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Quasi-Invariance of the Wiener Measure on Path Spaces: Noncompact Case[1]

Elton P. Hsu

Department of Mathematics, Northwestern University, Evanston, Illinois 60208
E-mail: elton@math.northwestern.edu

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For a geometrically and stochastically complete, noncompact Riemannian manifold, we show that the flows on the path space generated by the Cameron–Martin vector fields exist as a set of random variables. Furthermore, if the Ricci curvature grows at most linearly, then the Wiener measure (the law of Brownian motion on the manifold) is quasi-invariant under these flows.

Key Words: Path space; Wiener measure; Cameron–Martin vector fields; Quasi-invariance.

1. INTRODUCTION

Let M be a complete Riemannian manifold. The bundle of orthonormal frames of M is denoted by $\mathscr{O}(M)$ with the canonical projection $\pi : \mathscr{O}(M) \to M$. We use $P_o(M)$ to denote the pinned path space over M, namely the space of continuous maps from $[0, 1]$ to M starting from a fixed point $o \in M$. It is a metric space equipped with a standard filtration of σ-fields $\mathscr{B}_* = \{\mathscr{B}_s, 0 \leqslant s \leqslant 1\}$, where $\mathscr{B}_s$ is the σ-field generated by the canonical process up to time s. The last σ-field $\mathscr{B}_1$ coincides with the usual Borel σ-field on $P_o(M)$. Let $\mathbb{P}_o$ (denoted by $\mathbb{P}$ for simplicity) be the Wiener measure on $P_o(M)$, i.e., the law of a (Riemannian) Brownian motion starting from o. To ensure that $\mathbb{P}$ is indeed a probability measure on $(P_o(M), \mathscr{B})$, we make the standing assumption that Brownian motion is conservative, or equivalently the manifold M is stochastically complete. Analytically this means that, for the minimal heat kernel $p_M(t, x, y)$ of M,

$$\int_M p_M(t, x, y)\, dy = 1$$

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for some $(t, x) \in \mathbb{R}_+ \times M$ (and hence for all such pairs). We thus obtain a filtered probability space $(P_o(M), \mathscr{B}_*, \mathbb{P})$.

Let $\mathscr{H}$ be the Cameron–Martin space,

$$\mathscr{H} = \{h \in W_o(\mathbb{R}^n) : \dot{h} \in L^2([0,1], \mathbb{R}^n)\},$$

with the norm

$$|h|_{\mathscr{H}} = \left\{ \int_0^1 |\dot{h}_s|^2 \, ds \right\}^{1/2}.$$

The vector field D_h on $P_o(M)$ is defined by

$$D_h(X)_s = U(X)_s h_s, \qquad X \in P_o(M),$$

where $U(X)$ is the horizontal lift (whenever it is defined) of X to $\mathcal{O}(M)$ starting from a frame u_o, assumed to be fixed throughout the discussion. Thus $D_h(X)$ is a vector field along X. In the present work, we study the existence of the flow generated by D_h and the quasi-invariance of the Wiener measure $\mathbb{P}$ under this flow. More precisely, we seek a collection of measurable maps (or $P_o(M)$-valued random variables)

$$\zeta_h^t : P_o(M) \to P_o(M), \qquad t \in \mathbb{R},$$

$$\frac{d\zeta_h^t}{dt} = U(\zeta_h^t)\, h, \qquad \zeta_h^0 = I_{P_o(M)} \tag{1}$$

[$I_{P_o(M)}$ is the identity map on $P_o(M)$] and ask whether the measures $\mathbb{P}_h^t = \mathbb{P} \circ (\zeta^t)^{-1}$ (the law of ζ_h^t) are mutually absolutely continuous with respect to $\mathbb{P}$.

For a compact manifold, this problem was solved in [1]; see also [2, 3]. The purpose of the present work is to investigate the case of noncompact manifolds.

