*The Annals of Probability*
2003, Vol. 31, No. 3, 1305–1319


# BROWNIAN MOTION AND DIRICHLET PROBLEMS AT INFINITY[1]

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We discuss angular convergence of Riemannian Brownian motion on a Cartan–Hadamard manifold and show that the Dirichlet problem at infinity for such a manifold is uniquely solvable under the curvature conditions $-Ce^{(2-\eta)ar(x)} \le K_M(x) \le -a^2$ $(\eta > 0)$ and $-Cr(x)^{2\beta} \le K_M(x) \le -\alpha(\alpha-1)/r(x)^2$ $(\alpha > \beta + 2 > 2)$, respectively.

**1. Introduction.** A Cartan–Hadamard manifold is a complete, simply connected Riemannian manifold with nonpositive sectional curvature. We fix a reference point $o \in M$ once and for all. It is well known that the exponential map $\exp : T_oM \to M$ from the tangent space $T_oM$ based at $o$ is a diffeomorphism. This defines a polar coordinate system $(r, \theta)$ on $M$. Two geodesic rays $\gamma_1$ and $\gamma_2$ on $M$ are called equivalent if there is a constant $C$ such that $d(\gamma_1(t), \gamma_2(t)) \le C$ for all $t \ge 0$. It can be shown that this is an equivalence relation on the set of geodesic rays. The set of equivalence classes is the sphere at infinity $S_\infty(M)$. A basic fact of Cartan–Hadamard manifolds is that $\widehat{M} = M \cup S_\infty(M)$ with a properly defined topology (called the cone topology) is a compactification of $M$. For each $o \in M$, the sphere at infinity $S_\infty(M)$ can be identified homeomorphically with the unit sphere in the tangent space $T_oM$. If $(r, \theta)$ are the polar coordinates based at $o$, then a sequence of points $z_n \in M$ converges to a boundary point $\theta_0 \in S_\infty(M)$ if and only if $r(z_n) \to \infty$ and $\theta(z_n) \to \theta_0$ (see [5]).

Given a continuous function $f$ on $S_\infty(M)$, the Dirichlet problem at infinity is to find a function $u_f \in C^\infty(M) \cap C(\widehat{M})$ that is harmonic on $M$ and equal to $f$ on $S_\infty(M)$. We say that the Dirichlet problem at infinity is solvable for $M$ if for every $f \in C(S_\infty(M))$ there is a unique solution $u_f$. This property of a Cartan–Hadamard manifold can be obtained under certain conditions on the curvature of $M$ and can be approached analytically or probabilistically. For analytic methods, see [1, 3, 4, 6, 7]; for probabilistic methods, see [8–10, 14–16, 18]. The more difficult problem of identifying the Martin boundary with the boundary at infinity was discussed in [4] and [13]. We are mainly concerned with a probabilistic approach to the problem, which involves basically proving the angular convergence of transient Brownian motion.

In this paper, we will combine an improved version of the method used in [9] and an idea from [14] to prove the solvability of the Dirichlet problem at infinity

Received March 2001; revised July 2002.

[1]Supported in part by NSF Grant DMS-01-04079.

*AMS 2000 subject classifications.* Primary 58J65; secondary 60J60.

*Key words and phrases.* Brownian motion, Dirichlet problem, Cartan–Hadamard manifold.

under certain curvature growth conditions more generous than previously known. We consider two typical situations. In the first case, the sectional curvature is assumed to be bounded by a negative constant: $\mathrm{Sect}_x \le -a^2$. In the second case, we assume that $\mathrm{Sect}_x \le -c/r^2$ [$r = r(x) = d(x, o)$]. This second case is significant because it vanishes as $r \to \infty$. Let us now state our main theorems.

THEOREM 1.1. *Let $M$ be a Cartan–Hadamard manifold. Suppose that there exist a positive constant $a$ and a positive and nonincreasing function $h$ with $\int_0^\infty rh(r)\,dr < \infty$ such that*

$$-h(r)^2 e^{2ar} \le \mathrm{Ric}_x \quad \textit{and} \quad \mathrm{Sect}_x \le -a^2.$$

*Then the Dirichlet problem at infinity for $M$ is solvable.*

Early lower bounds of the form $Ce^{\lambda ar}$ were obtained in [6] with $\lambda < 1/3$ and in [14] with $\lambda < 1/2$. Our result represents a significant improvement in this respect.

THEOREM 1.2. *Let $M$ be a Cartan–Hadamard manifold. Suppose that there exist positive constants $r_0$, $\alpha > 2$ and $\beta < \alpha - 2$ such that*

$$-r^{2\beta} \le \mathrm{Ric}_x \quad \textit{and} \quad \mathrm{Sect}_x \le -\frac{\alpha(\alpha-1)}{r^2}$$

*for all $r = r(x) \ge r_0$. Then the Dirichlet problem at infinity for $M$ is solvable.*

Hsu and March [9] proved a lower bound of the form $-r^{2\beta}$ with $\beta < 1 - 2/\alpha < 1$. Our new result opens the possibility of $\beta \ge 1$.

The rest of this paper has three sections. In Section 2, we state some preliminary results needed for the proof of our main theorems. In Sections 3 and 4, we deal with the constant upper bound case and the vanishing upper bound case, respectively.

