ON THE PRINCIPLE OF NOT FEELING THE BOUNDARY FOR DIFFUSION PROCESSES

ELTON P. HSU

ABSTRACT

We derive the principle of not feeling the boundary for the transition density function of a diffusion process from its basic short-time logarithmic asymptotic relation. This allows us to extend this principle for more general diffusion processes.

1. *Introduction*

Let M be a smooth manifold and $p(t,x,y)$ the transition density function of a diffusion process (a strong Markov process with continuous sample paths) on M with respect to a smooth measure m on M. Let U be an open set in M. We use $p_U(t,x,y)$ to denote the transition density function on U of the diffusion process killed at the first exit time of U, that is

$$p_U(t,x,y)\,m(dy) = P_x\{\omega_t \in dy, t < \tau_U\},$$

where P_x denotes the law of the diffusion process in the path space $\Omega(M)$ (the space of continuous functions from $[0,\infty)$ to M) and $\tau_U = \tau_U(\omega) = \inf\{t > 0 : \omega_t \notin U\}$ is the first exit time from U. The principle of not feeling the boundary says in general that under certain geometric conditions, the short-time behaviour of $p_U(t,x,y)$ on U is comparable with that of the free transition density function $p(t,x,y)$. Since the diffusion has continuous sample paths, intuitively the diffusion particle starting from a point in U does not feel the presence of the boundary of U for small time. This principle can be useful in two ways. First, the study of $p(t,x,y)$ can be reduced to the study of $p_U(t,x,y)$ for a suitable choice of U, usually a smooth domain covered by a suitably chosen coordinate chart. We can then regard U as a subset of $\mathbb{R}^d$ (d is the dimension of M). Second, the study of $p_U(t,x,y)$ can be reduced to the study of $p(t,x,y)$ by suitably extending the diffusion generator on U to $\mathbb{R}^d$ so that we can simplify the computations involved by working in the whole space.

Let us now give a precise mathematical statement for this principle.

DEFINITION 1.1. We say that the principle of *not feeling the boundary* holds for the transition density function $p(t,x,y)$ at x, y in U if there exist two positive constants t_0, λ such that for all $t \leqslant t_0$,

$$1 - \frac{p_U(t,x,y)}{p(t,x,y)} \leqslant e^{-\lambda/t}. \tag{1.1}$$

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Note that the quantity on the left side is always nonnegative.

The principle of not feeling the boundary for the diagonal case $x = y$ was first pointed out in [5] in a weaker form for the euclidean Brownian motion. For the case $x \neq y$, the principle does not always hold, because x may be closer to the boundary ∂U than to y. In [9] S. R. S. Varadhan referred to [2] for early results in this direction for the case of euclidean Brownian motion and considered the case of diffusion processes on euclidean space generated by second order, uniformly elliptic operators with Hölder continuous coefficients. He proved that if all distance-minimizing geodesics joining x and y lie completely within U, then

$$\limsup_{t \downarrow 0} 2t \log |p(t,x,y) - p_U(t,x,y)| \leqslant -d_{\partial U}(x,y)^2, \tag{1.2}$$

where

$$d_{\partial U}(x,y) = \inf\{d(x,z) + d(z,y) : z \in \partial U\}.$$

From (1.2) and (1.3) below we see immediately that the principle of not feeling the boundary holds if $d_{\partial U}(x,y) > d(x,y)$. In [9], (1.2) follows from a path space large deviation upper bound for diffusions. The proof of the requisite large deviation upper bound uses, among other things, the logarithmic asymptotic behaviour for the transition density function

$$\lim_{t \downarrow 0} 2t \log p(t,x,y) = -d(x,y)^2, \tag{1.3}$$

where $d(x,y)$ is the Riemannian distance function determined by the second order differential operator generating the diffusion.

We aim in this paper to show that (1.2) holds for much more general diffusions as long as (1.3) holds for some distance function $d(\cdot,\cdot)$ under which the manifold M is complete. In other words, we shall prove that (1.3) implies (1.2). The necessity for such a result is twofold. First, there are diffusion processes for which the required large deviation upper bound is either unknown or technically very complicated but (1.3) holds for a suitably chosen distance function. See Remark 1.4 below. Our result shows that for such diffusions we have an appropriate principle of not feeling the boundary, adequate for studying short-time asymptotic behaviour of their transition density functions. Second, even for diffusions whose large deviation upper bounds are readily available, it is pedagogically more desirable to have a proof of the principle of not feeling the boundary independent of large deviation upper bounds, as is the case in [4].