Let $d(\cdot,\cdot)$ be the Riemannian distance function on M. From (1), for any path $X \in P_o(M)$,

$$d(\zeta^t(X)_s, X_s) \leqslant |\, t||h_s|.$$

[The reference to h is dropped from the notation for simplicity.] Therefore, we naturally expect that the flow generated by D_h exists as long as the manifold M is both geodesically and stochastically complete. In (1) the horizontal lift $U(\zeta^t)$ has to make sense; we therefore require that each ζ^t is an M-valued semimartingale. We will prove that there exists a unique family of semimartingales $\{\zeta^t\}$ satisfying (1). Although each map $\zeta^t : P_o(M) \to P_o(M)$

is well-defined $\mathbb{P}_o$-almost surely, this does not mean that $\{\zeta^t\}$ is a flow for the composition $\zeta^{t_1} \circ \zeta^{t_2}$ makes sense only when $\mathbb{P}^{t_2} \ll \mathbb{P}$, and, without proving the quasi-invariance, we cannot claim that $\zeta^{t_1} \circ \zeta^{t_2} = \zeta^{t_1+t_2}$.

In the compact case, the Radon–Nikodym derivative has the form

$$\frac{d\,\mathbb{P}^t}{d\,\mathbb{P}} = \exp\left\{ \int_0^t l(\zeta^{-\lambda}) d\lambda \right\},$$

$$l = \int_0^1 \left\langle \dot{h}_s - \frac{1}{2}\mathrm{Ric}_{U_s} h_s, dW_s \right\rangle.$$

Here U is the horizontal lift starting from u_o of the coordinate process X on $P_o(M)$ and $W = J^{-1} \circ X$ the anti-development of U (or X). Note that X is a Brownian motion from o under $\mathbb{P}$. The map $J : P_o(\mathbb{R}^n) \to P_o(M)$ is the Itô map, which sends an $\mathbb{R}^n$-valued semimartingale its (stochastic) development on M (see [5]). Let $r(\cdot) = d(\cdot, o)$ be the distance from the fixed point o. In view of the fact that

$$r(\zeta^{-\lambda}(X)_s) \leqslant |\lambda||h_s| + r(X_s) \leqslant |\lambda||h|_{\mathscr{H}} + r(X_s),$$

the formula for the Radon–Nikodym derivative suggests that a growth condition on the Ricci curvature together with an effective estimate of the size of $X_* \stackrel{\text{def}}{=} \max_{0 \leqslant s \leqslant 1} r(X_s)$ should be sufficient for the quasi-invariance of the Wiener measure. On the other hand, it is well known that the size of Brownian motion can also be controlled by a lower bound of the Ricci curvature. We will show that the growth condition

$$|\mathrm{Ric}_M(x)| \leqslant C\{1 + r(x)\}$$

is sufficient for this purpose. We note that a complete Riemannian manifold whose Ricci curvature satisfies the above growth condition is automatically stochastically complete; see [4].

2. EXISTENCE OF THE FLOW

We briefly recall how the existence of the flow $\{\zeta^t\}$ generated by D_h is proved when M is compact, following the exposition in [3]. For a frame $u \in \mathcal{O}(M)$ and $a \in \mathbb{R}^n$ we denote by $H_u a$ the horizontal vector at u such that $\pi_*(H_u a) = ua$. Thus $\{Ha, a \in \mathbb{R}^n\}$ are the fundamental horizontal vector fields on $\mathcal{O}(M)$. Let $(W_o(M), \mathscr{B}_*, \mathbb{P})$ be the pinned path space over M with the standard Wiener measure. The coordinate process X is a Brownian motion starting from o. Let $U = \{U_s, s \in [0, 1]\}$ be the horizontal lift of X

starting from a fixed frame u_o over o and $W = \{W_s, 0 \leqslant s \leqslant 1\}$ its anti-development. Then W is a euclidean Brownian motion, and U is the solution of the following stochastic differential equation on $\mathcal{O}(M)$:

$$dU_s = H_{U_s} \circ dW_s, \qquad U_0 = u_o.$$

The above equation makes sense for any $\mathbb{R}^n$-valued semimartingale W and we denote the projection πU of the solution by JW. If X is an M-valued semimartingale, we denote its anti-development by $J^{-1}X$, unique after choosing an initial frame over X_0.