**2. Preliminary results.** Let $M$ be a Riemannian manifold and $\widetilde{M} = M \cup \{\Delta\}$ its one-point compactification. The path space $W(M)$ based on $M$ is the space of continuous maps $X \in C([0,\infty); \widetilde{M})$ with the following property: if $X_t = \Delta$ for some $t$, then $X_s = \Delta$ for all $s \ge t$. The lifetime $e(X)$ is defined by $e(X) = \inf\{t : X_t = \Delta\}$. The path space $W(M)$ is equipped with the standard filtration $\mathscr{B}_* = \{\mathscr{B}_t\}$ and the lifetime $e : W(M) \to \mathbb{R}_+$ is a $\mathscr{B}_*$-stopping time. We use $\mathbb{P}_x$ to denote the law of Brownian motion on $M$ starting from $x$. It is a probability measure on $W(M)$.

Now let $M$ be a Cartan–Hadamard manifold and $\widehat{M} = M \cup S_\infty(M)$ its compactification by the sphere at infinity. A Brownian motion $X$ can be decomposed into the radial process $r_t = r(X_t)$ and the angular process $\theta_t = \theta(X_t)$. The probabilistic approach to the Dirichlet problem is based on the following well-known fact.

THEOREM 2.1. *Let $M$ be a Cartan–Hadamard manifold. Suppose that, for any $x \in M$,*

$$\mathbb{P}_x\left\{X_e = \lim_{t\uparrow e} X_t \text{ exists}\right\} = 1$$

*(in the cone topology of $\widehat{M}$) and, for any $\theta_0 \in S_\infty(M)$ and any neighborhood $U$ of $\theta_0$ in $S_\infty(M)$,*

$$\lim_{x\to\theta_0} \mathbb{P}_x\{X_e \in U\} = 1.$$

*Then the Dirichlet problem at infinity for $M$ is solvable. For any $f \in C(S_\infty(M))$, the function $u_f(x) = \mathbb{E}_x f(X_e)$ is the unique solution of the Dirichlet problem with boundary function $f$.*

PROOF. Since $u_f(x) = \mathbb{E}_x u_f(X_{\tau_D})$ for any relatively compact open set $D$ containing $x$, where $\tau_D$ is the first exit time of $D$, we see that $u$ is harmonic on $M$. For any $\varepsilon > 0$ and $\theta_0 \in S_\infty(M)$, choose a neighborhood $U$ of $\theta_0$ such that $|f(\theta) - f(\theta_0)| \le \varepsilon$ for $\theta \in U$. Then

$$\begin{aligned}|u_f(x) - f(\theta_0)| &\le \mathbb{E}_x|f(X_e) - f(\theta_0)| \\ &\le \varepsilon\mathbb{P}_x\{X_e \in U\} + 2\|f\|_\infty \mathbb{P}_x\{X_e \notin U\}.\end{aligned}$$

Letting $x \to \theta_0$, we have $\limsup_{x\to\theta_0} |u_f(x) - f(\theta_0)| \le \varepsilon$. This shows that $\lim_{x\to\theta_0} u_f(x) = f(\theta_0)$, as desired.

To prove the uniqueness, let $\{D_n\}$ be an exhaustion of $M$ and $u$ a solution of the Dirichlet problem at infinity with boundary function $f$. Then $\{u_f(X_{t\wedge\tau_{D_n}}), t \ge 0\}$ is a uniformly bounded martingale under $\mathbb{P}_x$; hence, $u(x) = \mathbb{E}_x u(X_{t\wedge\tau_{D_n}})$. Letting $t \uparrow \infty$ and then $n \uparrow \infty$, we have

$$u(x) = \mathbb{E}_x u(X_e) = \mathbb{E}_x f(X_e) = u_f(x). \qquad \square$$

REMARK 2.2. Ancona [2] constructed a Cartan–Hadamard manifold such that Brownian motion converges to a single point on the boundary at infinity. For such manifolds, the Dirichlet problem at infinity is clearly not solvable.

We end this section with a description of the general method for proving angular convergence of Brownian motion. Define a sequence of stopping times $\{\tau_n\}$ by $\tau_0 = 0$ and

$$\tau_n = \inf\{t \ge \tau_{n-1} : d(X_t, X_{\tau_{n-1}}) = 1\}.$$

Let $\Delta\tau_n = \tau_n - \tau_{n-1}$ be the amount of time for the $n$th step. The angular oscillation during the time interval $[\tau_{n-1}, \tau_n]$ is

$$\Delta\theta_n = \max_{\tau_{n-1}\le t\le\tau_n} \angle\big(\theta(X_{\tau_{n-1}}), \theta(X_t)\big).$$

PROPOSITION 2.3. *Let $M$ be a Cartan–Hadamard manifold on which Brownian motion is transient, that is,*

$$\mathbb{P}_x\{r_t \to \infty \text{ as } t \uparrow e\} = 1.$$

*The Dirichlet problem at infinity is solvable if, for any positive $\varepsilon$ and $\delta$, there is an $R$ such that, for all $z \in M$ with $r(z) \geq R$,*

$$\mathbb{P}_z\left\{\sum_{n=1}^{\infty} \Delta\theta_n \leq \delta\right\} \geq 1 - \varepsilon. \tag{2.1}$$

PROOF. First, we note that $\sum_{n=1}^{\infty} \Delta\theta_n < \infty$ implies that $\lim_{t\uparrow e} X_t = X_e$ exists. Let $x \in M$ and $\varepsilon > 0$. Choose $R \geq r(x)$ such that (2.1) holds (for $\delta = 1$, say). Let $\tau_R = \inf\{t : r_t = R\}$. Then

$$\begin{aligned}\mathbb{P}_x\left\{X_e = \lim_{t\uparrow e} X_t \text{ exists}\right\} &\geq \mathbb{P}_x\left\{\sum_{n=1}^{\infty} \Delta\theta_n < \infty\right\} \\ &= \mathbb{E}_x \mathbb{P}_{X_{\tau_R}}\left\{\sum_{n=1}^{\infty} \Delta\theta_n < \infty\right\} \\ &\geq 1 - \varepsilon.\end{aligned}$$

Since $\varepsilon$ is arbitrary, this shows that $\mathbb{P}_x\{X_e = \lim_{t\uparrow e} X_t \text{ exists}\} = 1$.

