parallel transport along the Brownian bridge $X_s = X_s^{x,x;\,t}$ and

$$dM_s^t = \frac{t}{2}\tau_s S(X_s)\,\tau_s^{-1} M_s^t ds, \quad M_0^t = I$$

($S = \Delta - \Delta^H$ is determined by Weitzenböck formula). Using the relation $p(t,x,x) \sim (2\pi t)^{-m/2}$ we obtain

$$\rho\left(e^{t\Delta/2}(x,x)\right) \sim \left(\frac{1}{2\pi t}\right)^{m/2} E\left[\rho(M_1^t \tau_1^t)\right].$$

By iterating the equation for M_s^t and by the exponential expansion of τ_1^t in the Lie group $O(E_x)$ we observe that M_1^t and τ_1^t have the following expansion

$$M_1^t = \sum_{i=0}^{N} M_1^{t,i} t^i + O(t^{N+1}) \qquad \tau_1^t = \sum_{i=0}^{N} \tau_1^{t,i} t^i + O(t^{N+1})$$

with $M_1^{t,i} \to 2^{-i} S^i$ and $\tau_1^{t,i} \to T_i$ as $t \to 0$. Operator T_i can be expressed in terms of a stochastic integral involving the curvature tensor at x and standard Brownian bridge in R^m. For our purpose, the actual form of T_i is not important. We stress at this point the crucial fact that the expansion of τ_1^t is in the powers of t rather than $\sqrt{t}$. It follows from a careful analysis of the equation for stochastic paralle transport on Brownian bridge.

Now it is a purely *algebraic* fact that

$$\rho(M_1^{t,i} \tau_1^{t,j}) = 0 \qquad \text{if} \quad i + 2j < m.$$

It now follows easily that the limit in (6.1) exists and is equal to

$$e(x) = \left(\frac{1}{4\pi}\right)^{m/2} \rho\left(S(x)^{m/2}\right)$$

if m is even and is equal to 0 if m is odd. It remains to use some differential geometry to express the last expression in terms of the curvature tensor at x and to identify it with the Euler characteristic.

It seems interesting to produce a parallel proof when M has a boundary.

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