A formal calculation shows that the pullback $p = J_*^{-1} D_h$ of the vector field D_h is given at $W \in P_o(\mathbb{R}^n)$ by

$$p(W)_s = h_s - \int_0^s K(W)_\tau \circ dW_\tau,$$

where

$$K(W)_s = \int_0^s \Omega_{U(JW)_\tau}(\circ\, dW_\tau, h_\tau).$$

Here Ω is the curvature form, which is by definition an $o(d)$-valued horizontal 2-form on $\mathcal{O}(M)$. We have written $\Omega_u(a, b)$ instead of more precise $\Omega_u(Ha, Hb)$ to simplify the notation. Finding the flow $\{\zeta^t\}$ generated by D_h on $P_o(M)$ is equivalent to finding the flow $\{\xi^t\}$ generated by the vector field p on $W_o(\mathbb{R}^n)$:

$$\frac{d\xi^t}{dt} = p(\xi^t), \qquad \xi^0 = I_{P_o(\mathbb{R}^n)}. \tag{2}$$

Once $\{\xi^t\}$ is found, the desired flow on $P_o(M)$ is given by $\zeta^t(X) = J\xi^t (J^{-1}X)$. Note that the right-hand side is well defined because $J^{-1}X$ is a euclidean Brownian motion and $\xi^t(J^{-1}X)$ is a semimartingale.

It turns out sufficient to seek solutions in the space of semimartingales of the form

$$z_s = \int_0^s A_\tau \, d\tau + \int_0^s O_\tau \, dW_\tau, \tag{3}$$

where O and A are, respectively, $O(d)$-valued and $\mathbb{R}^n$-valued processes, both being adapted to the canonical filtration $\mathscr{B}_*$. For such semimartingales, we introduce the norms

$$\|A\|^2 = \mathbb{E} \int_0^1 |A_s|^2 \, ds,$$
$$|O|^2 = \mathbb{E} \sup\nolimits_{0 \leqslant s \leqslant 1} |O_s|^2,$$
$$\langle z \rangle^2 = \|A\|^2 + |O|^2.$$

We have included the norm of O to accommodate the situation where O may not be $O(n)$-valued. In these norms, (2) can be solved by Picard's iteration. Let $\xi^{t,0} = W$, and

$$\xi^{t,n} = W + \int_0^t p(\xi^{\lambda,n-1})\, d\lambda.$$

If M is compact, there is a constant C depending only on $T > 0$ such that

$$\langle \xi^{t,n} - \xi^{t,n-1} \rangle \leqslant C \int_0^t \langle \xi^{\lambda,n-1} - \xi^{\lambda,n-2} \rangle \, d\lambda, \qquad |t| \leqslant T. \tag{4}$$

To prove this estimate we need to write p in Itô's form, if z is a semimartingale in the form (3), then

$$\begin{aligned} p(z)_s &= h_s - \frac{1}{2} \int_0^s \mathrm{Ric}_{U(Jz)_\tau} h_\tau \, d\tau - \int_0^s \langle K(z)_\tau, dz_\tau \rangle, \\ K(z)_s &= \int_0^s \Omega_{U(Jz)_\tau}(\circ\, dz_\tau, h_\tau) \\ &= \int_0^s \Omega_{U(Jz)_\tau}(dz_\tau, h_\tau) + \frac{1}{2} \int_0^s H_i \Omega_{U(Jz)_\tau}(e_j, h_\tau)\, d\langle z^i, z^j \rangle_\tau. \end{aligned}$$

Here $\{e_i\}$ is the canonical orthonormal basis of $\mathbb{R}^n$, and $H_i = He_i$. Note that by (3) the stochastic integrals with respect to $\{z^i\}$ and their co-variations in the above equations can be further reduced to stochastic integrals with respect to τ and dW_τ. If we write

$$\xi^t_s = \int_0^s A^t_\tau d\tau + \int_0^s O^t_\tau \, dW_\tau,$$

then the flow equation (2) is equivalent to the following system of equations:

$$\begin{cases} \xi^t_s = \int_0^s A^t_\tau \, d\tau + \int_0^s O^t_\tau \, dW_\tau, \\ O^t = I - \int_0^t K(\xi^\lambda) O^\lambda \, d\lambda, \\ A^t = O^t \int_0^t O^{\lambda *} \left[\dot{h} - \dfrac{1}{2} \mathrm{Ric}_{U(J\xi^\lambda)} h \right] d\lambda. \end{cases} \tag{5}$$

With this form of the flow equation, the proof of inequality (4) involves nothing more than routine bounds of stochastic integrals with respect to dW_τ by Doob's inequality and those with respect to $d\tau$ by taking absolute values under the integrals.

