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Quasi-invariance of the Wiener measure on the path space over a complete Riemannian manifold

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Abstract

We prove a generalization of the Cameron–Martin theorem for a geometrically and stochastically complete Riemannian manifold; namely, the Wiener measure on the path space over such a manifold is quasi-invariant under the flow generated by a Cameron–Martin vector field.
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1. Introduction

The Cameron–Martin theorem is a fundamental result in stochastic analysis. Let $P_o(\mathbb{R}) = C_o([0, 1]; \mathbb{R})$ be the (pinned) path space over $\mathbb{R}$, i.e., the space of continuous functions $w : [0, 1] \to \mathbb{R}$ such that $w_0 = o$, the origin. Let μ be the Wiener measure on $P_o(\mathbb{R})$. Denote by $w = \{w_s, \ 0 \leqslant s \leqslant 1\}$ the canonical coordinate process on $P_o(\mathbb{R})$. Under μ, the process w is a Brownian motion starting from the origin. Now consider the shifted Brownian motion $w^h = w + h$, where $h \in P_o(\mathbb{R})$ is a Cameron–Martin path, i.e., it has a distributional derivative $\dot{h}$ such that

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$$|h|^2_{\mathscr{H}} = \int_0^1 |\dot{h}_s|^2 \, ds < \infty. \tag{1.1}$$

The Cameron–Martin theorem (Cameron and Martin [1]) asserts that the law μ^h of w^h and the Wiener measure μ are mutually absolutely continuous. Furthermore, the Radon–Nikodym derivative is given by

$$\frac{d\mu^h}{d\mu} = \exp\left[\langle h, w \rangle_{\mathscr{H}} - \frac{1}{2}|h|^2_{\mathscr{H}} \right],$$

where

$$\langle h, w \rangle_{\mathscr{H}} = \int_0^1 \dot{h}_s \, dw_s$$

is the Itô stochastic integral of $\dot{h}$ with respect to the Brownian motion w. It is in this sense we say that the Wiener measure μ is quasi-invariant under a Cameron–Martin shift.

A more general form of the Cameron–Martin theorem is the Girsanov theorem (Girsanov [4]), a simple formulation of which is as follows. Suppose that $\mathscr{F}_* = \{\mathscr{F}_s, \ 0 \leqslant s \leqslant 1\}$ is a filtration of σ-algebras on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and W an $\mathscr{F}_*$-Brownian motion. Let $V = \{V_s, \ 0 \leqslant s \leqslant 1\}$ be an $\mathscr{F}_*$-adapted and progressively measurable process such that

$$e(s) = \exp\left[\int_0^s V_u \, dW_u - \frac{1}{2} \int_0^s |V_u|^2 \, du \right], \quad 0 \leqslant s \leqslant 1,$$

is a martingale. Let $\mathbb{Q}$ be a new probability measure defined by $d\mathbb{Q}/d\mathbb{P} = e(1)$. Then the process

$$X_s = W_s - \int_0^s V_u \, du, \quad 0 \leqslant s \leqslant 1,$$

is a Brownian motion under $\mathbb{Q}$.

The Cameron–Martin theorem has a generalization to the Wiener measure on the path space $P_o(M) = C_o([0, 1], M)$ over a compact Riemannian manifold M. Driver [2] found the correct analogue of the euclidean Cameron–Martin shift $w^h = w + h$ under which the Wiener measure is quasi-invariant. The shift should be embedded in a flow generated by a vector field D_h defined geometrically on the path space. In the euclidean case D_h is simply the constant vector field $D_h(\gamma) = h$ and the flow is given by $\zeta^t \gamma = \gamma + th$. For a Riemannian manifold M we define the vector field D_h as follows. Fix a point $o \in M$ and an orthonormal frame $u_o \in \mathscr{O}(M)$ at o. Let $U(\gamma)$ be the horizontal lift from u_o along a path $\gamma \in P_o(M)$. For each $s \in [0, 1]$,

$$U(\gamma)_s : \mathbb{R}^n \to T_{\gamma_s} M$$

is an isometry of the two indicated euclidean spaces. We define the vector field D_h on the path space $P_o(M)$ by $D_h(\gamma) = U(\gamma)h$. More precisely,

$$D_h(\gamma)_s = U(\gamma)_s h_s, \quad 0 \leqslant s \leqslant 1.$$

In the above cited paper, Driver showed that when the manifold M is compact and $h \in C^1[0, 1]$ the vector field D_h indeed generates a flow $\{\zeta^t, \ t \in \mathbb{R}\}$ in the path space $P_o(M)$ and the Wiener measure on $P_o(M)$ (the law of a Riemannian Brownian motion on M starting from o) is quasi-invariant under the flow. Later in Hsu [5] and Enchev and Stroock [3], the existence of the flow and the quasi-invariance of the Wiener measure were extended to all Cameron–Martin vector fields D_h, $h \in \mathscr{H}$. If we denote by μ^t the law of ζ^t under the Wiener measure μ, then its Radon–Nikodym derivative with respect to μ has the form

$$\frac{d\mu^t}{d\mu} = \exp\left[\int_0^1 \langle \theta_s^t, dw_s \rangle - \frac{1}{2} \int_0^1 |\theta_s^t|^2 \, ds \right] \overset{\text{def}}{=} e_t, \tag{1.2}$$

where θ^t can be expressed more or less explicitly in terms of the flow and the curvature tensor of M and $w \in P_o(\mathbb{R}^n)$ is the anti-development of $\gamma \in P_o(M)$. We have