Let us now be more precise. The general setting is as follows. Let M be a smooth manifold and $p(t,x,y)$ the transition density function of a diffusion process on M with respect to a smooth measure m on M. Let $d(\cdot,\cdot)$ be a distance function on M which generates the topology of M. We assume that M is complete under the distance function in the sense that every d-bounded set is relatively compact. We introduce the following condition.

(D) For every compact set K on M, (1.3) holds uniformly on $(x,y) \in K \times K$.

Here is our main result.

THEOREM 1.2. *Let M be a smooth manifold and $d(\cdot,\cdot)$ a distance function compatible with the topology of M under which M is complete. Let $p(t,x,y)$ be the transition density function of a diffusion process with respect to a smooth measure m on M such that Condition* (D) *holds. Then* (1.2) *holds for every open set U and every pair of points (x,y) in U.*

As in [9], the above theorem has the following immediate corollary.

COROLLARY 1.3. *Under the conditions of Theorem* 1.2, *the principle of not feeling the boundary holds for x, y in U such that $d(x,y) < d_{\partial U}(x,y)$.*

REMARK 1.4. As shown in [9], Condition (D) holds for diffusion processes generated by second order uniformly elliptic operators on $\mathbb{R}^d$ with Hölder coefficients. It also holds for the Riemannian Brownian motion on a complete Riemannian manifold, see [1, 3, 4]. The work in [6] implies that it holds for a hypoelliptic diffusion on a smooth manifold M if M is complete with respect to the control distance defined by the hypoelliptic operator which generates the diffusion.

Our study of (1.2) will not be complete without investigating conditions under which the corresponding limit exists and is equal to the right-hand side. We establish such a result in the most interesting case.

THEOREM 1.5. *Let M be a complete Riemannian manifold and U a smooth open set on M. Denote by $p(t,x,y)$ the transition density function of the Riemannian Brownian motion on M* (*that is, the heat kernel associated with the Laplace–Beltrami operator $\frac{1}{2}\Delta$ on M*) *and by $p_U(t,x,y)$ the transition density function of the same process killed at the first exit time of U* (*that is, the minimal heat kernel on U with the Dirichlet boundary condition*). *Then for any x,y in U we have*

$$\lim_{t\downarrow 0} 2t\log|p(t,x,y)-p_U(t,x,y)| = -d_{\partial U}(x,y)^2. \tag{1.4}$$

On a complete Riemannian manifold we say that an open set U is *strictly convex* if for each pair of points x,y in U, every distance-minimizing geodesic joining x and y lies completely inside U. The above theorem allows us to show that the principle of not feeling the boundary characterizes strictly convex domains.

COROLLARY 1.6. *Under the same hypotheses as in the preceding theorem, the principle of not feeling the boundary holds for all pairs of points x,y in U if and only if U is strictly convex.*

Proof. Since M is assumed to be complete, Condition (D) holds by Remark 1.4. Suppose that U is strictly convex. Fix a pair of points (x,y) in U. We have $d_{\partial U}(x,y) > d(x,y)$ for any x,y in U. Inequality (1.1) follows immediately from (1.3) and (1.4) for any positive $\lambda < \frac{1}{2}[d_{\partial U}(x,y)^2 - d(x,y)^2]$.

Conversely, suppose that (1.1) holds for all x,y in U and for some positive λ depending on x,y. Then (1.1) and (1.3) imply that

$$\limsup_{t\downarrow 0} 2t\log|p(t,x,y)-p_U(t,x,y)| \leqslant -d(x,y)^2-\lambda.$$

This together with (1.4) implies that $d_{\partial U}(x,y)^2 \geqslant d(x,y)^2+\lambda$, that is, $d_{\partial U}(x,y) > d(x,y)$. This last inequality shows that all distance-minimizing geodesics joining x and y must stay completely inside U. Therefore U is strictly convex. The corollary is proved.

