El. Math., 47, (1992)

0013-6018/92/040169-04$1.50 + 0.20/0

A Class of Singular Continuous Functions

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A nondecreasing continuous function $F(x)$ on the real line is *singular* if its derivative $F'(x)$ is zero almost everywhere (a.e.). The most famous singular continuous function is Cantor's function discussed in every textbook on real analysis. In this short note we construct a class of singular continuous functions which includes Cantor's function as a

Auch wenn es dem Mathematik-Studenten manchmal so vorkommen mag, die grundlegenden mathematischen Begriffe, wie "Funktion", "Konvergenz", "Stetigkeit", etc. sind *nicht* plötzlich in ihrer heutigen Eindeutigkeit und Schärfe vom Himmel gefallen. Vielmehr haben sie sich aus intuitiven Anfängen im Laufe langer Jahre langsam entwickelt: nur schrittweise haben sie an Schärfe gewonnen, erst nach und nach wurde ihr Verhältnis zur bestehenden Begriffswelt geklärt. Intensive Bemühungen führten schliesslich zu den heute gängigen ausgefeilten Definitionen dieser Konzepte. Ein Beispiel, an dem sich diese Entwicklung gut verfolgen lässt, ist der Begriff der Stetigkeit einer Funktion. Die Beziehungen der Eigenschaft der Stetigkeit zu anderen, wie etwa zur Differenzierbarkeit, und zur Geometrie waren während langer Zeit unklar. Immer wieder wurden Beispiele entdeckt, deren Eigenschaften die Fachwelt überraschten und die zur Korrektur von damals üblichen, aber eben irrigen Annahmen zwangen. — Pei Hsu geht in seinen Beitrag auf eine spezielle derartige Frage ein. Ausgehend von der bekannten Cantor-Funktion konstruiert er eine Klasse von monoton zunehmenden stetigen Funktionen, welche fast überall eine triviale Ableitung besitzen. Solche Funktionen, man nennt sie *singuläre stetige Funktionen*, widersprechen natürlich einer naiven Anschauung der Stetigkeit. — Eine sehr gut zugängliche Darstellung des ganzen in diesem Beitrag behandelten Themenkreises ist im Buch von Ralph P. Boas, jr. zu finden *A primer of real functions*, Carus Mathematical Monographs 13, Mathematical Association of America (1972). Für die Cantor-Funktion vergleiche man S. 135. *ust*

special case. Unlike Cantor's function, most functions in this class are strictly increasing on the interval [0, 1].

We construct these singular functions as the distribution functions of certain random variables. We also give an explicit, but less instructive, formula for these functions. Readers not familiar with probability theory may be more comfortable with the explicit formula.

Let $N \geq 2$ be a positive integer and $\mathbf{p} = [p_0, \dots, p_{N-1}]$ be an N-dimensional probability vector, namely,

$$p_i \geq 0, \qquad i = 0, \dots, N-1; \qquad \sum_{i=0}^{N-1} p_i = 1.$$

We exclude the trivial case where one of the p_i's is equal to 1. Next let $X_n, n = 1, 2, \dots$ be a sequence of independent, identically distributed random variables with the common distribution

$$P[X_1 = i] = p_i, \qquad i = 0, \dots, N-1.$$

Define a new random variable X with values in the interval [0, 1]:

$$X = \sum_{n=1}^{\infty} \frac{X_n}{N^n}.$$

Thus X is a number in [0, 1] whose digits in base N expansion $X = 0.X_1X_2\cdots$ are chosen randomly and independently according to the probability vector $\mathbf{p}$. Let $F(x; \mathbf{p})$ be the distribution function of X,

$$F(x; \mathbf{p}) = P[X \leq x].$$

We now derive an explicit formula for $F(x; \mathbf{p})$. In the following we simply write $F(x)$ instead of $F(x; \mathbf{p})$ if no confusion is possible.

It is clear that $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$. For $0 < x < 1$, suppose that $x = 0.x_1x_2\cdots$ is the expansion of x in base N. For definitiveness, when two expansions are possible, for example $0.13\overline{0} = 0.12\overline{N-1}$ ($\overline{k}$ means repeating the digit k indefinitely), unless indicated otherwise we always take the nonterminating one, namely, the one with only the digit $N-1$ from some digit on. Let $x^n = 0.x_1x_2\cdots x_n$ be the truncation of x at the nth place. Then since $P[X = x] = P[X \leq 0] = 0$, we have

$$F(x) = \sum_{n=0}^{\infty} P\left[x^n < X \leq x^{n+1}\right].$$

Now $x^n < X \leq x^{n+1}$ means that X must have the form $X = 0.x_1x_2\cdots x_nX_{n+1}X_{n+2}\cdots$, where X_{n+1} is equal to one of the digits $0, 1, \dots, x_{n+1}-1$. Note that we may assume that the expansion $X = 0.X_1X_2\dots$ is always nonterminating, for the probability that X has a terminating expansion is zero. Thus

$$P\left[x^n < X \leq x^{n+1}\right] = p_{x_1}\cdots p_{x_n}\left(p_0 + \cdots + p_{x_{n+1}-1}\right).$$

In the above inequality, the sum between the parentheses should be set equal to zero if $x_{n+1} = 0$. We now obtain

$$F(x;\mathbf{p}) = \sum_{n=0}^{\infty} p_{x_1} \cdots p_{x_n} \left(p_0 + \cdots + p_{x_{n+1}-1}\right).$$

This is an explicit formula for the function $F(x;\mathbf{p})$. Incidentally, the above explicit representation works when x is expressed in its terminating expansion as well. The special case $N = 3, \mathbf{p} = \left[\frac{1}{2}, 0, \frac{1}{2}\right]$ gives Cantor's function.