$$\left. \frac{d\theta_s^t}{dt} \right|_{t=0} = \dot{h}_s + \frac{1}{2} \text{Ric}_{U(\gamma)_s} h_s, \tag{1.3}$$

where $\text{Ric}_u : \mathbb{R}^n \to \mathbb{R}^n$ is the scalarized Ricci curvature tensor at an orthonormal frame $u \in \mathscr{O}(M)$. This relation and the flow equation

$$\frac{d\zeta^t}{dt} = D_h(\zeta^t), \qquad \zeta^0(\gamma) = \gamma \tag{1.4}$$

gives an integration by parts formula for the vector field D_h

$$\int_{P_o(M)} F(D_h G) \, d\mu = \int_{P_o(M)} G(D_h^* F) \, d\mu \tag{1.5}$$

on cylinder functions F and G on $P_o(M)$. The adjoint operator

$$D_h^* = -D_h + D_h^* 1$$

of the vector field (first order differential operator) D_h with respect to the Wiener measure μ is given by

$$D_h^* 1 = \int_0^1 \left\langle \dot{h}_s + \frac{1}{2} \text{Ric}_{U(\gamma)_s} h_s, dw_s \right\rangle.$$

The three objects crucial to our study of the Cameron–Martin theorem are the Radon–Nikodym derivative, the flow, and the integration by parts (through the divergence D_h^*1). They all appear in another representation of the Radon–Nikodym derivative

$$e_t = \exp\left[\int_0^t D_h^*1\left(\zeta^{-s}\right) ds \right]. \tag{1.6}$$

This formula can be verified directly from (1.2) and (1.3).

The question whether there is a complete generalization of the Cameron–Martin theorem for a general complete but possibly noncompact Riemannian manifold has been open for quite sometime. For this generalization we need to address two problems: the existence of the flow on the path space in an appropriate sense and the quasi-invariance of the Wiener measure under this flow. Since the vector field is $D_h(\gamma) = U(\gamma)h$, the flow equation (1.4) involves the horizontal lift $\{U(\zeta^t)_s,\ 0 \leqslant s \leqslant 1\}$ of the process $\zeta^t = \{\zeta_s^t,\ 0 \leqslant s \leqslant 1\}$, thus for each fixed t, the process ζ^t should be a semimartingale. In Hsu [6] the first author showed that if M is geometrically complete and its Ricci curvature has at most a linear growth

$$|\text{Ric}_x| \leqslant C\{1 + r(x)\},$$

where $r(x) = d(x, o)$, the Riemannian distance from o to x, then the flow $\{\zeta^t\}$ generated by the vector field D_h exists and the Wiener measure is quasi-invariant. It was also pointed out there that the existence of the flow, if properly interpreted, can be proved for any geometrically and stochastically complete Riemannian manifold. Note that a linear bound on the Ricci curvature implies stochastic completeness. The question was left open whether the quasi-invariance is true solely under the condition of geometric and stochastic completeness. These two completeness conditions are natural for our problem: geometric completeness for the existence of the flow and stochastic completeness for the Wiener measure to be a probability measure on the path space $P_o(M)$.

The purpose of this paper is to prove this quasi-invariance. The results of this work represent a complete generalization of the Cameron–Martin theorem to the Wiener measure on the path space of a Riemannian manifold.

We will prove the existence of the flow in very much the same way as in Hsu [6], but with more care of the details in view of later applications. The expression for the would-be Radon–Nikodym derivative (1.2) or (1.6) still makes sense. The density formula (1.2) represents the terminal value of a local exponential martingale on the time interval [0, 1]. The Wiener measure is quasi-invariant under the flow if we can show that the local exponential martingale is uniformly integrable, or equivalently,

$$\mathbb{E} \exp\left[\int_0^1 \langle \theta_s^t, dw_s \rangle - \frac{1}{2} \int_0^1 |\theta_s^t|^2\, ds \right] = 1.$$

A sufficient condition for the uniform integrability is the Novikov criterion

$$\mathbb{E} \exp\left[\frac{1}{2} \int_0^1 |\theta_s^t|^2 \, ds \right] < \infty.$$

In Hsu [6], this criterion was shown to hold for sufficiently small $|t|$ under the above mentioned linear growth restriction on the Ricci curvature. In the current paper, we avoid verifying the uniform integrability; instead, we take advantage of the fact that by our construction, the flow $\{\zeta^t\}$ is the limit of a sequence of flows for which the Wiener measure is quasi-invariant. For our argument to work, it is crucial that $\mu\{0 < e_t < \infty\} = 1$, which indeed holds under the assumption of stochastic completeness.

The rest of this paper has two sections. In Section 2 we show by an approximation argument that the flow equation has a unique solution. In Section 3, we show that the Wiener measure is quasi-invariant under the flow.

2. Existence of the flow

In this section we show that the Cameron–Martin vector field D_h generates a flow $\{\zeta^t\}$ in the path space $P_o(M)$. We assume that M is a geometrically and stochastically complete Riemannian manifold of dimension n. On a geometrically complete Riemannian manifold every bounded closed subset of M is compact, which ensures that the flow will be defined for all $t \in \mathbb{R}$. When M is stochastically complete, we have

$$\int_M p_M(s, x, y) \, dy = 1, \quad (x, s) \in M \times \mathbb{R}_+,$$

where $p_M(s, x, y)$ is the (minimal) heat kernel of M (the transition density function of Brownian motion on M). Under this condition Brownian motion on M does not explode and the Wiener measure μ is a probability measure. In particular, with probability 1 every Brownian path $\gamma[0, 1]$ is a compact subset of M.