REMARK 1.7. We have formulated our results about transition density functions in purely analytical terms and our proofs are probabilistic because we assume that these transition density functions come from diffusion processes, that is, strong Markov processes with continuous sample paths. Analytical conditions which guarantee that such a process exists for a given transition density function can be found in [7, Chapter XIV].

The proofs of Theorems 1.2 and 1.5 are carried out in Sections 3 and 4. In Section 2 we prove a lemma which will be used in the proofs.

2. *A useful lemma*

LEMMA 2.1. (a) *Suppose that τ is a nonnegative random variable such that*

$$\limsup_{t\downarrow 0} 2t\log P\{\tau \leqslant t\} \leqslant -c_1^2 \tag{2.1}$$

for some positive constant c_1. Then for any positive constant c_2

$$\limsup_{t\downarrow 0} 2t\log E\{e^{-c_2^2/2(t-\tau)}; \tau \leqslant t\} \leqslant -(c_1+c_2)^2. \tag{2.2}$$

(b) *Suppose that τ is a nonnegative random variable such that*

$$\liminf_{t\downarrow 0} 2t\log P\{\tau \leqslant t\} \geqslant -c_1^2 \tag{2.3}$$

for some positive constant c_1. Then for any positive constant c_2

$$\liminf_{t\downarrow 0} 2t\log E\{e^{-c_2^2/2(t-\tau)}; \tau \leqslant t\} \geqslant -(c_1+c_2)^2. \tag{2.4}$$

Proof. (a) Integrating by parts, we have

$$\begin{aligned} E\{e^{-c_2^2/2(t-\tau)}; \tau \leqslant t\} &= \int_0^t e^{-c_2^2/2(t-s)}\, dP\{\tau \leqslant s\} \\ &= \int_0^t \frac{c_2^2}{2(t-s)^2} e^{-c_2^2/2(t-s)}\, P\{\tau \leqslant s\}\, ds. \end{aligned} \tag{2.5}$$

Fix a small $\varepsilon \in (0,1)$. By (2.1) for sufficiently small s we have $P\{\tau \leqslant s\} \leqslant e^{-(1-\varepsilon)c_1^2/2s}$. On the other hand, $x^2e^{-x} \leqslant (8/e^2\varepsilon)\, e^{-(1-\varepsilon)x}$ for all $x \geqslant 0$. Hence from (2.5) we have

$$E\{e^{-c_2^2/2(t-\tau)}; \tau \leqslant t\} \leqslant \frac{8}{(ec_2\varepsilon)^2}\int_0^t \exp\left[-\frac{1-\varepsilon}{2}\left(\frac{c_1^2}{s}+\frac{c_2^2}{t-s}\right)\right] ds \leqslant \frac{8t}{(ec_2\varepsilon)^2} e^{-(1-\varepsilon)(c_1+c_2)^2/2t}.$$

In the last step we used the inequality

$$\frac{c_1^2}{s}+\frac{c_2^2}{t-s} \geqslant \frac{(c_1+c_2)^2}{t}.$$

This proves (3.1) since ε can be arbitrarily close to 0.

(b) From (2.5) and (2.3) we have

$$E\{e^{-c_2^2/2(t-\tau)};\tau \leqslant t\} \geqslant \frac{c_2^2}{2t^2}\int_0^t \exp\left[-\frac{1+\varepsilon}{2}\left(\frac{c_1^2}{s}+\frac{c_2^2}{t-s}\right)\right]ds.$$

Restricting the integral to the interval bounded by the points $(c_1 t/(c_1+c_2))(1\pm\varepsilon)$, we have

$$E\{e^{-c_2^2/2(t-\tau)};\tau \leqslant t\} \geqslant \frac{\varepsilon c_1 c_2^2}{(c_1+c_2)t^2}\exp\left[-\frac{1+\varepsilon}{1-\varepsilon}\frac{(c_1+c_2)^2}{2t}\right],$$

from which (2.4) follows immediately.

3. *Proof of Theorem* 1.2

We remind the reader that throughout this section Condition (D) stated in Section 1 is in force.

Let $\Omega_x(M)$ be the metric space of continuous paths $\omega:[0,\infty)\to M$ starting from x. We shall use P_x to denote the law on $\Omega_x(M)$ of the diffusion process starting from x.