Let $\theta_0 \in S_\infty(M)$ and $U$ a neighborhood of $\theta_0$ on $S_\infty(M)$ containing $\theta_0$. There is a $\delta > 0$ such that

$$\{\theta \in S_\infty(M) : \angle(\theta, \theta_0) \leq 2\delta\} \subset U.$$

We have

$$\angle(\theta_0, \theta(X_e)) \leq \angle(\theta_0, \theta(X_0)) + \sum_{n=0}^{\infty} \Delta\theta_n.$$

For any $\varepsilon > 0$, choose $R > 0$ such that (2.1) holds. Then, for all $x \in M$ such that $r(x) \geq R$ and $\angle(\theta(x), \theta_0) \leq \delta$, we have

$$\mathbb{P}_x\{X_e \in U\} \geq \mathbb{P}_x\{\angle(\theta_0, \theta(X_e)) \leq 2\delta\} \geq \mathbb{P}_x\left\{\sum_{n=0}^{\infty} \Delta\theta_n \leq \delta\right\} \geq 1 - \varepsilon.$$

This shows that

$$\lim_{x\to\theta_0} \mathbb{P}_x\{X_e \in U\} = 1.$$

By Theorem 2.1, the Dirichlet problem at infinity for $M$ is solvable. □

We use the following result to estimate the amount of time the Brownian motion spends for each step. Let

$$\tau_1 = \inf\{t > 0 : d(X_t, X_0) = 1\}.$$

PROPOSITION 2.4. *There are positive constants $C_1, C_2$ such that if the Ricci curvature on the geodesic ball $B(x; 1)$ of radius 1 centered at $x$ is bounded from below by a negative constant $-L^2 \le -1$, then*

$$\mathbb{P}_x\left\{\tau_1 \le \frac{C_1}{L}\right\} \le e^{-C_2 L}.$$

*In fact, we can take $C_1 = 1/8d$ and $C_2 = 1/2$.*

PROOF. This is Lemma 4 of [9]. We give a simpler proof here. Let $r_t = d(X_t, x)$ be the radial process. According to [11], there is a Brownian motion $\beta$ such that

$$r_t = \beta_t + \tfrac{1}{2}\int_0^t \Delta r(X_s)\,ds - L_t,$$

where $L$ is nondecreasing and increases only when $X_t$ is on the cut locus of $o$. By Itô's formula, we have

$$r_t^2 = 2\int_0^t r_s\,dr_s + \langle r\rangle_t.$$

Hence,

$$r_t^2 \le 2\int_0^t r_s\,d\beta_s + \int_0^t r_s\,\Delta r(X_s)\,ds + t. \tag{2.2}$$

By the Laplacian comparison theorem, we have, for all $z \in B(x; 1)$,

$$\Delta r(z) \le (d-1)L\coth Lr(z).$$

On the other hand, $l\coth l \le 1 + l$ for all $l \ge 0$. Hence, if $s \le \tau_1$, we have

$$r_s\,\Delta r(X_s) \le (d-1)Lr_s\coth Lr_s \le (d-1)(1+L).$$

We now let $t = \tau_1$ in (2.2) and obtain

$$1 \le 2\int_0^{\tau_1} r_s\,d\beta_s + 2dL\tau_1.$$

From the above inequality, we see that the event $\tau_1 \le 1/8dL$ implies

$$\int_0^{\tau_1} r_s\,d\beta_s \ge \tfrac{3}{8}.$$

By Lévy's criterion, there is a Brownian motion $W$ such that

$$\int_0^{\tau_1} r_s\,d\beta_s = W_\eta, \qquad \eta = \int_0^{\tau_1} r_s^2\,ds \le \frac{1}{8dL}.$$

Hence, $\tau_1 \le 1/8dL$ implies

$$\max_{0\le s\le 1/8dL} W_s \ge W_\eta \ge \tfrac{3}{8}.$$

The random variable on the left-hand side is distributed as $\sqrt{1/8dL}|W_1|$. It follows that

$$\mathbb{P}_x\left[\tau_1 \le \frac{1}{8dL}\right] \le \mathbb{P}_x\left[|W_1| \ge \sqrt{\frac{9L}{8}}\right] \le e^{-L/2}. \qquad \square$$

We will use the following geometric result to estimate the angle in a Cartan–Hadamard manifold. It is essentially Lemma 2 of [9], but we include a complete proof to clarify a few points.