Remark 2.1. Let $\mathscr{B}_* = \{\mathscr{B}_s, \ 0 \leqslant s \leqslant 1\}$ be the standard filtration of σ-fields on the path space $P_o(M)$. Then $\mathscr{B}_1 = \mathscr{B}(P_o(M))$, the Borel σ-field on $P_o(M)$ viewed as a metric space in the usual way. Throughout this paper, we often speak of the composition $F \circ G$ (or $F(G)$) of two measurable maps $F, G : P_o(M) \to P_o(M)$. For this composition to make sense, a measurable map such as F is always meant to be so in the Borel sense $F : (P_o(M), \mathscr{B}_1) \to (P_o(M), \mathscr{B}_1)$, i.e., $F^{-1}\mathscr{B}_1 \subset \mathscr{B}_1$.

Remark 2.2. Throughout this paper, an almost sure statement always refers to the Wiener measure μ.

Let $\mathscr{O}(M)$ be the orthonormal frame bundle of M and $\pi : \mathscr{O}(M) \to M$ the canonical projection. Let $H_i, i = 1, \ldots, n$, be the canonical horizontal vector fields on $\mathscr{O}(M)$. Fix an orthonormal frame $u_o \in \mathscr{O}_o(M)$ at o. Let w be the euclidean Brownian motion given by the coordinate process on the flat path space $(P_o(\mathbb{R}^n), \mathscr{B}_*, \nu)$. Consider the following stochastic differential equation

$$dU_s = \sum_{i=1}^n H_i(U_s) \circ dw_s^i, \qquad U_0 = u_o \tag{2.1}$$

of a horizontal process $U = U(w)$ on the orthonormal frame bundle $\mathscr{O}(M)$. The projection $\gamma = \pi U$ of its solution is the stochastic development of w. By the pathwise uniqueness of the above stochastic differential equation, the relation $Jw = \gamma$ defines the so-called Itô map $J : P_o(\mathbb{R}^n) \to P_o(M)$. The process U is the horizontal lift $U(\gamma)$ of γ to $\mathscr{O}(M)$ and at the same time the stochastic development $U(w)$ of w on the orthonormal frame bundle $\mathscr{O}(M)$. The resulting slight abuse of notation $U(\gamma) = U(w)$ should not cause any confusion. Note that $U(\gamma)$ is also the solution of a stochastic differential equation on $\mathscr{O}(M)$ driven by the Brownian motion γ on M. The line integral of the solder form θ on $\mathscr{O}(M)$ gives

$$
w_s = \int_0^s \theta(\circ dU_s).
$$

This procedure gives the inverse map $J^{-1} : P_o(M) \to P_o(\mathbb{R}^n)$ (see Hsu [7]).

Throughout the discussion we fix an $\mathbb{R}^n$-valued Cameron–Martin path h. The Cameron–Martin vector field on $P_o(M)$ is defined by $D_h(\gamma) = U(\gamma)h$ (see Driver [2]), where $U(\gamma)$ is the horizontal lift of γ. The equation of the flow generated by D_h is

$$
\frac{d\zeta^t}{dt} = U(\zeta^t)h, \qquad \zeta^0(\gamma) = \gamma. \tag{2.2}
$$

Here we assume that $\zeta^t = \{\zeta_s^t,\ 0 \leqslant s \leqslant 1\}$ is an M-valued semimartingale under the Wiener measure μ and $U(\zeta^t)$ is the horizontal lift of ζ^t. Note that the notation ζ^t plays the dual role as a process $\{\zeta_s^t,\ 0 \leqslant s \leqslant 1\}$ and as a map $\zeta^t : P_o(M) \to P_o(M)$. In the latter capacity it is a $P_o(M)$-valued random variable.

To prove the existence of the flow generated by D_h, we first convert the flow equation from the curved path space $P_o(M)$ to the flat path space $P_o(\mathbb{R}^n)$ by the Itô map $J : P_o(\mathbb{R}^n) \to P_o(M)$. This step has two purposes. First, we will introduce a cut-off function in the flow equation on the flat path space to deal with possible unboundedness of the curvature tensor and its derivatives. Doing so directly on the flow equation on the curved path space $P_o(M)$ will result in a much more complicated vector field when the equation with the cut-off function is mapped to the flat path space by the Itô map. Second, after introducing a cut-off function, we see easily that the flow equation in the flat path space has globally Lipschitz coefficients so that Picard's iteration can be applied.

The formal calculation of the pullback vector field $p = J_*^{-1} D_h$ is well known (see Driver [2] and Hsu [5]) and will not be repeated here. The result is

$$
p(w)_s = h_s - \int_0^s K(w)_\tau \circ dw_\tau,
$$

where

$$
K(w)_s = \int_0^s \Omega_{U(Jw)_\tau}(\circ dw_\tau, h_\tau).
$$

Here Ω is the curvature form, which is by definition a $o(n)$-valued horizontal 2-form on $\mathcal{O}(M)$, and $\circ\, dw$ denotes Stratonovich stochastic integration. To alleviate the notation, for $a, b \in \mathbb{R}^n$ we have written $\Omega_u(a, b)$ instead of more precise $\Omega_u(Ha, Hb)$ with $Ha = \sum_{i=1}^n H_i a_i$. Under the Wiener measure μ on $P_o(M)$, the anti-development $w = J^{-1}\gamma$ is a euclidean Brownian motion starting from the origin whose law is the Wiener measure ν on $P_o(\mathbb{R}^n)$. Conversely, under ν on $P_o(\mathbb{R}^n)$, the development $\gamma = Jw$ is a Brownian motion on M from o whose law is μ. Therefore studying the flow equation (2.2) under the measure μ is equivalent to that of the flow equation

$$\frac{d\xi^t}{dt} = p(\xi^t), \qquad \xi^0(w) = w \tag{2.3}$$

on $P_o(\mathbb{R}^n)$ under the measure ν. Once $\{\xi^t\}$ is found, the desired flow on $P_o(M)$ is simply $\zeta^t = J \circ \xi^t \circ J^{-1}$.