LEMMA 3.1. *Let U be a relatively compact open set in M and λ a positive number. Then, uniformly in all x, y in U such that $d(x,y) < d(y,\partial U)-\lambda$, we have*

$$\lim_{t\downarrow 0} 2t\log p_U(t,x,y) = -d(x,y)^2. \tag{3.1}$$

Proof. By the strong Markov property, we have the following first passage formula relating $p(t,x,y)$ and $p_U(t,x,y)$, namely,

$$p(t,x,y) = p_U(t,x,y) + E_x\{p(t-\tau_U,\omega_{\tau_U},y);\tau_U \leqslant t\}, \tag{3.2}$$

where $\tau_U = \inf\{t>0:\omega_t\notin U\}$. By Condition (D), for the second term on the right side of (3.2) we have

$$\limsup_{t\downarrow 0} 2t\log E_x\{p(t-\tau_U,\omega_{\tau_U},y);\tau_U \leqslant t\} \leqslant -d(y,\partial U)^2, \tag{3.3}$$

uniformly in $(x,y)\in U\times U$. Then (3.1) follows from (3.2), (3.3), (1.3) and the hypothesis that $d(y,\partial U) > d(x,y)+\lambda$.

LEMMA 3.2. *Suppose that K is a compact set in M and $\gamma>0$. Then there exists a $t_0>0$ such that for all $(x,y,t)\in M\times K\times(0,t_0)$ such that $d(x,y)\geqslant\gamma$, we have*

$$p(t,x,y)\leqslant 1. \tag{3.4}$$

Proof. Let $T=\inf\{t>0:\omega_t\in B_{\gamma/2}(y)\}$. We have by the strong Markov property at time T,

$$p(t,x,y) = E_x\{p(t-T,\omega_T,y);\ T<t\} \leqslant \sup\{p(s,z,y):s\leqslant t, y\in K, d(z,y)=\tfrac{1}{2}\gamma\}.$$

Then (3.4) follows immediately from Condition (D).

For a fixed $\gamma > 0$, define $\tau_\gamma = \inf\{t > 0 : d(\omega_0, \omega_t) = \gamma\}$. (Convention: $\inf \varnothing = \zeta$, the lifetime of the path ω.)

PROPOSITION 3.3. *For any fixed $x \in M$ and $\gamma \geqslant 0$ we have*

$$\limsup_{t\downarrow 0} 2t \log P_x\{\tau_\gamma \leqslant t\} \leqslant -\gamma^2. \tag{3.5}$$

Proof. We prove (3.5) in three steps:

(a) for any compact K, there exists $\gamma_0 > 0$ such that (3.5) holds uniformly for all $x \in K$ and all $\gamma \leqslant \gamma_0$;
(b) if (3.5) holds for $\gamma_1 \geqslant 0$, then there exists $\gamma_2 > \gamma_1$ such that (3.5) holds for $\gamma \in [\gamma_1, \gamma_2]$;
(c) if (3.5) holds for all $\gamma < \gamma_3$, then (3.5) holds also for $\gamma = \gamma_3$.

Clearly (b) and (c) imply that the set of γ for which (3.5) holds is both open and closed in $[0, \infty)$, which implies that (3.5) holds for all $\gamma \geqslant 0$. Step (a) is needed for the proof of Step (b).

(a) There exists a small $\gamma_0' > 0$ such that for all $\gamma \in (0, \gamma_0')$ and $\varepsilon > 0$ we have

$$\sigma(\gamma, \varepsilon) \overset{\text{def}}{=} \inf\{m[B_\gamma(x)^c \cap B_\varepsilon(z)] : x \in K, z \in \partial B_\gamma(x)\} > 0.$$

Note that if γ is too large, the above inequality may not hold. This is the case if, for example, M is compact and γ is greater than the diameter of M. In this case $B_\gamma(x)^c$ is empty.

