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Logarithmic Sobolev Inequalities on Path Spaces Over Riemannian Manifolds*

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Abstract: Let $W_o(M)$ be the space of paths of unit time length on a connected, complete Riemannian manifold M such that $\gamma(0) = o$, a fixed point on M, and ν the Wiener measure on $W_o(M)$ (the law of Brownian motion on M starting at o). If the Ricci curvature is bounded by c, then the following logarithmic Sobolev inequality holds:

$$\int_{W_o(M)} F^2 \log |F| d\nu \le e^{3c} \|DF\|^2 + \|F\|^2 \log \|F\|.$$

1. Introduction

Logarithmic inequalities were introduced in Gross [5] as a tool for studying hypercontractivity of symmetric Markov semigroups. Let (X, ν) be a probability space and $\mathcal{E}$ a densely defined nonnegative quadratic form on $L^2(X, \nu)$. We say that the logarithmic Sobolev inequality holds for $\mathcal{E}$ if

$$\int_X F^2 \log |F| d\nu \le \mathcal{E}(F, F) + \|F\|^2 \log \|F\|, \qquad \forall F \in \mathrm{Dom}(\mathcal{E}).$$

Gross [5] proved that it holds for the standard gaussian measure on $\mathbb{R}^N$ for any N with

$$\mathcal{E}(F, F) = \int_{\mathbb{R}^N} |\nabla F|^2 d\nu.$$

This implies immediately by a simple argument that the logarithmic Sobolev inequality holds for the quadratic form

$$\mathcal{E}(F, F) = \int_{W_o(\mathbb{R}^N)} |DF|^2 d\nu,$$

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on the probability space $(W_o(\mathbb{R}^N), \nu)$, where $W_o(\mathbb{R}^N)$ is the path space on $\mathbb{R}^N$ and ν the Wiener measure, and D is the gradient operator on $W_o(\mathbb{R}^n)$ (see the definition below). It was proved in Gross [6] that the logarithmic Sobolev inequality holds for the Wiener measure on $W_o(G)$, where G is a connected Lie group with the gradient operator derived from the Cartan ($\pm$)-connection. Note that for these connections the curvature vanishes. It has been conjectured by the author that a logarithmic Sobolev inequality on the path space over a general complete, connected Riemannian manifold holds with a bounding constant which can be estimated in terms of the Ricci curvature. This conjecture is completely borne out by the main result of the present work.

Let M be a complete, connected Riemannian manifold of dimension n. Throughout this work we assume that M has bounded Ricci curvature. We write $|\mathrm{Ric}_M| \le c$ if

$$\sup \{|\mathrm{Ric}(v,v)| : v \in T_x M, |v| = 1, x \in M\} \le c.$$

Fix a point $o \in M$ and let

$$W_o(M) = \{\gamma \in C([0,1], M) : \gamma(0) = o\},$$

be the space of pinned paths from o. We will work with the Wiener measure ν on $W_o(M)$, which can be defined as follows. Let $O(M)$ be the bundle of orthonormal frames over M and $\pi : O(M) \to M$ the canonical projection. Fix an orthonormal frame u_o at o and let $\{U_s\}$ be the solution of the following stochastic differential equation on $O(M)$:

$$dU_s = H_{U_s} \circ d\omega_s, \quad U_0 = u_o, \tag{1.1}$$

where $\{\omega_s\}$ is an $\mathbb{R}^n$-valued Brownian motion. Here $H = \{H_i, 1 \le i \le n\}$ are the canonical horizontal vector fields on $O(M)$. The projected process $\gamma_s = \pi U_s$ is a Brownian motion on M starting from o. The Wiener measure ν is just the law of the Brownian motion $\{\gamma_s\}$.