In terms of Itô integrals the vector field on $P_o(\mathbb{R}^n)$ is given by

$$p(z)_s = h_s + \frac{1}{2}\int_0^s \mathrm{Ric}_{U(Jz)_\tau} h_\tau \, d\tau - \int_0^s \langle K(z)_\tau, dz_\tau \rangle,$$

$$K(z)_s = \int_0^s \Omega_{U(Jz)_\tau}(dz_\tau, h_\tau) + \frac{1}{2}\int_0^s H_i \Omega_{U(Jz)_\tau}(e_j, h_\tau)\, d\langle z^i, z^j \rangle_\tau. \tag{2.4}$$

Here $\{e_i\}$ is the canonical orthonormal basis of $\mathbb{R}^n$.

In order that the switch between the path spaces $P_o(M)$ and $P_o(\mathbb{R}^n)$ work properly it is crucial that all stochastic processes involved are semimartingales with respect to the Wiener measures ν or μ. It turns out sufficient to seek solutions in the space of semimartingales of the special form

$$z_s = \int_0^s A_\tau \, d\tau + \int_0^s O_\tau \, dw_\tau, \quad 0 \leqslant s \leqslant 1, \tag{2.5}$$

where A and O are, respectively, $\mathbb{R}^n$- and $O(n)$-valued processes, both being adapted to the canonical Borel filtration $\mathscr{B}_*$ on $P_o(\mathbb{R}^n)$. Suppose that

$$\xi_s^t = \int_0^s A_\tau^t \, d\tau + \int_0^s O_\tau^t \, dw_\tau. \tag{2.6}$$

Then the flow equation (2.3) becomes

$$\begin{cases} O^t = I - \displaystyle\int_0^t K(\xi^\lambda) O^\lambda \, d\lambda, \\[4mm] A^t = O^t \displaystyle\int_0^t O^{\lambda *}\left[\dot{h} + \frac{1}{2}\mathrm{Ric}_{U(J\xi^\lambda)} h\right] d\lambda. \end{cases} \tag{2.7}$$

In order to solve these equations by Picard's iteration we need to introduce an appropriate norm on a semimartingale of the form (2.5) (see Hsu [5]):

$$\|z\|^2 = \mathbb{E} \int_0^1 |A_s|^2 \, ds + \mathbb{E} \sup_{0 \leqslant s \leqslant 1} |O_s|^2. \tag{2.8}$$

If the manifold M is compact, the components of the curvature tensor Ω and their derivatives are uniformly bounded, hence the coefficients of (2.7) are globally Lipschitz with respect to the above norm. In this case one can directly apply Picard's iteration.

For a geometrically and stochastically complete Riemannian manifold M, we will use a cut-off function defined on M to truncate the curvature tensor. Let $\phi : M \to \mathbb{R}$ be a smooth function with compact support. We define a new vector field on $P_o(\mathbb{R}^n)$:

$$p^\phi(z)_s = h_s + \frac{1}{2} \int_0^s \phi\big((Jz)_\tau\big) \mathrm{Ric}_{U(Jz)_\tau} h_\tau \, d\tau - \int_0^s \big\langle K^\phi(z)_\tau, dz_\tau \big\rangle,$$

$$K^\phi(z)_s = \int_0^s \phi\big((Jz)_\tau\big) \Omega_{U(Jz)_\tau}(dz_\tau, h_\tau) + \frac{1}{2} \int_0^s \phi\big((Jz)_\tau\big) H_i \Omega_{U(Jz)_\tau}(e_j, h_\tau) \, d\big\langle z^i, z^j \big\rangle_\tau.$$

This definition should be compared with (2.4). The new vector field uses only the values of the curvature tensor components and their derivatives on a fixed compact region, namely, on the support of the cut-off function ϕ. We also note that $K_s^{\phi,t} \in o(n)$. Consider the flow equation

$$\frac{d\xi^{\phi,t}}{dt} = p^\phi\big(\xi^{\phi,t}\big), \qquad \xi^{\phi,0}(w) = w,$$

or equivalently, consider the integral equation

$$\xi^{\phi,t} = w + \int_0^t p^\phi\big(\xi^{\phi,\lambda}\big) \, d\lambda. \tag{2.9}$$

We write as before

$$\xi_s^{\phi,t} = \int_0^s A_u^{\phi,t} \, du + \int_0^s O_u^{\phi,t} \, dw_u. \tag{2.10}$$

In terms of the pair $\{A^{\phi,t}, O^{\phi,t}\}$ the flow equation becomes

$$
\begin{cases}
O^{\phi,t} = I - \displaystyle\int_0^t K^\phi(\xi^{\phi,\lambda}) O^{\phi,\lambda}\, d\lambda, \\[4mm]
A^{\phi,t} = O^{\phi,t} \displaystyle\int_0^t O^{\phi,\lambda*}\left[\dot{h} + \frac{1}{2}\phi(J\xi^{\phi,\lambda})\mathrm{Ric}_{U(J\xi^{\phi,\lambda})} h\right] d\lambda.
\end{cases}
\tag{2.11}
$$

These equations can be solved by Picard's iteration as if the manifold is compact. The crucial step is to show that in the norm $\|z\|$ defined in (2.8) the vector field is globally Lipschitz continuous.