Now, inequality (4) implies that the limit $\xi^t = \lim_{n\to\infty} \xi^{t,n}$ exists and is the solution to (2). The uniqueness is clear because we are dealing with a Volterra-type integral equation. The flow on $P_o(M)$ is now obtained by $\zeta^t(X) = J\xi^t(J^{-1}X)$.

Remark 2.1. What we have said so far is still valid if we stop at a $\mathscr{B}_*$-stopping time $\sigma \leqslant 1$ in the s-direction.

Because the process K takes values in $o(n)$, the space of anti-symmetric matrices, from (5) we see that O^t takes value in $O(n)$, the space of orthogonal matrices. If M is compact, the Ricci curvature is uniformly bounded, hence

$$|\mathrm{Ric}_{U(J\xi^t)_s} h_s| \leqslant C|h_s| \leqslant C|h|_{\mathscr{H}}.$$

It follows from (5) that

$$|A_s^t| \leqslant C \mid t|\{|\dot{h}_s| + |h|_{\mathscr{H}}\}$$

for some constant C. Let $\mathbb{Q}$ be the Wiener measure on $P_o(\mathbb{R}^n)$. Girsanov's theorem and the hypothesis that $h \in \mathscr{H}$ imply that $\mathbb{Q} \circ (\xi^t)^{-1}$ (the law of ξ^t) is mutually absolutely continuous with respect to $\mathbb{Q}$, namely, the Wiener measure is quasi-invariant under the flow $\{\xi^t\}$ on $P_o(\mathbb{R}^n)$. Transporting this result to the space $P_o(M)$ by the Itô map J, we obtain the quasi-invariance of the Wiener measure $\mathbb{P}$ under the flow $\{\zeta^t\}$ generated by D_h.

We now turn to a complete, but not necessarily compact Riemannian manifold. Estimate (4) may not hold because the curvature Ω and its derivatives $H_i\Omega, H_iH_j\Omega$ may not be uniformly bounded. To overcome this difficulty, we will truncate the vector field D_h to zero whenever the path are outside a large compact set. Let $\phi : M \to [0,1]$ be a cut-off function on M, vanishing outside a compact subset of M. Consider a modified flow equation

$$\frac{d\zeta^{t,\phi}}{dt} = \phi(\zeta^{t,\phi})D_h(\zeta^{t,\phi}) = U(\zeta^{t,\phi})\big[\phi(\zeta^{t,\phi})h\big], \qquad \zeta^{0,\phi} = I_{P_o(M)}.$$

We rewrite this equation on $W_o(\mathbb{R}^n)$. Define

$$p_s^\phi = \phi(X_s)h_s - \int_0^s K^\phi(W) \circ dW_s, \qquad X = JW,$$

$$K^\phi(W) = \int_0^s \Omega_{U(X)_\tau}(\circ\, dW_\tau, \phi(X_\tau)h_\tau).$$

Note that these definitions are obtained from the old p and K by replacing h with the semimartingale $\phi(X)h$. The equation for $\xi^{t,\phi}(W) = J^{-1}\zeta^{t,\phi}(JW)$ is

$$\frac{d\xi^{t,\phi}}{dt} = p^\phi(\xi^{t,\phi}), \qquad \xi^{0,\phi} = I_{P_o(\mathbb{R}^n)}.$$

Equations (5) become

$$\begin{cases} \xi_s^{t,\phi} = \int_0^s A_\tau^{t,\phi}\, d\tau + \int_0^s O_\tau^{t,\phi}\, dW_\tau, \\ O^{t,\phi} = I - \int_0^t K^\phi(\xi^\lambda) O^\lambda\, d\lambda, \\ A^{t,\phi} = O^{t,\phi} \int_0^t O^{\lambda,\phi *} \left\{ \dot{h} - \frac{1}{2} \operatorname{Ric}_{U(J\xi^{\lambda,\phi})} \phi(\zeta^{\lambda,\phi}) h \right\} d\lambda. \end{cases} \tag{6}$$