Let U be a relatively compact open set containing K. We choose a positive γ_0 less than $\min\{\gamma_0', \frac{1}{3}d(K, \partial U)\}$. For any $\gamma \leqslant \gamma_0$, by the strong Markov property at τ_γ we have

$$P_x\{\omega_t \in B_\gamma(x)^c, t < \tau_U\} = E_x\{F(t - \tau_\gamma, \omega_{\tau_\gamma}); \tau_\gamma \leqslant t\}, \tag{3.6}$$

where $F(u, z) = P_z\{\omega_u \in B_\gamma(x)^c, u < \tau_U\}$. For any $z \in \partial B_\gamma(x)$ and a fixed positive $\varepsilon \leqslant \frac{1}{3}d(K, \partial U)$ and for all sufficiently small u, by Lemma 3.1 we have

$$\begin{aligned} F(u, z) &\geqslant P_z\{\omega_u \in B_\gamma(x)^c \cap B_\varepsilon(z), u < \tau_U\} \\ &= \int_{B_\gamma(x)^c \cap B_\varepsilon(z)} p_U(u, z, z_1)\, dz_1 \geqslant m[B_\gamma(x)^c \cap B_\varepsilon(z)]\, e^{-\varepsilon^2/u}. \end{aligned}$$

Using this estimate in (3.6) for any fixed $\lambda \in (0, 1)$ and sufficiently small t we have

$$\begin{aligned} P_x\{\omega_t \in B_\gamma(x)^c, t < \tau_U\} &\geqslant \sigma(\gamma, \varepsilon)\, E_x\{e^{-2\varepsilon^2/(t - \tau_\gamma)}; \tau_\gamma \leqslant (1 - \lambda)\, t\} \\ &\geqslant \sigma(\gamma, \varepsilon)\, e^{-2\varepsilon^2/\lambda t}\, P_x\{\tau_\gamma \leqslant (1 - \lambda)\, t\}. \end{aligned}$$

On replacing t by $t/(1 - \lambda)$, we see that the above inequality is equivalent to

$$P_x\{\tau_\gamma \leqslant t\} \leqslant \frac{e^{2\varepsilon^2(1-\lambda)/\lambda t}}{\sigma(\gamma, \varepsilon)} P_x\{\omega_{t/(1-\lambda)} \in B_\gamma(x)^c, t/(1 - \lambda) < \tau_U\}. \tag{3.7}$$

On the other hand, since $p_U(t, x, y) \leqslant p(t, x, y)$, we have

$$P_x\{\omega_t \in B_\gamma(x)^c, t < \tau_U\} = \int_{B_\gamma(x)^c} p_U(t, x, y)\, dy \leqslant m(U) \max_{y \in U \setminus B_\gamma(x)} p(t, x, y). \tag{3.8}$$

Note that $m(U) < \infty$ because U is relatively compact. From (3.7), (3.8), and (1.3) we have immediately

$$\limsup_{t\downarrow 0} 2t \log P_x\{\tau_\gamma \leqslant t\} \leqslant (1-\lambda)\left(\frac{4\varepsilon^2}{\lambda} - \gamma^2\right).$$

Letting $\varepsilon \downarrow 0$ and then $\lambda \uparrow 1$, we obtain (3.5).

(b) We assume that (3.5) holds for γ_1, that is,

$$\limsup_{t\downarrow 0} 2t \log P_x\{\tau_{\gamma_1} \leqslant t\} \leqslant -\gamma_1^2. \tag{3.9}$$

Let γ_0 be the one in Part (a) chosen for the compact set K which is equal to the closure of $B_{\gamma_1}(x)$, and $\gamma_2 = \gamma_1 + \gamma_0$. For any $\gamma \in [\gamma_1, \gamma_2]$ we have

$$P_x\{\tau_\gamma \leqslant t\} = E_x\{G(\omega_{\tau_{\gamma_1}}, t - \tau_{\gamma_1}), \tau_{\gamma_1} \leqslant t\}, \tag{3.10}$$

where $G(z,u) = P_z\{\tau_{B_\gamma(x)} \leqslant u\}$. Clearly $z \in \partial B_{\gamma_1}(x)$ implies that $\tau_{B_\gamma(x)} \geqslant \tau_{\gamma-\gamma_1}$, P_z-a.s. Hence $G(z,u) \leqslant P_z\{\tau_{\gamma-\gamma_1} \leqslant u\}$. Using Part (a) we have

$$\limsup_{u\downarrow 0} 2u \log G(z,u) \leqslant -(\gamma-\gamma_1)^2 \tag{3.11}$$

uniformly in $z \in \partial B_{\gamma_1}(x)$. By (3.9) to (3.11) and Lemma 2.1(a) we have immediately (3.5).