We now introduce the gradient operator D on the path space. Let $\mathbb{H}$ be the $\mathbb{R}^n$-valued Cameron-Martin space, i.e., the space of $\mathbb{R}^n$-valued functions h such that $h_0 = 0$ and $\dot{h} \in L^2([0,1]; \mathbb{R}^n)$. It is a Hilbert space with the norm

$$|h|^2_{\mathbb{H}} = \int_0^1 |\dot{h}_s|^2_{\mathbb{R}^n} ds.$$

For each $h \in \mathbb{H}$, let D_h be the vector field on the path space $W_o(M)$ defined by $D_h(\gamma)_s = U(\gamma)_s h_s$, where $U(\gamma)$ is the horizontal lift of γ with initial value u_o. Let $\left\{\zeta_h^t, t \in \mathbb{R}^1\right\}$ be the flow on the path space $W_o(M)$ generated by the vector field D_h. Let F be a real-valued function on $W_o(M)$. We define

$$D_h F(\gamma) = \lim_{t \to 0} \frac{F(\zeta_h^t \gamma) - F(\gamma)}{t},$$

if the limit exists in $L^2(\nu)$. The gradient DF is defined to be the $\mathbb{H}$-valued function DF such that $\langle DF, h\rangle_{\mathbb{H}} = D_h F$ for all $h \in \mathbb{H}$. Suppose that F is a cylindrical function given by

$$F(\gamma) = f(\gamma_{s_1}, \cdots, \gamma_{s_l}), \tag{1.2}$$

where $f : M \times \cdots \times M \to \mathbb{R}^1$ is smooth and $0 \le s_1 < \cdots < s_l \le 1$. Then it is easy to verify that

$$DF(\gamma) = \sum_{i=1}^{l} (s \wedge s_i) U(\gamma)_{s_i}^{-1} \nabla^{(i)} F(\gamma), \tag{1.3}$$

where $\nabla^{(i)}F$ denotes the gradient of f with respect to the i^{th} variable.

From definition we have for F in (1.2)

$$|DF|^2_{\mathbb{H}} = \sum_{i=1}^{l}(s_i - s_{i-1})\left|\sum_{j=i}^{l} U_{s_i}^{-1}\nabla^{(i)}F\right|^2. \tag{1.4}$$

This formula will be useful later.

We state the main result of the present work.

Theorem 1.1. *Suppose that M is a complete, connected manifold such that $|\mathrm{Ric}_M| \le c$. Then the following logarithmic Sobolev inequality on the path space $W_o(M)$:*

$$\int_{W_o(M)} F^2 \log|F| d\nu \le e^{3c}\|DF\|^2 + \|F\|^2 \log\|F\|, \qquad \forall F \in \mathrm{Dom}(D).$$

We conclude this introduction by stating a few well known consequences of the logarithmic Sobolev inequality (see Gross [7]). The self-adjoint operator $L = -D^*D$ is a generalization of the usual Ornstein-Uhlenbeck operator for a euclidean path space. Let $Q_t = e^{Lt/2}$ be the associated Markovian semigroup.

Theorem 1.2. *Let M be a complete, connected Riemannian manifold such that $|\mathrm{Ric}_M| \le c$. Let $\lambda_M = e^{-3c}$.*
(i) The semigroup $\{Q_t\}$ on the path space $W_o(M)$ is hypercontractive:

$$\|Q_t\|_{L^p(\nu)\to L^q(\nu)} \le 1 \quad \text{if} \quad e^{\lambda_M t} \ge \frac{q-1}{p-1}.$$

(ii) The spectral gap of L exists and is at least λ_M, namely the following Poincaré inequality holds: if $F \in \mathrm{Dom}(D)$, then

$$\|F - EF\|^2 \le \lambda_M^{-1}\|DF\|^2.$$

(iii) If $F \in L^2(\nu)$, then

$$\|Q_tF - EF\| \le e^{-\lambda_M t/2}\|F\|.$$

Remark 1.3. Using a Clark-Haussman-Ocone formula for path spaces, Fang [4] proved directly the existence of a spectral gap for the Ornstein-Uhlenbeck operator L on the path space over a connected, compact Riemannian manifold.

2. Gradient of a Wiener Functional

The key to the present proof to the logarithmic Sobolev inequality is a formula for $\nabla E_x F$, the gradient of the expected value of a cylindrical function F.