By inspecting the definition of p^ϕ we see that these equations involve only the curvature and its derivatives on the support of ϕ. Therefore, we can apply Picard's iteration just as we did before for the compact case and claim that it has a unique solution $\{\xi^{t,\phi}\}$. Again the solution on $P_o(M)$ is obtained by $\zeta^{t,\phi}(X) = J\xi^{t,\phi}(J^{-1}X)$.

After these preliminary remarks, we are ready to prove our first main result.

THEOREM 2.2. *Let M be a geodesically and stochastically complete Riemannian manifold. Let $h \in \mathscr{H}$. Then there exists a unique set of measurable maps ($P_o(M)$-valued random variables)*

$$\zeta^t : P_o(M) \to P_o(M), \qquad t \in \mathbb{R}$$

with the following properties:

(1) $\{\zeta_s^t, s \in [0,1]\}$ *is an M-valued $\mathscr{B}(P_o(M))_*$-semimartingale for each* $t \in \mathbb{R}$;

(2) *for $\mathbb{P}$-almost all X, $t \mapsto \zeta^t(X)_s$ is C^∞ for fixed s and satisfies*

$$\frac{d\zeta^t(X)_s}{dt} = U(\zeta^t(X))_s h_s.$$

Proof. Let $\phi_N : M \to [0,1]$ be a smooth function such that $\phi_N = 1$ on $B(o;N)$, the geodesic ball of radius N, and $\phi_N = 0$ on $B(o;2N)^c$. Let $\xi^{t,N} = \xi^{t,\phi_N}$ and $\zeta^{t,N} = \zeta^{t,\phi_N}$ for simplicity. From the equation for $\zeta^{t,N}$, we see that

$$d(\zeta_s^{t,N} X, X_s) \leqslant |t||h_s| \leqslant |t||h|_{\mathscr{H}}.$$

This holds for all N. Recall that $r(\cdot)$ is the distance function from the reference point o. Now for a fixed positive L let

$$\sigma_L = \inf\{s \leqslant 1 \colon r(X_s) = L\}.$$

Then for all $s \leqslant \sigma_L$, all N and $|t| \leqslant T$ we have

$$r(\zeta_s^{t,N} X) \leqslant d(\zeta_s^{t,N} X, X_s) + r(X_s) \leqslant T|h|_{\mathscr{H}} + L.$$

In particular, if $N \geqslant T|h|_{\mathscr{H}} + L$, then $r(\zeta_s^{t,N} X) \leqslant N$. Thus the stopped process $\zeta_s^{t,N;L} \stackrel{\text{def}}{=} \zeta_{s\wedge\sigma_L}^{t,N}$ does not wander out of the geodesic ball $B(o;N)$, on which the cut-off function $\phi^N = 1$. It follows that $\{\zeta^{t,N;L}\}$ satisfies the same equation for all $N \geqslant T|h|_{\mathscr{H}} + L$. Note that in the s-direction, the equation only runs up to time σ_L, which may be strictly less than 1, see Remark 2.1. By the uniqueness, for all sufficiently large N we have $\zeta_s^{t,N;L} = \zeta_s^{t,N+1;L}$ for all $s \in [0,1]$, or equivalently, $\zeta_s^{t,N} = \zeta_s^{t,N+1}$ for all $s \leqslant \sigma_L$. Now because M is stochastically complete,

$$\mathbb{P}[\sigma_L = 1] \uparrow 1 \qquad \text{as} \quad L \uparrow \infty.$$

We define

$$\zeta_s^t = \lim_{N\to\infty} \zeta_s^{t,N} \qquad \text{on} \quad \{\sigma_L = 1\}.$$

The properties of $\{\zeta^t\}$ stated in the theorem are inherited from the corresponding properties of $\{\zeta^{t,N}\}$. The proof is completed. ∎