(c) For any $\gamma < \gamma_3$ we have $P_x\{\tau_{\gamma_3} \leqslant t\} \leqslant P_x\{\tau_\gamma \leqslant t\}$. Since (3.5) holds for γ, we have

$$\limsup_{t\downarrow 0} 2t \log P\{\tau_{\gamma_3} \leqslant t\} \leqslant -\gamma^2.$$

Letting $\gamma \uparrow \gamma_3$ we see that (3.5) holds for $\gamma = \gamma_3$. The proof of the proposition is completed.

Proof of Theorem 1.2. By (3.2) it is enough to show that

$$\limsup_{t\downarrow 0} 2t \log E_x\{p(t-\tau_U, \omega_{\tau_U}, y); \tau_U \leqslant t\} \leqslant -d_{\partial U}(x,y)^2. \tag{3.12}$$

We denote the expectation in (3.12) by $I_U(t,x,y)$. We shall use Condition (D) to estimate $p(t-\tau_U, \omega_{\tau_U}, y)$. But Condition (D) cannot be applied directly because ω_{τ_U} is not confined to a compact set. So we fix a large $R > d(x,y)$ and split $I_U(t,x,y)$ into two parts as follows:

$$\begin{aligned} I_U(t,x,y) &= E_x\{p(t-\tau_U, \omega_{\tau_U}, y); \omega_{\tau_U} \in B_R(x), \tau_U \leqslant t\} \\ &\quad + E_x\{p(t-\tau_U, \omega_{\tau_U}, y); \omega_{\tau_U} \notin B_R(x), \tau_U \leqslant t\} \\ &\overset{\text{def}}{=} J_U(t,x,y) + K_U(t,x,y). \end{aligned} \tag{3.13}$$

The term $K_U(t,x,y)$ is easy to estimate. By Lemma 3.2 we have

$$K_U(t,x,y) \leqslant P_x\{\omega_{\tau_U} \notin B_R(x), \tau_U \leqslant t\} \leqslant P_x\{\tau_R \leqslant t\}.$$

Hence from Proposition 3.3 we have

$$\limsup_{t\downarrow 0} 2t \log K_U(t,x,y) \leqslant -R^2. \tag{3.14}$$

For the term $J_U(t,x,y)$, the exit position ω_{τ_U} is confined to the relatively compact set $B_R(x)$ and we can use Condition (D) to estimate $p(t-\tau_U, \omega_{\tau_U}, y)$. For any given positive ε, there exists a positive t_1 depending on R and ε such that for all $t \leqslant t_1$ we have

$$p(t-\tau_U, \omega_{t_U}, y) \leqslant \exp\left[-\frac{\{d(\omega_{\tau_U}, y)-\varepsilon\}^2}{2(t-\tau_U)}\right]. \tag{3.15}$$

Next, we cover the set $\partial U \cap B_R(x)$ by N balls $B_\varepsilon(z_i)$ of radius ε and centres $z_i \in \partial U$. The number of balls N depends on R and ε but is finite. From the definition of $J_U(t,x,y)$ and (3.15) we have

$$\begin{aligned} J_U(t,x,y) &\leqslant \sum_{j=1}^{N} E_x\left\{\exp\left[-\frac{\{d(\omega_{\tau_U}, y)-\varepsilon\}^2}{2(t-\tau_U)}\right]; \omega_{\tau_U} \in B_\varepsilon(z_j), \tau_U \leqslant t\right\} \\ &\leqslant \sum_{j=1}^{N} E_x\left\{\exp\left[-\frac{\{d(z_j, y)-2\varepsilon\}^2}{2(t-\tau_{\gamma_j})}\right]; \tau_{\gamma_j} \leqslant t\right\} \stackrel{\text{def}}{=} \sum_{j=1}^{N} J_U^j(t,x,y), \end{aligned}$$

where $\gamma_j = d(z_j, x) - \varepsilon$. In the last step we used the fact that $\omega_{\tau_U} \in B_\varepsilon(z_j)$ implies that $d(\omega_{\tau_U}, y) \geqslant d(z_j, y) - \varepsilon$ and $\tau_U \geqslant \tau_{\gamma_i}$. From Proposition 3.3 for $j = 1, \dots, N$ we have

$$\limsup_{s\downarrow 0} 2s \log P_x\{\tau_{\gamma_j} \leqslant s\} \leqslant -\gamma_j^2.$$