Proposition 2.3. *There is a constant C such that for any semimartingales z_i of the form (2.5) with the norm defined in (2.8), we have*

$$
\left\| p^\phi(z_1) - p^\phi(z_2) \right\| \leqslant C\|z_1 - z_2\|.
$$

Let ξ_i^t and η_i^t be semimartingales of the form (2.5) such that

$$
\xi_i^t = \xi_i^0 + \int_0^t p^\phi(\eta_i^\lambda)\, d\lambda.
$$

Then

$$
\left\| \xi_1^t - \xi_2^t \right\| \leqslant \left\| \xi_1^0 - \xi_2^0 \right\| + C \int_0^t \left\| \eta_1^\lambda - \eta_2^\lambda \right\| d\lambda.
$$

Proof. Since the vector field p^ϕ only uses the components of the curvature tensor and their derivatives on a compact subset of the manifold, the Lipschitz continuity of p^ϕ can be proved in exactly the same way as in the case of a compact manifold, which involves nothing more than routine bounds of stochastic integrals with respect to dw_τ by Doob's inequality and those with respect to $d\tau$ by taking absolute value under the integrals. The details can be found in Hsu [5] and will not be repeated here, but we point out an important fact, namely, even after introducing the cut-off function, the new K^ϕ still takes values in the space $o(n)$ of anti-symmetric matrices and the corresponding $O^{\phi,t}$ takes values in the space $O(n)$ of orthogonal matrices $O(n)$ (see the first equation in (2.11)). As a consequence, $O^{\phi,t}$ is always uniformly bounded. $\square$

Proposition 2.4. *There exists a unique family of semimartingales $\{\xi^{\phi,t},\ t \in \mathbb{R}\}$ of the form (2.5) such that with probability 1:*

(a) $\xi^{\phi,0}(w) = w$;
(b) $p^\phi(\xi^{\phi,t})_s$ *is jointly continuous in* $(t,s) \in \mathbb{R} \times [0,1]$;
(c) $\xi_s^{\phi,t}$ *is jointly C^1 in $t \in \mathbb{R}$ and continuous in $s \in [0,1]$;*
(d) $(d/dt)\xi_s^{\phi,t} = p^\phi(\xi^{\phi,t})_s$.

Proof. We only outline the proof here, the technical details being mostly contained in Hsu [5]. The cut-off function ϕ is fixed in the course of this proof. For simplicity, we drop the superscript ϕ wherever no confusion is likely to occur.

Let $\xi^{t,0}(w) = w$ and

$$\xi^{t,n} = \xi^{t,0} + \int_0^t p\left(\xi^{\lambda,n-1}\right) d\lambda.$$

From Proposition 2.3 we have

$$\left\| \xi^{t,n} - \xi^{t,n-1} \right\| \leqslant C \int_0^t \left\| \xi^{\lambda,n-1} - \xi^{\lambda,n-2} \right\| d\lambda.$$

This inequality implies that the limit $\xi^t = \lim_{n\to\infty} \xi^{t,n}$ exists and is the solution to (2.11). The uniqueness is clear because we are dealing with a Volterra type integral equation. This shows the existence and uniqueness of the solution of the flow equation (2.9) in the class of semimartingales of the form (2.5).

From (2.9) we have (a) immediately. The joint continuity claimed in (b) follows from Kolmogorov's sample path continuity criterion (for processes with two time parameters). Using this we see that (c) and (d) follow again from (2.9). $\quad\square$

Proposition 2.5. *Let* $\{\xi^{\phi,t}\}$ *be the flow in Proposition* 2.4.

(a) *For each fixed* t *the law* $\nu^{\phi,t}$ *of* $\xi^{\phi,t}$ *and the Wiener measure* ν *on* $P_o(\mathbb{R}^n)$ *are mutually absolutely continuous. The Radon–Nikodym derivative is given by*

$$\frac{d\nu^{\phi,t}}{d\nu} = \exp\left[\int_0^1 O_s^{\phi,t*} A_s^{\phi,t} \, dw_s - \frac{1}{2} \int_0^1 \left|A_s^{\phi,t}\right|^2 ds \right]. \tag{2.12}$$

We have

$$\frac{d\nu^{\phi,t}}{d\nu} = \exp \int_0^t l_h^\phi\left(\xi^{\phi,-\lambda}\right) d\lambda,$$

where

$$l_h^\phi(w) = \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\phi\left((Jw)_s\right)\mathrm{Ric}_{U(Jw)_s} h_s, dw_s \right\rangle.$$

(b) *For all* $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$ *we have almost surely,*

$$\xi^{\phi,t_1} \circ \xi^{\phi,t_2} = \xi^{\phi,t_1+t_2}. \tag{2.13}$$

Proof. From (2.6) and the fact that $O_s^{\phi,t} \in O(n)$ the assertion in (a) and (2.12) immediately from Girsanov's theorem. The second formula for the Radon–Nikodym derivative can be obtained from differentiating (2.9) with respect to t and use the flow equation (2.11). For (b), both sides of (2.13) (with t_1 as time variable) are the solution of the flow equation with initial value ξ^{ϕ,t_2} at $t_1 = 0$, hence the equality holds by the uniqueness of solutions of the flow equation (see Proposition 2.3). $\quad\square$

For each fixed t we define the semimartingale $\zeta^{\phi,t}$ on the probability space $(P_o(M), \mathscr{B}_*, \mu)$ by

$$\zeta^{\phi,t} = J \circ \xi^{\phi,t} \circ J^{-1}. \tag{2.14}$$

The maps J and J^{-1} send semimartingales to semimartingales, therefore $\zeta^{\phi,t} = \{\zeta_s^{\phi,t}, 0 \leqslant s \leqslant 1\}$ is an M-valued semimartingale for each fixed t.