LEMMA 2.5. *Let $M$ be a Cartan–Hadamard manifold. Suppose that there are positive constants $\alpha \ge 1$ and $r_0 \ge 1$ such that*

$$\mathrm{Sect}_x \le -\frac{\alpha(\alpha-1)}{r(x)^2}, \qquad r(x) \ge r_0.$$

*Let $x, y \in M$ be such that*

$$r(x) \ge 2r_0, \qquad r(y) \ge 2r_0, \qquad d(x,y) \le 1.$$

*Then there is a constant $C$ independent of $x$ and $y$ such that the angle between the geodesic rays to $x$ and $y$ satisfies*

$$\angle(\theta(x), \theta(y)) \le \frac{C}{r(x)^\alpha}.$$

PROOF. Without loss of generality, we assume $r(x) \le r(y)$. Let

$$K(r) = \min\left\{-\sup_{r(x)\le r} \mathrm{Sect}_x, \frac{\alpha(\alpha-1)}{r^2}\right\}.$$

Let $G$ be the unique solution of the Jacobi equation

$$G''(r) - K(r)G(r) = 0, \qquad G(0) = 0, \qquad G'(0) = 1.$$

Since $K(r) = \alpha(\alpha-1)/r^2$ for $r \ge r_0$, we have $G(r) = c_1 r^\alpha + c_2 r^{1-\alpha}$. Hence,

$$G(r) \sim c_1 r^\alpha, \qquad \frac{G'(r)}{G(r)} \sim \frac{\alpha}{r} \qquad \text{as } r \uparrow \infty. \tag{2.3}$$

In particular, $G(r) \ge C^{-1} r^\alpha$ for some $C$ and all $r \ge r_0$. Now let $N$ be the rotationally symmetric manifold with the metric $ds_N^2 = dr^2 + G(r)^2 d\theta^2$. In $N$, consider the geodesic triangle $AOB$ such that

$$d(O,A) = r(x), \qquad d(O,B) = r(y), \qquad \angle(\theta(A), \theta(B)) = \angle(\theta(x), \theta(y)).$$

By the Rauch comparison theorem, we have $d_N(A,B) \le d(x,y)$. Hence,

$$1 \ge d_N(A,B) \ge G(r(x))\angle(\theta(A), \theta(B)) = G(r(x))\angle(\theta(x), \theta(y)).$$

This implies that $\angle(\theta(x), \theta(y)) \le C/r(x)^\alpha$. $\square$

When the sectional curvature is bounded from above by a negative constant, we have the following analogue of the above lemma.

LEMMA 2.6. *Let $M$ be a Cartan–Hadamard manifold. Suppose that there is a positive constant $a$ such that* $\mathrm{Sect}_x \le -a^2$. *Let $x, y \in M$ be such that $r(x) \le r(y)$ and $d(x,y) \le 1$. Then*

$$\angle(\theta(x), \theta(y)) \le \frac{a}{\sinh ar(x)} \le \left[\frac{1}{r(x)} + 2a\right] e^{-ar(x)}.$$

PROOF. Let $G(r) = \sinh ar/a$ and follow the proof of the preceding lemma. □

**3. Constant upper bound.** In this section, we consider the case of a constant upper bound on the sectional curvature of $M$. We first give an estimate on the probability that Brownian motion starting at $r(x) = R$ will ever return to $r = R \le r(x)$.

LEMMA 3.1. *Suppose that* $\mathrm{Sect}_x \le -a^2$. *For any $R \ge 0$, we have, for $r(x) \ge R$,*

$$\mathbb{P}_x\{r_t \le R \textit{ for some } t \ge 0\} \le \cosh^{1-d} a(r - R). \tag{3.1}$$

PROOF. There is a Brownian motion $\beta$ such that

$$r_t = r_0 + \beta_t + \tfrac{1}{2}\int_0^t \Delta r(X_t)\, dt.$$

By the Laplacian comparison theorem, we have $\Delta r \ge (d-1)a \coth ar$. If we define $r^*$ by

$$r_t^* = r_0 + \beta_t + \frac{d-1}{2}\int_0^t a \coth ar_s^*\, ds,$$

then a comparison theorem for stochastic differential equations shows that $r_t \ge r_t^*$. Thus, it is enough to prove the estimate for $r^*$.

The following argument is well known. Let

$$l(r) = \int_r^\infty (\sinh au)^{1-d}\, du$$

and $\sigma_R = \inf\{t : r_t^* = R\}$. If $r(x) \ge R$, then $\{l(r_{t\wedge\sigma_R}^*)\}$ is a uniformly bounded martingale. Letting $t \uparrow \infty$, we have

$$l(r) = \mathbb{E}_x l(r_{t\wedge\sigma_R}^*) = l(R)\mathbb{P}_x\{\sigma_R < \infty\}.$$

Hence,

$$\mathbb{P}_x\{r_t^* \le R \text{ for some } t \ge 0\} = \mathbb{P}_x\{\sigma_R < \infty\} = \frac{l(r)}{l(R)}.$$

On the other hand,

$$\begin{aligned}\frac{l(r(x))}{l(R)} &= \frac{\int_r^\infty (\sinh au)^{1-d}\,du}{\int_R^\infty (\sinh au)^{d-1}\,du}\\ &\le \sup_{u\ge R}\left[\frac{\sinh a(u+r-R)}{\sinh au}\right]^{1-d}\\ &\le \cosh^{1-d} a(r-R).\end{aligned}$$

In the last step, we have used

$$\frac{\sinh(x+y)}{\sinh x} = \frac{\sinh x\cosh y + \cosh x\sinh y}{\sinh x} \ge \cosh y.$$

The result follows. □

Next, we consider the rate of escape for Brownian motion.