3. QUASI-INVARIANCE OF THE WIENER MEASURE

Throughout this section we assume that there is a constant C such that

$$|\mathrm{Ric}_M(x)| \leqslant C\{1 + r(x)\}. \tag{7}$$

Let $\{\zeta^t\}$ be the semimartingale solution of the flow equation for D_h on $P_o(M)$ constructed in Section 2 and $\{\xi^t\}$ the corresponding solution on $W_o(\mathbb{R}^n)$. We will show that the law of ζ^t is mutually absolutely continuous with respect to the Wiener measure $\mathbb{P}$ on $P_o(M)$, namely

$$\mathbb{P}^t \stackrel{\text{def}}{=} \mathbb{P} \circ (\zeta^t)^{-1} \approx \mathbb{P}.$$

This is equivalent to showing that

$$\mathbb{Q}^t \stackrel{\text{def}}{=} \mathbb{Q} \circ (\xi^t)^{-1} \approx \mathbb{Q},$$

where $\mathbb{Q}$ is the Wiener measure on $W_o(\mathbb{R}^n)$. We need the following criterion.

PROPOSITION 3.1. *Let* $z = \{z_s, s \in [0,1]\}$ *be a semimartingale on the filtered probability space* $(W_o(\mathbb{R}^n), \mathscr{B}_*, \mathbb{Q})$ *such that*

$$z_s = \int_0^s A_\tau \, d\tau + \int_0^s O_\tau \, dW_\tau,$$

where A is a $\mathscr{B}_*$-adapted, $\mathbb{R}^n$-valued integrable process and O is a $\mathscr{B}_*$-adapted, $O(n)$-valued process. If

$$\mathbb{E}^{\mathbb{Q}}\exp\left[\frac{1}{2}\int_0^1 |A_s|^2\,ds\right] < \infty, \tag{8}$$

then the law $\mathbb{Q}^z \stackrel{\text{def}}{=} \mathbb{Q}\circ z^{-1}$ is mutually absolutely continuous with respect to the Wiener measure $\mathbb{Q}$.

Proof. Define the local exponential martingale

$$e_s = \exp\left[\int_0^s \langle -A_\tau, O_\tau\,dW_\tau\rangle - \frac{1}{2}\int_0^s |A_\tau|^2\,d\tau\right].$$

Then it is well known that (8) implies $\mathbb{E}e_1 = 1$ (see [6, p. 152]). Define a new probability measure $\tilde{\mathbb{Q}}$ on $W_o(\mathbb{R}^n)$ by $d\tilde{\mathbb{Q}}/d\mathbb{Q} = e_1$. By Girsanov's theorem, z is a Brownian motion under $\tilde{\mathbb{Q}}$. Let $\mathscr{B}^z = z^{-1}(\mathscr{B}_1)$ be the σ-field generated by z. General measure theory guarantees the existence of a measurable function $Q\colon W_o(\mathbb{R}^n) \to [0,\infty)$ such that

$$\mathbb{E}^{\tilde{\mathbb{Q}}}[e_1^{-1}|\mathscr{B}^z] = Q(z).$$

Now for any nonnegative measurable function f on $W_o(\mathbb{R}^n)$,

$$\mathbb{E}^{\mathbb{Q}^z}f = \mathbb{E}^{\mathbb{Q}}f(z) = \mathbb{E}^{\tilde{\mathbb{Q}}}[f(z)e_1^{-1}] = \mathbb{E}^{\tilde{\mathbb{Q}}}[f(z)Q(z)] = \mathbb{E}^{\mathbb{Q}}[fQ].$$

The last equality holds because the law of z under $\tilde{\mathbb{Q}}$ is $\mathbb{Q}$. This shows that $\mathbb{Q}^z \approx \mathbb{Q}$ and in fact $d\,\mathbb{Q}^z/d\,\mathbb{Q} = Q$. ∎

In the following, we will use C to denote a constant depending on h and M, whose value may differ from one appearance to another. From (5), (7), and $|h_s| \leqslant |h|_{\mathscr{H}}$ we have

$$|A_s^t| \leqslant \int_0^t \{|\dot{h}_s| + C + Cr(\zeta_s^\lambda)\}\,d\lambda.$$