We will prove two properties of these functions.

(I) *$F(x;\mathbf{p})$ is strictly increasing unless one of the p_i's is equal to zero, and is Hölder continuous with the exponent*

$$\alpha = \frac{\log \frac{1}{r}}{\log N},$$

where $r = \max\{p_0, \ldots, p_{N-1}\}$.

(II) *$F(x;\mathbf{p})$ is singular continuous except when $\mathbf{p}$ is the uniform distribution:*

$$\mathbf{p} = \left[\frac{1}{N}, \ldots, \frac{1}{N}\right],$$

in which case $F(x;\mathbf{p}) = x$ *on* $[0, 1]$.

Proof of (I). If x, y in [0, 1] have the same first n digits, then from the explicit formula for $F(x)$ we have

$$|F(x) - F(y)| \le \sum_{l=n}^{\infty} r^l \le \frac{r^n}{1-r}.$$

Let x, y be two distinct numbers in [0, 1]. Pick a nonnegative n such that $N^{-(n+2)} \le |x-y| < N^{-(n+1)}$. Then in the nonterminating expansions of x and y either they have the same first n digits or $x^n = y^n + N^{-n}$. In the latter case $y^n = 0.x_1 x_2 \cdots x_n \overline{N-1}$ and x^n have the same first n digits. Therefore we have

$$\begin{aligned}|F(x) - F(y)| &\le |F(x) - F(x^n)| + |F(x^n) - F(y^n)| + |F(y^n) - F(y)| \\ &\le \frac{3r^n}{1-r}.\end{aligned}$$

It follows that for any x, y in [0,1] with $|x - y| \le N^{-1}$ we have

$$|F(x) - F(y)| \le \frac{3}{r^2(1-r)} |x-y|^{\alpha}.$$

This shows that $F(x)$ is Hölder continuous with exponent α.

Let $x = 0.x_1x_2\ldots$ (in base N) be a point in $[0,1]$. Let $x^n = 0.x_1x_2\ldots x_n$ as before. Then

$$x^n < x \leq x^n + \frac{1}{N^n}.$$

Writing $x^n + N^{-n}$ as $0.x_1 \cdots x_n\overline{N-1}$, from the explicit formula for $F(x)$ we have

$$F\left(x^n + N^{-n}\right) - F\left(x^n\right) = p_{x_1} \cdots p_{x_n}.$$

This shows that F is strictly increasing on $(0,1)$ if none of the p_i's is equal to zero.

Proof of (II). From the above equality we see that

$$D_nF(x) \stackrel{\text{def}}{=} N^n \left[F\left(x^n + N^{-n}\right) - F\left(x^n\right)\right] = Np_{x_1} \cdots Np_{x_n}.$$

Since $F(x)$, being a monotone function, is a.e. differentiable, we have a.e.,

$$\lim_{n\to\infty} D_nF(x) = F'(x) \text{ and } F'(x) \text{ is finite.}$$

It follows that

$$F'(x) = \prod_{n=1}^{\infty} Np_{x_n}, \qquad \text{a.e.}$$

If all of p_i's are equal to N^{-1}, then $F'(x) = 1$ a.e. on $[0, 1]$. We have $F(x) = x$ on $[0,1]$.

If at least one of the p_i's is not equal to N^{-1}, say, $p_1 \neq N^{-1}$, then for those x whose expansion has infinitely many 1's the general term Np_{x_n} of the above infinite product does not converge to 1. Therefore, by an elementary theorem on infinite products, the value of the infinite product, if it exists and is finite, must be 0. On the other hand, the reader can verify easily that the numbers whose expansions have only finitely many 1's form a set of Lebesgue measure zero. Therefore we have proved $F'(x) = 0$ a.e. and $F(x)$ is a singular continuous function.

For readers familiar with the strong law of large numbers in probability theory the following argument may be more appealing. By the strong law of large numbers, almost all numbers in [0, 1] are normal in base N, namely, if $s_n(x;i)$ is the number of the digit i among the first n digits of the expansion of x, then for almost all $x \in [0,1]$,

$$\text{for all } i = 0,\ldots,N-1: \qquad \frac{s_n(x;i)}{n} \to \frac{1}{N} \qquad \text{as } n \to \infty.$$

It follows that, if at least one of the p_i's is not equal to N^{-1}, then for almost all $x \in [0,1]$

$$\sqrt[n]{D_nF(x)} = \prod_{i=0}^{N-1} (Np_i)^{s_n(x;i)/n} \to N\ \sqrt[N]{p_0 \cdots p_{N-1}} < 1.$$

This implies immediately $F'(x) = 0$, a.e.

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