Proposition 2.6. *Let $\zeta^{\phi,t} = J \circ \xi^{\phi,t} \circ J^{-1}$. The following assertions hold.*

(a) *For each fixed t the law $\mu^{\phi,t}$ of $\zeta^{\phi,t}$ and the Wiener measure μ are mutually absolutely continuous and the Radon–Nikodym derivative is give by*

$$\frac{d\mu^{\phi,t}}{d\mu} = \frac{dv^{\phi,t}}{dv} \circ J^{-1}.$$

(b) *For all fixed $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$ we have almost surely,*

$$\zeta^{\phi,t_1} \circ \zeta^{\phi,t_2} = \zeta^{\phi,t_1+t_2}. \tag{2.15}$$

Proof. These properties are inherited from the corresponding properties of $\xi^{\phi,t}$ proved in Proposition 2.5. $\quad\square$

Now we come to the main result of this section. Since the manifold is assumed to be geometrically complete, we can gradually remove the effect of the cut-off function and construct a flow for the Cameron–Martin vector field D_h.

Theorem 2.7. *Let M be a geometrically and stochastically complete Riemannian manifold. There exists a family of semimartingales $\{\zeta^t, t \in \mathbb{R}\}$ such that with probability 1:*

(a) $\zeta^0(\gamma) = \gamma$;
(b) $U(\zeta^t)_s$ *is jointly continuous in* $(t, s) \in \mathbb{R} \times [0, 1]$;
(c) ζ_s^t *is jointly* C^1 *in* $t \in \mathbb{R}$ *and continuous in* $s \in [0, 1]$;
(d) $d\zeta_s^t/dt = U(\zeta^t)_s h_s$;
(e) *For all fixed* $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$, *we have almost surely*

$$\zeta^{t_1} \circ \zeta^{t_2} = \zeta^{t_1+t_2}.$$

Proof. Without loss of generality we assume that $\|h\|_\infty \leqslant 1$. We choose a sequence of cut-off functions $\{\phi_N\}$ as follows. Denote by B_R the geodesic ball of radius R centered at o. For a

positive integer N, we let ϕ_N be a bounded (uniformly in N) smooth function on M such that $\phi_N(x) = 1$ for $x \in B_{2N}$ and $\phi_N(x) = 0$ for $x \notin B_{3N}$ (see Remark 2.8 below). Consider the flow $\zeta^{N,t} = \zeta^{\phi_N,t}$ in Proposition 2.6. These flows will not feel the presence of the cut-off function as long as they stay within the geodesic ball B_{2N}. More precisely, define an increasing sequence of stopping times

$$\sigma_N = \inf\{s \leqslant 1 \colon r(\gamma_s) = N\}$$

(with the convention that $\inf \emptyset = 1$) and let $C_N = \{\sigma_N = 1\}$. Then

$$C_N = \{\gamma \in P_o(M) \colon \gamma_s \in B_N \text{ for all } s \in [0,1]\}.$$

If $s \leqslant \sigma_N$, then the initial path $\zeta^{N,0} = \{\gamma_s, \ 0 \leqslant s \leqslant 1\}$ of the flow $\{\zeta^{N,t}\}$ lies within the geodesic ball B_N. Since $\phi_N = 1$ on B_{2N}, from (2.2) and

$$\big|U\big(\zeta^\lambda\big)_s h_s\big| \leqslant |h_s| \leqslant 1$$

we have $d(\zeta_s^{N,t}, \gamma_s) \leqslant |t|$ for small time t and

$$r\big(\zeta_s^{N,t}\big) \leqslant d\big(\zeta_s^{N,t}, \gamma_s\big) + r(\gamma_s) \leqslant |t| + N.$$

By a routine open-closed argument with the above inequality we see that the above inequalities will hold for all $s \leqslant \sigma_N$ and $|t| \leqslant N$. As a consequence, for $M \geqslant N$, the flows $\zeta_s^{M,t}$ and $\zeta_s^{N,t}$ satisfy the same equation for all $|t| \leqslant N$ and $s \leqslant \sigma_N$. By uniqueness we must have the consistency

$$\zeta^{M,t}(\gamma)_s = \zeta^{N,t}(\gamma)_s, \quad M \geqslant N, \ s \leqslant \sigma_N, \ |t| \leqslant N.$$

Note that since $C_N = \{\sigma_N = 1\}$, the above equality also holds for $0 \leqslant s \leqslant 1$ if $\gamma \in C_N$. We can now define

$$\zeta^t(\gamma)_s = \zeta^{N,t}(\gamma)_s, \quad 0 \leqslant s \leqslant \sigma_N, \ |t| \leqslant N. \tag{2.16}$$

This defines $\zeta^t = \{\zeta_s^t, \ 0 \leqslant s \leqslant 1\}$ for all $\gamma \in C_N$. By stochastic completeness Brownian motion on M does not explode. With probability one the path $\gamma[0,1]$ is a compact subset of M, hence $\mu\{C_N\} \uparrow 1$. This shows that (2.16) defines a family $\{\zeta^t\}$ of semimartingales such that

$$\zeta^t(\gamma) = \zeta^{N,t}(\gamma), \quad \gamma \in C_N, \ |t| \leqslant N.$$