LEMMA 3.2. *Suppose that* $\mathrm{Sect}_x \le -a^2$. *For any* $\lambda < (d-1)a/2$, *we have*

$$\lim_{r(x)\to\infty} \mathbb{P}_x\big\{r_t \ge \max\{\lambda t, r(x)/2\},\ \forall t\ge 0\big\} = 1.$$

PROOF. Again, it is enough to show the result for the $r_t^*$ in the proof of the preceding lemma. Fix a $\lambda_1 \in (\lambda, (d-1)a/2)$ and take $R$ such that

$$[(d-1)a/2]\coth ar \ge \lambda_1, \qquad r\ge R/2.$$

Suppose that $\varepsilon > 0$. By Lemma 3.1, we can take $R$ even larger such that, for all $x\in M$ with $r(x)\ge R$,

$$\mathbb{P}_x\{r_t^* \ge r(x)/2,\ \forall t\ge 0\} \ge 1-\varepsilon. \tag{3.2}$$

By the law of iterated logarithm,

$$\liminf_{t\uparrow\infty} \frac{\beta_t}{\sqrt{2t\log\log t}} = -1.$$

Hence, there is an even larger $R$ (independent of $x$) such that

$$\mathbb{P}_x\{\beta_t \ge -(\lambda-\lambda_1)t - R,\ \forall t\ge 0\} \ge 1-\varepsilon. \tag{3.3}$$

If the events in (3.2) and (3.3) happen simultaneously, then

$$\begin{aligned} r_t^* &= r_0^* + \beta_t + \frac{d-1}{2}\int_0^t a\coth ar_s^*\,ds\\ &\ge R - (\lambda_1-\lambda)t - R + \lambda_1 t\\ &= \lambda t.\end{aligned}$$

It follows that for all $x \in M$ with $r(x) \geq R$ we have

$$\mathbb{P}_x\{r_t^* \geq \max\{\lambda_1 t, r(x)/2\},\ \forall t \geq 0\} \geq 1 - 2\varepsilon.$$

This proves the lemma. □

We now estimate the total angular variation. Suppose that $r_t \geq r(x)/2$ for all $t \geq 0$ with large $r(x)$. Recall that in Section 2 we have defined

$$\begin{aligned} \tau_n &= \inf\{t \geq \tau_{n-1} : d(X_t, X_{\tau_{n-1}}) = 1\}, \qquad \tau_0 = 0, \\ \Delta\tau_n &= \tau_n - \tau_{n-1}, \\ \Delta\theta_n &= \max_{\tau_{n-1} \leq t \leq \tau_n} \angle(\theta(X_{\tau_{n-1}}), \theta(X_t)). \end{aligned}$$

From Lemma 2.6, we have $\Delta\theta_n \leq Ce^{-ar_{\tau_n}}$. Hence,

$$\sum_{n=1}^{\infty} \Delta\theta_n \leq C \sum_{n=1}^{\infty} e^{-ar_{\tau_n}}.$$

Next, let $J_k$ be the total number of steps in the geodesic ball of radius $k$, that is,

$$J_k = \#\{n : r_{\tau_n} \leq k\}.$$

We have

$$\sum_{n=1}^{\infty} \Delta\theta_n \leq C \sum_{k=1}^{\infty} (J_k - J_{k-1}) e^{-a(k-1)} \leq C_0 \sum_{k=1}^{\infty} J_k e^{-ak}. \tag{3.4}$$

Thus, the problem is reduced to finding a good estimate for $J_k$.

REMARK 3.3. The idea of studying $J_k$ is due to Leclercq [14].

THEOREM 3.4. *Let $M$ be a Cartan–Hadamard manifold whose sectional curvature is bounded from above by $-a^2$. Suppose that the Ricci curvature satisfies the lower bound*

$$\mathrm{Ric}_x \geq -h(r)^2 e^{2ar},$$

*where $h$ is a positive and nonincreasing function such that $\int_0^\infty rh(r)\,dr < \infty$. Then the Dirichlet problem at infinity for $M$ is solvable.*

PROOF. Fix a constant $\lambda < (d-1)a/2$ and let

$$A = \{r_t \geq \max\{\lambda t, r(x)/2\},\ \forall t \geq 0\}.$$

By Lemma 3.2, there is an $R$ such that, for $r(x) \geq R$,

$$\mathbb{P}_x\{A\} \geq 1 - \frac{\varepsilon}{2}.$$

Let $\tau_{n_l}$ be the $l$th time such that $r_{\tau_{n_l}} \le k-1$. Then

$$\{\tau_{n_l} \le t\} = \left\{ \sum_{n=1}^{\infty} I_{\{r_{\tau_n} \le k-1, \tau_n \le t\}} \ge l \right\},$$

from which it is clear that $\tau_{n_l}$ is a stopping time.