Now let

$$X_* = \max_{0\leqslant s\leqslant 1} r(X_s)$$

be the maximum distance traveled by the Brownian motion X. We have

$$r(\zeta_s^t X) \leqslant t|h_s| + r(X_s) \leqslant t|h|_{\mathscr{H}} + X_*.$$

Hence for $|t| \leqslant 1$,

$$|A_s^t| \leqslant Ct\{|\dot{h}_s| + |h|_{\mathscr{H}} + X_*\}$$

and

$$\frac{1}{2}\int_0^1 |A_s^t|^2 ds \leqslant Ct^2\{1 + X_*^2\}. \tag{9}$$

In order to apply Proposition 3.1 we need to investigate the exponential integrability of X_*.

LEMMA 3.2. *Under assumption* (7) *on the Ricci curvature we have*

$$\mathbb{E}\, e^{X_*^2/10} < \infty.$$

Proof. It is a well-known fact in stochastic analysis that the radial process $r_s = r(X_s)$ has the decomposition (see [7])

$$r_s = \beta_s + \frac{1}{2}\int_0^s \Delta_M r(X_\tau)\, d\tau - L_s,$$

where β is a one-dimensional Brownian motion, L is a nondecreasing process which increases only when X_s is on the cut-locus of o, and Δ_M is the Laplace–Beltrami operator on M. By Itô's formula, we have

$$r_s^2 = 2\int_0^s r_\tau\, dr_\tau + \langle r\rangle_s.$$

Hence, noting that L is nondecreasing we have

$$r_s^2 \leqslant 2\int_0^s r_\tau\, d\beta_\tau + \int_0^s r_\tau \Delta_M r(X_\tau)\, d\tau + s. \tag{10}$$

Fix a $K \geqslant 1$ and let

$$\sigma_K = \inf\{s : r_s = K\}.$$

If $r(x) \leqslant K$, then the Ricci curvature is bounded by $C(1+K)$. By the Laplacian comparison theorem (see [5]),

$$\Delta_M r(x) \leqslant (n-1)\sqrt{C(1+K)}\coth\sqrt{C(1+K)}r(x).$$

For $\tau \leqslant \sigma_K$, using the inequality $c\coth c \leqslant 1 + c$ for all $c \geqslant 0$ and the fact that $r_\tau \leqslant K$, we have

$$r_\tau \Delta_M r(X_\tau) \leqslant 1 + \sqrt{C(1+K)}r_\tau \leqslant C_1 K^{3/2}.$$

Letting $s = \sigma_K$ in (10), we have

$$K^2 \leqslant 2 \int_0^{\sigma_K} r_\tau \, d\beta_\tau + CK^{3/2}\sigma_K.$$

From this inequality we see that $\sigma_K \leqslant 1$ implies

$$2 \int_0^{\sigma_K} r_\tau \, d\beta_\tau \geqslant K^2 - CK^{3/2} \geqslant \frac{K^2}{2}$$

for sufficiently large K. On the other hand, by Lévy's criterion there is a one-dimensional Brownian motion W such that

$$\int_0^{\sigma_K} r_\tau \, d\beta_\tau = W_\eta, \qquad \eta = \int_0^{\sigma_K} r_\tau^2 \, d\tau \leqslant K^2.$$

It follows that $\sigma_K \leqslant 1$ implies

$$\max_{0 \leqslant s \leqslant K^2} W_s \geqslant W_\eta \geqslant \frac{K^2}{4}.$$

Since $\max_{0 \leqslant s \leqslant K^2} W_s$ has the same distribution as $K|W_1|$, we have

$$\mathbb{P}\{X_* \geqslant K\} \leqslant \mathbb{P}\{\sigma_K \leqslant 1\} \leqslant \mathbb{P}\left\{|W_1| \geqslant \frac{K}{4}\right\} \leqslant Ce^{-K^2/8}.$$

This implies immediately that $\mathbb{E}\, e^{X_*^2/10}$ is finite. ∎

Remark 3.3. In the above proof, we only used the lower bound $\mathrm{Ric}_M(x) \geqslant -C\{1 + r(x)\}$. The exponential integrability of X_*^2 can be proved under much more relaxed growth condition on the Ricci curvature; for example, it holds when $\mathrm{Ric}_M(x) \geqslant -C\{1 + r(x)^2\}$ (see [8, p. 128]). But this does not seem to lead to any improvement on our final result in the next theorem.