We now prove the properties of the flow $\{\zeta^t\}$ listed in the statement of the theorem. To start with, (a) follows from the fact that $\zeta^{N,0}(\gamma) = \gamma$. The joint continuity of $U(\zeta^{N,t})_s$ can be proved again by Kolmogorov's sample path continuity criterion for processes with two time parameters. The assumption of geometric completeness is needed for this step of the proof; see Remark 2.8 below. From $\zeta_s^t = J(\xi^t)_s$ and the definition of the Itô map J (see (2.1)) we have (d) as the equality of two semimartingales for each fixed t. This together with (b) implies (c) and also (d) for all s and t.

For the composition property (e), by (2.15) we have first

$$\zeta^{2N,t_1}\left(\zeta^{2N,t_2}\gamma\right) = \zeta^{2N,t_1+t_2}\gamma.$$

If $|t_1| + |t_2| \leqslant N$, then

$$\zeta^{2N,t_2}\gamma = \zeta^{t_2}\gamma \quad \text{and} \quad \zeta^{2N,t_1+t_2}\gamma = \zeta^{t_1+t_2}\gamma$$

for all $\gamma \in C_N$. But it is clear that $\zeta^{t_2}\gamma \in C_{2N}$, hence

$$\zeta^{2N,t_1}\left(\zeta^{2N,t_2}\gamma\right) = \zeta^{t_1}\left(\zeta^{t_2}\gamma\right).$$

It follows that $\zeta^{t_1}(\zeta^{t_2}\gamma) = \zeta^{t_1+t_2}\gamma$ for all $\gamma \in C_N$, hence for almost all $\gamma \in P_o(M)$ because $\mu\{C_N\} \uparrow 1$. $\quad\square$

Remark 2.8. The cut-off functions used in the proof of Theorem 2.7 exist by the assumption that M is geometrically complete. However, even without the geometric completeness (but retaining the stochastic completeness), we can construct a sequence of cut-off functions $\{\phi_N\}$ such that the curvature tensor and its derivatives are uniformly bounded on $\{\phi_N > 0\}$ for each fixed N and that the sequence of increasing interiors $\{\phi_N = 1\}^\circ$ exhausts M. Using such a sequence of the cut-off functions, we can show by properly modifying the proof given above that a jointly continuous flow $\{\zeta_s^t(\gamma), \ e^-(\gamma, s) < t < e^+(\gamma, s)\}$ can be constructed up to its natural lifetime in t. This means that $\lim_{t \to e^\pm(\gamma,s)} \zeta^t(\gamma, s) = \partial_M$ in the one-point compactification $M_\partial = M \cup \{\partial_M\}$. There are also manifolds not geometrically complete which nevertheless has a sequence of cut-off functions having all the properties used in the proof of Theorem 2.7. In this case we can construct a global *measurable* flow $\{\zeta^t, \ t \in \mathbb{R}\}$ as the solution of the corresponding integral equation in the space of $P_o(M)$-valued random variables. However, this flow may not have a version jointly continuous in the two parameters t and s, for Kolmogorov's sample path continuity theorem requires that the state space M be a complete metric space (see, e.g., Kallenberg [8, p. 313]). A case in point is $M = \mathbb{R}^2$ with a single point removed and $h_s = (0, s)$. While the removed point is not needed for the existence of the measurable flow $\zeta^t(\gamma) = \gamma_s + ts$ (for each fixed t), it is needed if we want to have a version of the flow that is jointly continuous in t and s.

Finally we show that the flow $\{\zeta^t\}$ is infinitely differentiable in the t-direction.

Proposition 2.9. *There is a version of the flow $\{\zeta^t, \ t \in \mathbb{R}\}$ such that with probability one the path $t \mapsto \zeta_s^t(\gamma)$ is smooth for every $s \in [0, 1]$. More precisely,*

$$\mathbb{P}\left\{\gamma \in P_o(M) \colon \zeta(\gamma) \in C^{\infty,0}\left(\mathbb{R} \times [0, 1]; M\right)\right\} = 1.$$

Proof. The proof involves repeatedly use of Kolmogorov's sample path continuity criterion. To start with, we have shown in Theorem 2.7 that $U(\zeta^t)_s$ has a version jointly continuous in t and s, the flow ζ_s^t has a version jointly C^1 in t and continuous in s, and the identity $d\zeta_s^t/dt = U(\zeta^t)_s h_s$ holds. Once we know that ζ^t is C^1 in t, we can differentiate the stochastic differential equation (in the s-direction) satisfied by $U(\zeta^t)_s h_s$ with respect to t and use Kolmogorov's criterion to conclude that $U(\zeta^t)_s h_s$ has a version jointly C^1 in t and continuous in s. The identity $d\zeta_s^t/dt = U(\zeta^t)_s h_s$ then shows that ζ_s^t has a version jointly C^2 in t and continuous in s. This argument

can be repeated indefinitely and shows that ζ_s^t has a version jointly C^m in t and continuous in s for any positive integer m. For each m, there is an exceptional set Ω_m of probability zero of the path space $P_o(M)$ on which the assertion does not hold. By excluding from the path space the union $\bigcup_{m=1}^\infty \Omega_m$, which still has probability zero, we see that there is version of the flow jointly continuous in s and infinitely differentiable in t. $\square$

3. Quasi-invariance of the Wiener measure

Let M be a geometrically and stochastic complete Riemannian manifold and $\{\zeta^t\}$ the flow of the Cameron–Martin vector field D_h on the path space $P_o(M)$ constructed in the preceding section. In this section we show that for each $t \in \mathbb{R}$, the law μ^t of the semimartingale ζ^t is mutually absolutely continuous with respect to the Wiener measure μ.