By this inequality and Lemma 2.1(a), for $j = 1, \dots, N$ we have

$$\limsup_{t\downarrow 0} 2t \log J_U^j(t,x,y) \leqslant -\{(d(x,z_j) + d(z_j,y) - 3\varepsilon)\}^2.$$

But $d(x,z_j) + d(z_j,y) \geqslant d_{\partial U}(x,y)$ for all $j = 1, \dots, N$. It follows that for all $t \leqslant t_3$ we have

$$\limsup_{t\downarrow 0} 2t \log J_U(t,x,y) \leqslant -\{d_{\partial U}(x,y) - 2\varepsilon\}^2. \tag{3.16}$$

Now from (3.13), (3.14), and (3.16) we have

$$\limsup_{t\downarrow 0} 2t \log I_U(t,x,y) \leqslant -\min\{d_{\partial U}(x,y) - 2\varepsilon, R\}^2.$$

Letting $\varepsilon \downarrow 0$ and $R \uparrow \infty$, we obtain the desired inequality (3.12). This completes the proof of Theorem 1.2.

4. *Proof of Theorem* 1.5

The proof of Theorem 1.5 depends on a more delicate geometric argument. Since we have shown in Theorem 1.2 that (1.2) holds, it suffices to prove that

$$\liminf_{t\downarrow 0} 2t \log |p(t,x,y) - p_U(t,x,y)| \geqslant -\{d(x,z) + d(z,y)\}^2 \tag{4.1}$$

for all $z \in \partial U$. We introduce the following condition.

$T(U; z)$ There exists a distance-minimizing geodesic ϕ joining x and z such that ϕ lies entirely in U and that ϕ is transversal to the boundary ∂U at z.

We shall first prove (4.1) under the above transversality condition $T(U;z)$ and then remove this condition. From (3.2) and (1.3) we have

$$\begin{aligned} p(t,x,y)-p_U(t,x,y) &= E_x\{p(t-\tau_U,\omega_{\tau_U},y);\tau_U \leqslant t\} \\ &\geqslant E_x\{p(t-\tau_U,\omega_{\tau_U},y);\omega_{\tau_U}\in B_\varepsilon(z),\tau_U \leqslant t\} \\ &\geqslant E_x\left\{\exp\left[-\frac{\{d(z,y)+2\varepsilon\}^2}{2(t-\tau_U)}\right];\omega_{\tau_U}\in W_\varepsilon,\tau_U \leqslant t\right\}, \end{aligned} \tag{4.2}$$

where $W_\varepsilon = B_\varepsilon(z) \cap \partial U$. We estimate the probability $P_x\{\omega_{\tau_U}\in W_\varepsilon, \tau_U \leqslant s\}$. Let $\varepsilon_1 < \varepsilon$ and $V_{\varepsilon_1} = B_{\varepsilon_1}(z) \cap U^c$. Since $\omega_s \in V_{\varepsilon_1}$ implies that $\tau_U \leqslant s$, by the strong Markov property at τ_U we have

$$\begin{aligned} P_x\{\omega_s\in V_{\varepsilon_1}\} &= E_x\{P_{\omega_{\tau_U}}\{\omega_{s-\tau_U}\in V_{\varepsilon_1}\};\omega_{\tau_U}\in W_\varepsilon,\tau_U\leqslant s\} \\ &\quad + E_x\{P_{\omega_{\tau_U}}\{\omega_{s-\tau_U}\in V_{\varepsilon_1}\};\omega_{\tau_U}\notin W_\varepsilon,\ \tau_U\leqslant s\}. \end{aligned}$$

Hence the probability we want to estimate

$$P_x\{\omega_{\tau_U}\in W_\varepsilon,\tau_U\leqslant s\} \geqslant L_U(s,x,z)-M_U(s,x,z), \tag{4.3}$$

where

$$L_U(s,x,z) = P_x\{\omega_s\in V_{\varepsilon_1}\} = \int_{V_{\varepsilon_1}} p(s,x,z_1)\,dz_1$$

and

$$M_U(s,x,z) = E_x\{P_{\omega_{\tau_U}}\{\omega_{s-\tau_U}\in V_{\varepsilon_1}\};\omega_{\tau_U}\notin W_\varepsilon,\tau_U\leqslant s\}.$$