We are ready to prove the second main result of this paper.

THEOREM 3.4. *Let M be a complete Riemannian manifold. Suppose that there is a constant C such that*

$$|\mathrm{Ric}_M(x)| \leqslant C\{1 + r(x)\}.$$

Then for any $h \in \mathscr{H}$, the Wiener measure is quasi-invariant under the flow $\{\zeta_h^t\}$ generated by D_h, namely $\mathbb{P}^t \stackrel{\text{def}}{=} \mathbb{P} \circ (\zeta_h^t)^{-1}$ and $\mathbb{P}$ are mutually absolutely continuous.

Proof. From (9) and Lemma 3.2 we have

$$\mathbb{E} \exp\left[\frac{1}{2}\int_0^1 |A_s^t|^2\, ds\right] < \infty$$

for all sufficiently small $|t|$. By Lemma 3.1 we have $\mathbb{Q}^t \approx \mathbb{Q}$ and $\mathbb{P}^t \approx \mathbb{P}$ for small $|t|$. It remains to show that this implies that the equivalence holds for all t.

Suppose that $\mathbb{P}^t \approx \mathbb{P}$ for all $0 \leqslant t \leqslant t_0$. The composition $\zeta^{t-t_0} \circ \zeta^{t_0}$ now makes sense as a $P_o(M)$-valued random variable because ζ^{t-t_0} is $\mathbb{P}$-almost everywhere defined and $\mathbb{P}^{t_0} \approx \mathbb{P}$. If we define $\tilde{\zeta}^t = \zeta^t$ for $0 \leqslant t \leqslant t_0$ and $\tilde{\zeta}^t = \zeta^{t-t_0} \circ \zeta^{t_0}$ for $t_0 \leqslant t \leqslant 2t_0$, then it is easy to see that $\{\tilde{\zeta}^t, 0 \leqslant t \leqslant 2t_0\}$ satisfies the same flow equation as $\{\zeta^t, 0 \leqslant t \leqslant 2t_0\}$. By the uniqueness we have $\tilde{\zeta}^t = \zeta^t$ for $0 \leqslant t \leqslant 2t_0$, or $\zeta^t = \zeta^{t-t_0} \circ \zeta^{t_0}$ for $t_0 \leqslant t \leqslant 2t_0$. Now $\mathbb{P}^t \approx \mathbb{P}$ for $t_0 \leqslant t \leqslant 2t_0$ follows from $\mathbb{P}^{t_0} \approx \mathbb{P}$ and $\mathbb{P}^{t-t_0} \approx \mathbb{P}$. In fact, if we let $R : P_o(M) \to \mathbb{R}$ be a measurable function such that

$$\mathbb{E}\left[\frac{d\mathbb{P}^{t_0}}{d\mathbb{P}}\middle|\mathscr{B}^{\zeta^{t-t_0}}\right] = R(\zeta^{t-t_0}),$$

where $\mathscr{B}^{\zeta^{t-t_0}} \stackrel{\text{def}}{=} (\zeta^{t-t_0})^{-1}(\mathscr{B}_1)$, then

$$\frac{d\mathbb{P}^t}{d\mathbb{P}} = \left[\frac{d\mathbb{P}^{t-t_0}}{d\mathbb{P}}\right] R, \qquad t_0 \leqslant t \leqslant 2\, t_0.$$

This method can be continued to show that $\mathbb{P}^t \approx \mathbb{P}$ for all t. ∎

We end this paper with two remarks.

(1) The need for an upper bound of the Ricci curvature runs counter to intuition: it seems that a complete manifold whose Ricci curvature is bounded from below by, say, a negative constant $-K$ should be more likely to have the quasi-invariance property than the simply connected manifold of constant curvature $-K$.

(2) It is unlikely that the quasi-invariance property holds for all geodesically complete and stochastically complete manifolds without any further restrictions.

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