We have shown in Proposition 2.6 that the law $\mu^{N,t}$ of $\zeta^{N,t}$ and μ are mutually absolutely continuous and the density function is

$$e_t^N = \exp \int_0^t l_h^N(\zeta^{N,-\lambda}) \, d\lambda,$$

where

$$l_h^N(\gamma) = \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\phi_N(\gamma_s)\mathrm{Ric}_{U(\gamma)_s} h_s, dw_s \right\rangle.$$

Here w is the stochastic anti-development of γ. Recall that

$$C_N = \left\{ \gamma \in P_o(M) : \max_{0 \leqslant s \leqslant 1} r(\gamma_s) \leqslant N \right\}.$$

For $\gamma \in C_N$ and $|\lambda| \leqslant N$ we have

$$l_h^N(\gamma) = \int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U(\gamma)_s} h_s, dw \right\rangle \overset{\mathrm{def}}{=} l_h$$

and $\zeta^{N,-\lambda} = \zeta^{-\lambda}$, hence for $|t| \leqslant N$

$$e_t^N = \exp \int_0^t l_h(\zeta^{-\lambda}) \, d\lambda \overset{\mathrm{def}}{=} e_t \quad \text{on } C_N. \tag{3.1}$$

Since M is geometrically and stochastically complete, the flow does not explode and

$$\mu\{0 < e_t < \infty\} = 1.$$

Now we prove the main result of this section.

Theorem 3.1. *The laws μ^t of ζ^t and the Wiener measure μ are mutually absolutely continuous and the Radon–Nikodym derivative is given by*

$$\frac{d\mu^t}{d\mu} = \exp \int\limits_0^t l_h\left(\zeta^{-\lambda}\right) d\lambda.$$

Proof. Write $X = P_o(M)$ for simplicity. Fix $N \geqslant |t|$. Then $\zeta^t(\gamma) = \zeta^{N,t}(\gamma)$ and $e_t(\gamma) = e_t^N(\gamma)$ for $\gamma \in C_N$. For a nonnegative bounded measurable function F on $P_o(M)$ we have

$$\int\limits_X F\left(\zeta^t\right) d\mu \geqslant \int\limits_{C_N} F\left(\zeta^{N,t}\right) d\mu$$

$$\geqslant \int\limits_X F\left(\zeta^{N,t}\right) d\mu - \|F\|_\infty \mu(X\backslash C_N)$$

$$= \int\limits_X F e_t^N \, d\mu - \|F\|_\infty \mu(X\backslash C_N)$$

$$\geqslant \int\limits_{C_N} F e_t \, d\mu - \|F\|_\infty \mu(X\backslash C_N)$$

$$\to \int\limits_X F e_t \, d\mu.$$

Therefore we have

$$\int\limits_X F e_t \, d\mu \leqslant \int\limits_X F\left(\zeta^t\right) d\mu. \tag{3.2}$$

By the monotone convergence theorem, the above inequality holds for all nonnegative measurable F. From Theorem 2.6 we have $\zeta^{N,t}(\zeta^{N,-t}\gamma) = \gamma$. Let $G = F(\zeta^{N,t})e_t(\zeta^{N,t})$. Then

$$\int\limits_X F e_t \, d\mu = \int\limits_X G\left(\zeta^{N,-t}\right) d\mu$$

$$= \int\limits_X G e_{-t}^N \, d\mu$$

$$\geqslant \int\limits_{C_N} F\left(\zeta^t\right) e_t\left(\zeta^t\right) e_{-t} \, d\mu$$

$$\to \int\limits_X F\left(\zeta^t\right) e_t\left(\zeta^t\right) e_{-t} \, d\mu.$$

Therefore

$$\int_X F e_t \, d\mu \geqslant \int_X F(\zeta^t) e_t(\zeta^t) e_{-t} \, d\mu. \tag{3.3}$$

On the other hand, from the definition (3.1) of e_t and the identity $\zeta^{-\lambda} \circ \zeta^t = \zeta^{t-\lambda}$ it is easy to verify that

$$e_t(\zeta^t) e_{-t} = 1.$$

This identity together with (3.2) and (3.3) implies

$$\int_X F e_t \, d\mu = \int_X F(\zeta^t) \, d\mu = \int_X F \, d\mu^t.$$

In view of $\mu\{0 < e_t < \infty\} = 1$ this shows that μ and μ^t are mutually absolutely continuous and $d\mu^t/d\mu = e_t$. $\square$

Remark 3.2. We have restricted our discussion in this work to the Wiener measure on the path space over a geometrically and stochastically complete Riemannian manifold equipped with the Levi-Civita connection. Our methods and the results remain valid for a general non-degenerate diffusion measure on a differentiable manifold provided that (1) the diffusion process is stochastically complete (i.e., with probability 1 it has infinite lifetime); (2) the manifold is geometrically complete under the Riemannian metric defined by the diffusion measure; (3) the connection (needed for defining the Cameron–Martin vector fields) is compatible with the Riemannian metric and has an anti-symmetric torsion. In particular, our results remain valid for canonical Brownian motions on Lie groups equipped with the Cartan connections.

Acknowledgments

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