We have by a proof similar to that of Theorem 1.2

$$\limsup_{s\downarrow 0} 2s\log M_U(s,x,z) \leqslant -d_{\partial U\setminus W_\varepsilon}(x, V_{\varepsilon_1})^2. \tag{4.4}$$

On the other hand by (1.3)

$$\liminf_{s\downarrow 0} 2s\log L_U(s,x,z) \geqslant -d(x, V_{\varepsilon_1})^2. \tag{4.5}$$

Now Condition $T(U;z)$ implies that for sufficiently small $\varepsilon_1 < \varepsilon$

$$d(x, V_{\varepsilon_1}) < d_{\partial U\setminus W_\varepsilon}(x, V_{\varepsilon_1}). \tag{4.6}$$

From (4.3) to (4.6) we have immediately

$$\liminf_{s\downarrow 0} P_x\{\omega_{\tau_U}\in W_\varepsilon,\tau_U\leqslant s\} \geqslant -d(x, V_{\varepsilon_1}). \tag{4.7}$$

From (4.2), (4.7) and Lemma 2.1(b) we have

$$\begin{aligned} \liminf_{t\downarrow 0} 2t\log|p(t,x,y)-p_U(t,x,y)| &\geqslant -[d(x, V_{\varepsilon_1})+d(z,y)+2\varepsilon]^2 \\ &\geqslant -[d(x,z)+d(z,y)+3\varepsilon]^2, \end{aligned}$$

which proves (4.1) under Condition $T(U;z)$.

We now remove the transversality condition $T(U;z)$. Let $z \in \partial U$ and let ϕ be a distance-minimizing geodesic from x to z. Let z' be the first point at which ϕ intersects ∂U. Then by the triangle inequality we have

$$d(x,z')+d(z',y) \leqslant d(x,z')+d(z',z)+d(z,y) \leqslant d(x,z)+d(z,y). \tag{4.8}$$

In the last step we used the fact that z' lies on a distance-minimizing geodesic ϕ from x to z. For any fixed positive ε we choose a point $z_1 \in B_\varepsilon(z') \cap \partial U$ and a small $\varepsilon_1 \in (0, \frac{1}{2}\varepsilon)$ such that the closure of $B_{2\varepsilon_1}(z_1)$ does not intersect $C(x)$, the cutlocus of x. This is possible because $C(x)$ is a closed set of measure zero. Take any point $z_2 \in \partial B_{\varepsilon_1}(z_1) \cap U^c$ such that the unique geodesic ϕ^2 joining x and z_2 is transversal to $\partial B_{\varepsilon_1}(z_1)$ at z_2. Now choose a smooth open set $\tilde{U}$ such that

(i) $\tilde{U}$ contains U;
(ii) $\tilde{U}$ contains ϕ^2;
(iii) $\partial \tilde{U}$ near z_2 coincides with $\partial B_{\varepsilon_1}(z_1)$ in a small neighbourhood of z_2.

Such $\tilde{U}$ obviously exists. Now Condition $T(\tilde{U}; z_2)$ is satisfied. Hence by what we have proved under the transversality condition,

$$\liminf_{t \downarrow 0} 2t \log |p(t,x,y) - p_{\tilde{U}}(t,x,y)| \geqslant -[d(x,z_2)+d(z_2,y)]^2. \tag{4.9}$$

Since $\tilde{U} \supset U$ we have by the maximum principle for the Dirichlet heat kernel $p_{\tilde{U}}(t,x,y) \geqslant p_U(t,x,y)$. Hence

$$p(t,x,y)-p_U(t,x,y) \geqslant p(t,x,y)-p_{\tilde{U}}(t,x,y) \geqslant 0. \tag{4.10}$$

On the other hand by the choice of z_2 we have $d(z',z_2) \leqslant 2\varepsilon$, hence by (4.8)

$$d(x,z_2)+d(z_2,y) \leqslant d(x,z)+d(z,y)+4\varepsilon. \tag{4.11}$$

The desired inequality (4.1) follows immediately from (4.9) to (4.11). This completes the proof of Theorem 1.5.

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Department of Mathematics
Northwestern University
Evanston
Illinois 60208
USA