Define the matrix-valued process $\{\phi_s\}$ by

$$\phi_s = I - \frac{1}{2}\int_0^s \phi_\tau \mathrm{Ric}_{U_\tau} d\tau, \tag{2.1}$$

where $\mathrm{Ric}_u : \mathbb{R}^n \to \mathbb{R}^n$ denotes the Ricci curvature transform read at the frame u and I is the identity matrix.

Proposition 2.1. *Let F be a cylindrical function given by (1.2). Then*

$$\nabla E_x F = U_x E_x \left\{ \sum_{i=1}^{l} \phi_{s_i} U_{s_i}^{-1} \nabla^{(i)} F \right\}. \tag{2.2}$$

Proof. The case $l = 1$ is due to Bismut (see Bismut [2], p.82). We give a proof here based solely on Itô's formula for the horizontal Brownian motion $\{U_s\}$, see (1.1). Let f be a smooth function on M and consider the function $J(\tau, u) = E_{\pi u} f(\gamma_\tau)$. It satisfies the equation

$$\partial_\tau J(s-\tau, u) + \frac{1}{2}\Delta^H J(s-\tau, u) = 0, \tag{2.3}$$

where $\Delta^H = \sum_{i=1}^n H_i^2$ is Bochner's horizontal Laplacian on $O(M)$. This implies that $\{J(s-\tau, U_\tau), 0 \le \tau \le s\}$ is a martingale. We now apply Itô's formula to the horizontal gradient $\nabla^H J(s-\tau, U_\tau)$, using the fact that $\{U_\tau\}$ is a diffusion generated by Δ^H:

$$\begin{aligned}
& d_\tau \nabla^H J(s-\tau, U_\tau) \\
= & \partial_\tau \nabla^H J(s-\tau, U_\tau) d\tau + \langle \nabla^H \nabla^H J(s-\tau, U_\tau), d\omega_\tau \rangle \\
& + \frac{1}{2} \Delta^H \nabla^H J(s-\tau, U_\tau) d\tau \\
= & \langle \nabla^H \nabla^H J(s-\tau, U_\tau), d\omega_\tau \rangle + \frac{1}{2} \left[\Delta^H, \nabla^H \right] J(s-\tau, \gamma_\tau) d\tau \\
& + \nabla^H \left\{ \partial_\tau J(s-\tau, U_\tau) + \frac{1}{2} \Delta^H J(s-\tau, U_\tau) \right\} d\tau \\
= & \langle \nabla^H \nabla^H J(s-\tau, U_\tau), d\omega_\tau \rangle + \frac{1}{2} \mathrm{Ric}_{U_\tau} \nabla^H J(s-\tau, U_\tau) d\tau.
\end{aligned}$$

Here we have used (2.3) and Bochner's formula

$$\left[\Delta^H, \nabla^H\right] J(u) = \mathrm{Ric}_u \nabla^H J(u)$$

for the lift J of a function on M. It follows that

$$\left\{ \phi_\tau \nabla^H J(s-\tau, U_\tau), 0 \le \tau \le s \right\}$$

is a martingale. Equating the expected values at $\tau = 0$ and $\tau = s$ we obtain

$$\nabla^H J(s, u_o) = E\left\{ \phi_s \nabla^H J(0, U_s) \right\},$$

This is equivalent to what we wanted because by definition $\nabla^H J(\tau, u) = u^{-1} \nabla E_{\pi u} f(\gamma_\tau)$.

By the Markov property and the case $l = 1$ we have

$$\nabla E_x F = \nabla E_x g(\gamma_{s_1}) = U_x E_x \left\{ \phi_{s_1} U_{s_1}^{-1} \nabla g(\gamma_{s_1}) \right\}, \tag{2.4}$$

where

$$g(y) = E_y \left\{ f(y, \gamma_{s_2 - s_1}, \cdots, \gamma_{s_l - s_1}) \right\}.$$

Using the induction hypothesis we have

$$\begin{aligned}
\nabla g(y) = & E_y \left\{ \nabla^{(1)} f(y, \gamma_{s_2-s_1}, \cdots, \gamma_{s_l - s_1}) \right\} \\
& + U_y \sum_{i=2}^{l} E_y \left\{ \phi_{s_i - s_1} U_{s_i - s_1}^{-1} \nabla^{(i)} f(y, \gamma_{s_2 - s_1}, \cdots, \gamma_{s_l - s_1}) \right\},
\end{aligned}$$

where U_y is any orthonormal frame at y and $\{U_s\}$ starts at U_y. Substituting this into (2.4) and using the Marko property again we obtain the desired equality. □

The following corollary to Proposition 2.1 is known and appeared as a special case of Theorem 6.4(iii) in Donnelly and Li [3].