For a fixed $k$, denote for the time being

$$L_k = C_1 h(k) e^{ak}, \qquad N_k = \frac{(k+1)L_k}{\lambda C_1}.$$

Without loss of generality, we may assume that $h(k) \ge e^{-ak/2}$ [otherwise, just add $e^{-ar/2}$ to $h(r)$] and $L_k \ge 1$. Consider the length of time $\Delta\tau_{n_l}$ for the next step. Let

$$B_l = \left\{ \Delta\tau_{n_l} \le \frac{C_1}{L_k}, \tau_{n_l} < \infty \right\}, \qquad C_{N_k} = B_1 \cup B_2 \cup \cdots \cup B_{N_k}.$$

By Proposition 2.4 and the fact that $\tau_{n_l}$ is a stopping time,

$$\mathbb{P}_x B_l = \mathbb{E}_x \left\{ \mathbb{P}_{X_{\tau_{n_l}}} \left[ \tau_1 \le \frac{C_1}{L_k} \right], \tau_{n_l} < \infty \right\} \le e^{-C_2 L_k}. \tag{3.5}$$

Recall that $J_{k-1}$ is the total number of steps such that $r_{\tau_n} \le k-1$. We have $\{J_{k-1} \ge N_k\} = \{\tau_{n_{N_k}} < \infty\}$. Now

$$\{J_{k-1} \ge N_k\} \cap A = \left\{\tau_{n_{N_k}} < \infty\right\} \cap A \cap C_{N_k} + \left\{\tau_{n_{N_k}} < \infty\right\} \cap A \cap C_{N_k}^c. \tag{3.6}$$

On $A$, we have $r_t \ge \lambda t$ for all $t \ge 0$. This means that

$$|\{t : r_t \le k\}| \le \frac{k}{\lambda}.$$

But on $\{\tau_{n_{N_k}} < \infty\} \cap C_{N_k}^c$,

$$|\{t : r_t \le k\}| \ge \sum_{l=1}^{N_k} \Delta\tau_{n_l} \ge N_k \frac{C_1}{L_k} = \frac{k+1}{\lambda}.$$

This shows that $\{\tau_{n_{N_k}} < \infty\} \cap A \cap C_{N_k}^c = \varnothing$ and we have, from (3.6),

$$\{J_{k-1} \ge N_k\} \cap A \subseteq C_{N_k} = B_1 \cup B_2 \cup \cdots \cup B_{N_k}.$$

By (3.5),

$$\mathbb{P}_x\{J_{k-1} \ge N_k, A\} \le N_k e^{-C_2 L_k} \le C_3 k e^{ak - C_2 e^{ak/2}}.$$

Using the definition of $L_k$, we see from the above inequality that, for any $\varepsilon > 0$, there is a sufficiently large $R$ such that, for $r(x) \ge R$,

$$\sum_{k \ge r(x)/2}^{\infty} \mathbb{P}_x\{J_k \ge C_4 k h(k) e^{ak}, A\} \le \frac{\varepsilon}{2}.$$

On $A$, we have $r_t \geq r(x)/2$ for all $t$. This means that $J_k = 0$ for $k \leq r(x)/2$. It follows that, for $r(x) \geq R$,

$$\mathbb{P}_x\left\{J_k = 0, k \leq \frac{r(x)}{2}; J_k \leq C_4 k h(k) e^{ak}, k \geq \frac{r(x)}{2}\right\} \geq \mathbb{P}_x A - \frac{\varepsilon}{2} \geq 1 - \varepsilon.$$

If the event in the above inequality holds, then, by (3.4),

$$\sum_{n=1}^{\infty} \Delta\theta_n \leq C_4 \sum_{k \geq r(x)/2} k h(k).$$

This can be made arbitrarily small because the $\sum_{k=1}^{\infty} k h(k)$ converges by hypothesis. Therefore, we have shown that for any positive $\varepsilon$ and $\delta$, there is an $R$ such that, for all $x \in M$ with $r(x) \geq R$,

$$\mathbb{P}_x\left\{\sum_{n=1}^{\infty} \Delta\theta_n \leq \delta\right\} \geq 1 - \varepsilon.$$

By Proposition 2.3, this implies the solvability of the Dirichlet problem at infinity for $M$. □

**4. Vanishing upper bound.** In this section, we assume that $M$ is a Cartan–Hadamard manifold whose curvature satisfies the following condition: there are positive constant $r_0$, $\alpha > 2$ and $\beta < \alpha - 2$ such that, for all $r(x) \geq r_0$,

$$-r(x)^{2\beta} \leq \mathrm{Ric}_x \quad \text{and} \quad \mathrm{Sect}_x \leq -\frac{\alpha(\alpha - 1)}{r(x)^2}.$$

The proof for this case is completely parallel to that in the previous section, so we will be brief.

LEMMA 4.1. *There is a constant $C$ such that, for all $R \geq 1$ and $x \in M$ with $r(x) \geq R$,*

$$\mathbb{P}_x\{r_t \leq R \textit{ for some } t \geq 0\} \leq C\left[\frac{R}{r(x)}\right]^{(d-1)\alpha - 1}.$$

PROOF. Define the function $G$ as in the proof of Lemma 2.5. As before, we may assume that $M$ is rotationally symmetric with metric $ds^2 = dr^2 + G(r)^2\, d\theta^2$. In this case, by the same argument as in Lemma 3.1, we have

$$\mathbb{P}_x\{r_t \leq R \text{ for some } t \geq 0\} = \frac{\int_{r(x)}^{\infty} G(s)^{1-d}\, ds}{\int_R^{\infty} G(s)^{1-d}\, ds}.$$

The result follows immediately from the fact that $G(r) \sim c_1 r^{\alpha}$ as $r \uparrow \infty$. □

In the proof of the next lemma, we need the following fact (see [17]): let $Y^a$ be the Bessel process of index $q > 1$ from $a \geq 0$:

$$Y_t^a = a + \beta_t + \frac{q}{2}\int_0^t \frac{ds}{Y_s^a}, \tag{4.1}$$

where $\beta$ is a one-dimensional Brownian motion. Then for any $\lambda > 0$ we have

$$\mathbb{P}\left\{\lim_{t\uparrow\infty} \frac{Y_t^a}{t^{1/2-\lambda}} = \infty\right\} = 1. \tag{4.2}$$

Note that $Y_t^a \leq Y_t^b$ if $a \leq b$.