Corollary 2.2. *Suppose that* $\mathrm{Ric}_M \geq -c$ *for a nonnegative constant* c*. Then*

$$|\nabla E_x f(\gamma_s)| \leq e^{cs/2} E_x |\nabla f(\gamma_s)|.$$

Proof. This follows from the preceding proposition (with $l = 1$) and the inequality $|\phi_s| \leq e^{cs/2}$, which is a consequence of the assumption on the Ricci curvature. □

3. A Finite Dimensional Case

Let μ be the gaussian measure on $\mathbb{R}^n$ given by

$$\mu_s(dx) = \left(\frac{1}{2\pi s}\right)^{n/2} e^{-|x|^2/2s} dx.$$

Gross [6] proved the following logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^n} f^2 \log |f| d\mu_s \leq s \int_{\mathbb{R}^n} |\nabla f|^2 d\mu_s + \|f\|^2_{\mu_s} \log \|f\|_{\mu_s}.$$

The first step towards proving a logarithmic Sobolev inequality in path space is to generalize the above result to Riemannian manifolds with the gaussian measure μ_s replaced by $\nu_{s,z}(dx) = p(s, z, x)dx$, where $p(s, z, x)$ is the heat kernel. Note that $\nu_{s,z}$ is the distribution of Brownian motion (starting at z) at time s. The main result of this section is Theorem 3.1 below. As before, let

$$P_s f(z) = \nu_{s,z}(f) = \int_M f(x) p(s, z, x) dx$$

be the heat semigroup.

Theorem 3.1. *Suppose that* $\mathrm{Ric}_M \geq -c$ *for a nonnegative constant* c*. Then for any smooth function* f *on* M*,*

$$\int_M f^2 \log |f| d\nu_{s,z} \leq \frac{e^{cs} - 1}{c} \|\nabla f\|^2_{\nu_{s,z}} + \|f\|^2_{\nu_{s,z}} \log \|f\|_{\nu_{s,z}}.$$

Proof. We may assume that f is strictly greater than a fixed positive constant on M. Otherwise consider the function $f_\epsilon = \sqrt{f^2 + \epsilon^2}$ and let $\epsilon \to 0$. Let $g = f^2$ and consider the function $H_r = P_r \phi(P_{s-r} g)$, where $\phi(t) = 2^{-1} t \log t$. Differentiating with respect to r and noting that Δ commutes with P_r we have

$$\begin{aligned}
\frac{dH_r}{dr} &= \frac{1}{2} P_r \Delta \phi(P_{s-r} g) - \frac{1}{2} P_r \left\{\phi'(P_{s-r} g) \Delta P_{s-r} g\right\} \\
&= \frac{1}{2} P_r \left\{\phi'(P_{s-r} g) \Delta P_{s-r} g + \phi''(P_{s-r} g) \left|\nabla P_{s-r} g\right|^2\right\} \\
&\quad - \frac{1}{2} P_r \left\{\phi'(P_{s-r} g) \Delta P_{s-r} g\right\} \\
&= \frac{1}{2} P_r \left\{\phi''(P_{s-r} g) \left|\nabla P_{s-r} g\right|^2\right\}.
\end{aligned}$$

Now using Corollary 2.2 and then the inequality

$$\{P_{s-r}|\nabla g|\}^2 \le 4P_{s-r}gP_{s-r}|\nabla f|^2,$$

we have

$$\begin{aligned}\frac{dH_r}{dr} &\le \frac{1}{4}e^{c(s-r)}P_r\left\{\frac{\left(P_{s-r}|\nabla g|\right)^2}{P_{s-r}g}\right\}\\ &\le e^{c(s-r)}P_r\left\{P_{s-r}|\nabla f|^2\right\}\\ &= e^{c(s-r)}P_s|\nabla f|^2.\end{aligned}$$

Integrating over r from 0 to s we obtain the desired inequality. □

4. Proof of the Main Theorem

We are in a position to prove our main result Theorem 1.1. We divide the proof into two steps, Lemma 4.1 and Lemma 4.3 below.

Lemma 4.1. *Let M be a Riemannian manifold such that $\mathrm{Ric}_M \ge -c$ for a nonnegative constant c. Suppose that F is a cylindrical F given by (1.2). Define $\{\phi_{s_0,s}, s \ge s_0\}$ by*

$$\frac{d}{ds}\phi_{s_0,s} = -\frac{1}{2}\phi_{s_0,s}\mathrm{Ric}_{U_s}, \qquad \phi_{s_0,s_0} = I. \tag{4.1}$$

Then

$$\int_{W_o(M)} F^2\log|F|d\nu \le e^c\sum_{i=1}^{l}(s_i - s_{i-1})E\left|\sum_{j=i}^{l}\phi_{s_i,s_j}U_{s_j}^{-1}\nabla^{(j)}F\right|^2 + \|F\|^2\log\|F\|.$$

Proof. For the sake of simplicity we assume that

$$F(\gamma) = f(\gamma_{s_1}, \gamma_{s_2}, \gamma_{s_3}).$$

Using the Markov property and Theorem 3.1 we have

$$\begin{aligned}&\|F\|^2\log\|F\|\\ =\ &\frac{1}{2}EE_{\gamma_{s_1}}f(\gamma_0,\gamma_{s_2-s_1},\gamma_{s_3-s_1})^2\log EE_{\gamma_{s_1}}f(\gamma_0,\gamma_{s_2-s_1},\gamma_{s_3-s_1})^2\\ =\ &\frac{1}{2}Ef_1(\gamma_{s_1})^2\log Ef_1(\gamma_{s_1})^2\\ \ge\ &-\frac{e^{cs_1}-1}{c}E|\nabla f_1(\gamma_{s_1})|^2 + Ef_1(\gamma_{s_1})^2\log|f_1(\gamma_{s_1})|,\end{aligned} \tag{4.2}$$

where $f_1(x) = \sqrt{E_xf(\gamma_0,\gamma_{s_2-s_1})^2}$. Let $g(x) = E_xf(\gamma_0,\gamma_{s_2-s_1},\gamma_{s_3-s_1})^2$. We have

$$|\nabla f_1|^2 = \frac{|\nabla g|^2}{4g}.$$

Now compute ∇g by Proposition 3.1 and use the Cauchy-Schwarz inequality. We have

$$E|\nabla f_1(\gamma_{s_1})|^2 \le E\left|U_{s_1}^{-1}\nabla^{(1)}F + \phi_{s_1,s_2}U_{s_2}^{-1}\nabla^{(2)}F + \phi_{s_1,s_3}U_{s_3}^{-1}\nabla^{(3)}F\right|^2.$$

Using this and the inequality $(e^{cs}-1)/c \le se^c$ for $c \ge 0$ and $0 \le s \le 1$ in (4.2) we have

$$\begin{aligned} &\|F\|^2 \log \|F\| \\ \ge & -e^c s_1 E\left|U_{s_1}^{-1}\nabla^{(1)}F + \phi_{s_1,s_2}U_{s_2}^{-1}\nabla^{(2)}F + \phi_{s_1,s_3}U_{s_3}^{-1}\nabla^{(3)}F\right|^2 \\ &+ Ef_1(\gamma_{s_1})^2 \log|f_1(\gamma_{s_1})|. \end{aligned} \tag{4.3}$$