LEMMA 4.2. *For any* $\lambda > 0$, *we have*

$$\lim_{r(x)\to\infty} \mathbb{P}_x\{r_t \geq \max\{t, r(x)\}^{1/2-\lambda},\ \forall t \geq 0\} = 1.$$

PROOF. Again, it is enough to assume that $M$ is rotationally symmetric, as in Lemma 4.1. The radial process is given by

$$r_t = r_0 + \beta_t + \frac{d-1}{2}\int_0^t \frac{G'(r_s)}{G(r_s)}\,ds.$$

Now take a $q \in (1, (d-1)\alpha)$. By (2.3), there is an $r_1 \geq 1$ such that

$$(d-1)\frac{G'(r)}{G(r)} \geq \frac{q}{r}, \qquad r \geq r_1.$$

Let $Y^a$ be the Bessel process of index $q$ defined by (4.1). If $r(x) \geq r_1$, then we have

$$r_t \geq Y_t^{r(x)} \geq Y_t^{r_1} \geq Y_t^1, \qquad t \leq \sigma_{r_1},$$

where $\sigma_{r_1}$ is the first time $r_t$ reaches $r_1$. For any $\varepsilon > 0$, there is an $R \geq r_1$ (independent of $x$) such that

$$\mathbb{P}_x\{Y_t^1 \geq t^{1/2-\lambda},\ \forall t \geq R\} \geq 1 - \varepsilon.$$

Hence, using Lemma 4.1, we have, for $r(x) \geq R \geq 1$,

$$\begin{aligned}
&\mathbb{P}_x\{r_t \geq \max\{t, r(x)\}^{1/2-\lambda},\ \forall t \geq 0\}\\
&\quad\geq \mathbb{P}_x\{r_t \geq t^{1/2-\lambda},\ \forall t \geq r(x)\} - \mathbb{P}_x\{r_t \leq r(x)^{1/2-\lambda} \text{ for some } t \geq 0\}\\
&\quad\geq \mathbb{P}_x\{Y_t^1 \geq t^{1/2-\lambda},\ \forall t \geq R\} - Cr(x)^{-(\lambda+1/2)[(d-1)\alpha-1]}\\
&\quad\geq 1 - \varepsilon - Cr(x)^{-(\lambda+1/2)[(d-1)\alpha-1]}.
\end{aligned}$$

It follows that for all sufficiently large $r(x)$ we have

$$\mathbb{P}_x\{r_t \geq \max\{t, r(x)\}^{1/2-\lambda},\ \forall t \geq 0\} \geq 1 - 2\varepsilon. \qquad \square$$

THEOREM 4.3. *Suppose that $M$ is a Cartan–Hadamard manifold. Suppose that there exist positive constants $r_0$, $\alpha > 2$ and $\beta < \alpha - 2$ such that*

$$-r(x)^{2\beta} \le \mathrm{Ric}_x \quad \textit{and} \quad \mathrm{Sect}_x \le -\frac{\alpha(\alpha-1)}{r(x)^2} \qquad \textit{for } r \ge r_0.$$

*Then the Dirichlet problem at infinity is solvable for $M$.*

PROOF. We define $\tau_n$, $\Delta\tau_n$, $\Delta\theta_n$, $\tau_{n_l}$ and $J_k$ as in the previous section. Under the current upper bound of the sectional curvature, we have $\Delta\theta_n \le C/r_{\tau_n}^{\alpha}$ by Lemma 2.5. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \Delta\theta_n &\le C_0 J_1 + C_0 \sum_{k=1}^{\infty} \frac{J_{k+1} - J_k}{k^{\alpha}} \\ &\le C_0 J_1 + C_1 \sum_{k=1}^{\infty} \frac{J_k}{k^{\alpha+1}} + C_0 \liminf_{k\uparrow\infty} \frac{J_k}{k^{\alpha}}.\end{aligned} \tag{4.3}$$

We will now estimate the size of $J_k$. By Proposition 2.4, we have

$$\mathbb{P}_x\{\Delta\tau_{n_l} \le C_1 k^{-\beta}, \tau_{n_l} < \infty\} \le e^{-C_1 k^{\beta}}.$$

Choose a positive $\lambda$ such that $\beta + 2/(1-2\lambda) < \alpha$. Let

$$A = \{r_t \ge \max\{t, r(x)\}^{1/2-\lambda}, \ \forall t \ge 0\}.$$

Fix an arbitrary $\varepsilon > 0$. By Lemma 4.2, $\mathbb{P}_x A \ge 1 - \varepsilon/2$ for sufficiently large $r(x)$. By the same argument as in Theorem 3.4, we have

$$\mathbb{P}_x\{J_k \ge (C_1+1)k^{\beta+2/(1-2\lambda)}, A\} \le C_3 k^{\beta+2/(1-2\lambda)} e^{-C_2 k^{\beta}}.$$

On $A$, we have $|\{t : r_t \le k\}| \le k^{2/(1-2\lambda)}$ and $J_k = 0$ for $k \le r(x)^{1/2-\lambda}$. Hence, as in the proof of Theorem 3.4, we have, for sufficiently large $r(x)$,

$$\begin{aligned}&\mathbb{P}_x\{J_k = 0, k \le r(x)^{1/2-\lambda}; J_k \le C_4 k^{\beta+2/(1-2\lambda)}, k \ge r(x)^{1/2-\lambda}\} \\ &\quad \ge \mathbb{P}_x A - C_3 \sum_{k \ge r(x)^{1/2-\lambda}} k^{\beta+2/(1-2\lambda)} e^{-C_2 k^{\beta}} \\ &\quad \ge 1 - \varepsilon.\end{aligned}$$

If the event in the above inequality is true, then $J_k/k^{\alpha} \to 0$ as $k \uparrow \infty$ and, by (4.3),

$$\begin{aligned}\sum_{n=1}^{\infty} \Delta\theta_n &\le C_4 \sum_{k \ge r(x)^{1/2-\lambda}} k^{-(\alpha+1)+\beta+2/(1-2\lambda)} \\ &\le C_5 r(x)^{-(\alpha-\beta)(1-2\lambda)/2+1}.\end{aligned}$$

By our choice of $\lambda$, the exponent is negative. Hence, we have shown that for any positive $\varepsilon$ and $\delta$, there is an $R$ such that, for $r(x) \geq R$,

$$\mathbb{P}_x\left\{\sum_{n=1}^{\infty} \Delta\theta_n \leq \delta\right\} \geq 1 - \varepsilon.$$

The theorem now follows from Proposition 2.3. □

REMARK 4.4. For the Bessel process $Y^a$ in (4.1), we have

$$\mathbb{P}\left\{\liminf_{t\to\infty} \frac{Y_t^a}{\sqrt{t}\psi(t)} \geq 1\right\} = 1$$

if $\psi$ is a positive nonincreasing function such that $\int_0^\infty \psi(t)^{q-1}\,dt < \infty$. Using this rate instead of $t^{1/2-\lambda}$ in (4.2), we can improve the lower bound in the above theorem. For example, it can be shown that the Dirichlet problem is solvable if the Ricci curvature is bounded from below by $-r^{2(\alpha-2)}/(\ln r)^{2l}$ for $l > (d\alpha - \alpha + 1)/(d\alpha - \alpha - 1)$.

**Acknowledgment.** The author gratefully acknowledges the anonymous referee's careful reading of the paper.

## REFERENCES

[1] ANCONA, A. (1987). Negatively curved manifolds, elliptic operators and the Martin boundary. *Ann. of Math.* **125** 495–536.

[2] ANCONA, A. (1994). Convexity at infinity and Brownian motion on manifolds with unbounded negative curvature. *Rev. Mat. Iberoamericana* **10** 189–220.

[3] ANDERSON, M. T. (1983). The Dirichlet problem at infinity for manifolds of negative curvature. *J. Differential Geom.* **18** 701–711.

[4] ANDERSON, M. T. and SCHOEN, R. (1985). Positive harmonic functions on complete manifolds of negative curvature. *Ann. of Math. (2)* **121** 429–461.

[5] BISHOP, R. I. and O'NEILL, B. (1969). Manifolds of negative curvature. *Trans. Amer. Math. Soc.* **145** 1–49.

[6] BORBELY, A. (1993). A note on the Dirichlet problem at infinity for manifolds of negative curvature. *Proc. Amer. Math. Soc.* **118** 205–210.

[7] CHOI, H. I. (1984). Asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds. *Trans. Amer. Math. Soc.* **281** 691–716.

[8] HSU, P. and KENDALL, W. S. (1992). Limiting angles of certain two-dimensional Riemannian Brownian motion. *Ann. Fac. Sci. Toulouse Math. (6)* **1** 169–186.

[9] HSU, P. and MARCH, P. (1985). The limiting angle of certain Riemannian Brownian motion. *Comm. Pure Appl. Math.* **38** 755–768.

[10] KENDALL, W. S. (1984). Brownian motion on a surface of negative curvature. *Seminar on Probability XVIII. Lecture Notes in Math.* **1059**.

[11] KENDALL, W. S. (1987). The radial part of Brownian motion on a manfiold: A semimartingale property. *Ann. Probab.* **15** 1491–1500.

[12] KIFER, YU. (1976). Brownian motion and harmonic functions on manifolds of negative curvature. *Theory Probab. Appl.* **21** 755–768.

[13] KIFER, YU. (1985). Brownian motion and positive harmonic functions on complete manifolds of nonpositive curvature. In *From Local Times to Global Geometry, Control and Physics* (K. D. Elworthy, ed.) 187–232. Wiley, New York.

[14] LECLERCQ, É. (1997). The asymptotic Dirichlet problem with respect to an elliptic operator on a Cartan–Hadamard manifold with unbounded curvatures. *C. R. Acad. Sci. Sér. I. Math.* **325** 857–862.

[15] PINSKY, M. A. (1978). Stochastic Riemannian geometry. In *Probabilistic Analysis and Related Topics* (A. T. Bharucha-Reid, ed.) **1** 199–236. Academic, New York.

[16] PRAT, J. J. (1975). Etude asymptotique et convergence angulaire du mouvement brownien sur un variété à courbure negative. *C. R. Acad. Sci. Paris* **290** 1539–1542.

[17] SHIGA, T. and WATANABE, S. (1973). Bessel diffusions as a one-parameter family of diffusion processes. *Z. Wahrsch. Verw. Gebiete* **27** 37–46.

[18] SULLIVAN, D. (1983). The Dirichlet problem at infinity for a negatively curved manifold. *J. Differential Geom.* **18** 723–732.

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