Repeating the same calculation for the last term

$$f_1(x)^2 \log|f_1(x)| = \frac{1}{2}E_x f(\gamma_0, \gamma_{s_2-s_1}, \gamma_{s_3-s_1})^2 \log E_x f(\gamma_0, \gamma_{s_2-s_1}, \gamma_{s_3-s_1})^2,$$

we have

$$\begin{aligned} &Ef_1(\gamma_{s_1})^2 \log|f_1(\gamma_{s_1})| \\ \ge & -e^c(s_2-s_1)E\left|U_{s_2}^{-1}\nabla^{(2)}F + \phi_{s_2,s_3}U_{s_3}^{-1}\nabla^{(3)}F\right|^2 \\ &+ Ef_2(\gamma_{s_1},\gamma_{s_2})^2 \log|f_2(\gamma_{s_1},\gamma_{s_2}), \end{aligned} \tag{4.4}$$

where $f_2(x,y) = \sqrt{E_xE_y f(x,y,\gamma_{s_3-s_2})^2}$. We have for the same reason

$$\begin{aligned} &Ef_2(\gamma_{s_1},\gamma_{s_2})^2 \log|f_2(\gamma_{s_1},\gamma_{s_2})| \\ \ge & -e^c(s_3-s_2)E\left|U_{s_3}^{-1}\nabla^{(3)}F\right|^2 + E\left\{F^2\log|F|\right\}. \end{aligned} \tag{4.5}$$

The desired inequality follows immediately from (4.3)–(4.5). $\square$

Remark 4.2. The above proof is reminiscent of the Federbush–Faris–Gross addivity property of logarithmic Sobolev inequalities for product measure spaces (see Gross [7]). The independence property needed in the original proof is replaced by the weaker property of Markov dependence. The same idea appeared in the works of Stroock and Zegarlinski [9, 10], especially p. 118 in the second article.

Lemma 4.3. *Suppose that M is a Riemannian manifold such that $|\mathrm{Ric}_M| \le c$. Then*

$$\sum_{i=1}^{l}(s_i - s_{i-1})\left|\sum_{j=i}^{l}\phi_{s_i,s_j}U_{s_j}^{-1}\nabla^{(j)}F\right|^2 \le e^{2c}|DF|_{\mathbb{H}}^2. \tag{4.6}$$

Proof. Let $Z_i = \sum_{j=i}^{l} U_{s_j}^{-1}\nabla^{(j)}F$. From (4.1) and the assumption on the Ricci curvature we have

$$\|\phi_{s_i,s_j} - \phi_{s_i,s_{j-1}}\| \le \frac{c}{2}\int_{s_{j-1}}^{s_j} e^{c(s-s_i)/2}ds.$$

Note that this is the only place we have to use the absolute bound of the Ricci curvature instead of just the lower bound. Now we have

$$\left|\sum_{j=i}^{l} \phi_{s_i,s_j} U_{s_j}^{-1} \nabla^{(i)} F\right|^2 = \left|Z_i + \sum_{j=i+1}^{l} \left(\phi_{s_i,s_j} - \phi_{s_i,s_{j-1}}\right) Z_j\right|^2$$

$$\leq \left\{|Z_i| + \frac{c}{2} \sum_{j=i+1}^{l} |Z_j| \int_{s_{j-1}}^{s_j} e^{c(\tau - s_i)/2} d\tau\right\}^2$$

$$\leq (1+\lambda)\,|Z_i|^2 + \left(1 + \frac{1}{\lambda}\right) \frac{c^2}{4} \left|\int_{s_i}^{1} e^{c(\tau - s_i)} g_\tau d\tau\right|^2,$$

where $g_s = |Z_j|$ if $s \in [s_{j-1}, s_j)$. It follows that

the left-hand side of (4.6)

$$\leq (1+\lambda) \int_0^1 g_s^2 ds + \left(1 + \frac{1}{\lambda}\right) \frac{c^2}{4} \int_0^1 \left|\int_s^1 e^{c(\tau - s)/2} g_\tau d\tau\right|^2 ds$$

$$\leq \left\{1 + \lambda + \frac{1}{4}\left(1 + \frac{1}{\lambda}\right)(ce^c - e^c + 1)\right\} \int_0^1 g_s^2 ds.$$

Note that $|DF|_{\mathbb{H}}^2 = \int_0^1 g_s^2 ds$ by (1.4). We complete the proof by using the inequality $ce^c - e^c + 1 \leq c^2 e^c$ and choosing $\lambda = (c/2)e^{c/2}$. □

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