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Quasi-Invariance of the Wiener Measure on the Path Space over a Compact Riemannian Manifold*

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We study a quasi-invariance property of the Wiener measure on the path space over a compact Riemannian manifold which generalizes the well-known Cameron–Martin theorem for euclidean space. This property is used to prove an integration by parts formula for the gradient operator. We use the integration by parts formula to compute explicitly the Ornstein–Uhlenbeck operator in the path space.

1. INTRODUCTION

The general setting of this paper is as follows. Let M be a compact Riemannian manifold M of dimension d. We use $O(M)$ to denote the bundle of orthonormal frames over M. Let $o \in M$ be a fixed point on M and $u_o \in O(M)$ a fixed orthonormal frame over o. We will use $W_o(M)$ and $W_o(O(M))$ to denote the (pinned) path spaces based on M and $O(M)$, namely the spaces of continuous functions from the unit interval $[0, 1]$ to M and $O(M)$ starting from o and u_o respectively. The notations $W_o^\infty(M)$ and $W_o^\infty(O(M))$ denote the subset of smooth paths of $W_o(M)$ and $W_o(O(M))$ respectively. Similar notations apply when M is replaced by $\mathbb{R}^d$, in which case o is taken to be the origin.

Let $\gamma \in W_o^\infty(M)$. An element $h \in W_o(\mathbb{R}^d)$ determines a vector field $D_h(\gamma)$ along γ by letting $D_h(\gamma)_s = U(\gamma)_s h_s$, where $s \mapsto U(\gamma)_s$ is the horizontal lift of γ to $O(M)$ with the initial condition $U(\gamma)_0 = u_o$. Thus each $h \in W_o(\mathbb{R}^d)$ defines a vector field D_h on the smooth path space $W_o^\infty(M)$.

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Let F be a real-valued function on the smooth path space $W_o^\infty(M)$. We define the directional derivative at γ along the direction h by

$$D_h F(\gamma) = \lim_{t \to 0} \frac{F(\zeta_h^t \gamma) - F(\gamma)}{t} \tag{1.1}$$

if the limit exists. Here $\{\zeta_h^t \gamma, t \geqslant 0\}$ is an integral curve of the vector field D_h starting from γ, namely,

$$\frac{d\zeta_h^t \gamma}{dt} = D_h(\gamma), \qquad \zeta_h^0 \gamma = \gamma.$$

The gradient $DF(\gamma)$ at γ is defined as follows. Let $\mathbb{H}$ be the $\mathbb{R}^d$-valued Cameron–Martin space, namely, the completion of the space of smooth paths in $W_o(\mathbb{R}^d)$ with respect to the Hilbert norm

$$|h|_{\mathbb{H}} = \sqrt{\sum_{i=1}^{d} \int_0^1 |\dot{h}^i(s)|^2 \, ds}.$$

The gradient $DF(\gamma)$, if it exists, is the unique element in $\mathbb{H}$ satisfying the condition

$$\langle DF(\gamma), h \rangle_{\mathbb{H}} = D_h F(\gamma) \qquad \text{for all} \quad h \in \mathbb{H}. \tag{1.2}$$

The directional derivative operator D_h and the gradient operator D, properly extended to a closed operator on $L^2(W_o(M), \nu)$ with ν the Wiener measure, will play a role similar to the usual gradient operator on a finite-dimensional manifold and will be used to define the Ornstein–Uhlenbeck operator on $W_o(M)$, which generalizes the usual Ornstein–Uhlenbeck operator on euclidean path spaces. Our analysis on the path space $W_o(M)$ is based on the Wiener measure ν, the law of the Riemannian Brownian motion on M starting from o. We will show in what sense the vector field D_h (with $h \in \mathbb{H}$) generates a flow $\{\zeta_h^t, t \in \mathbb{R}^1\}$ on the path space $W_o(M)$. For a successful integration of the gradient operator D and the Wiener measure ν into an analytical theory of the path space $W_o(M)$, the quasi-invariance of ν under the flow $\{\zeta_h^t, t \in \mathbb{R}^1\}$ is a highly desirable property. We say that the Wiener measure ν is quasi-invariant under the flow $\{\zeta_h^t, t \in \mathbb{R}^1\}$ if for all $t \in \mathbb{R}^1$, the measures $\nu_h^t = \nu \circ (\zeta_h^t)^{-1}$ and ν are mutually absolutely continuous. It is helpful to point out at this point that the existence and the quasi-invariance of the flow are two closely related problems and have to be dealt with simultaneously. In the case where the base manifold $M = \mathbb{R}^d$, the quasi-invariance property is the well-known Cameron–Martin theorem for the euclidean Wiener measure.

The problem of existence and quasi-invariance of the flow generated by D_h has a long history, but the first significant progress for a general compact Riemannian manifold M was made by Driver [2], who proved the quasi-invariance property of the flow $\{\zeta_h^t, t \in \mathbb{R}^1\}$ for all Lipschitz h. To extend this quasi-invariance property to its natural domain, namely for all $h \in \mathbb{H}$, is the main task of the present work. For the history of the problems discussed here, see the relevant passages and the references in Driver [2].

The quasi-invariance property of the Wiener measure can be used to prove an integration by parts formula for the gradient operator $D: L^2(\nu) \to L^2(\mathbb{H}; \nu)$ on the path space $W_o(M)$ in a natural way, where $L^2(\nu)$ and $L^2(\mathbb{H}; \nu)$ are the spaces of $\mathbb{R}^1$-valued and $\mathbb{H}$-valued square integrable functions on $W_o(M)$ respectively. We will define the directional derivative operator D_h as in (1.1) and the gradient operator D as in (1.2) on cylindrical functions and we will show that D_h and D are closable. We will prove integration by parts formulas for them by computing explicitly their adjoint D_h^* and D^* in terms of stochastic integrals. The closability of D implies the same for the associated Dirichlet form

$$\mathscr{E}(F, F) = \int_{W_o(M)} |DF(\gamma)|_{\mathbb{H}}^2 \, \nu(d\gamma). \tag{1.3}$$

As an application of the explicit formula for the adjoint operator D^*, we will derive a formula for the Ornstein–Uhlenbeck operator, namely the self-adjoint operator corresponding to the Dirichlet form (1.3).

For general discussions on stochastic and geometric analysis on path and loop spaces, see Fang and Malliavin [5], Malliavin [10], and Malliavin and Malliavin [11] and the literature cited there. We point out that directional derivatives D_h and their adjoint D_h^* for $h \in \mathbb{H}$ are studied in Driver [2] via approximation of h by a sequence of smooth functions. The closability of the gradient operator and the integration by parts formula were proved in Fang and Malliavin [5] without using the quasi-invariance property of the Wiener measure. The closability of the Dirichlet form (1.3) was proved in Driver and Röckner [4], where the existence of the Ornstein–Uhlenbeck process on a path space was also proved.

The approach we adopt in the present work is as follows. The Itô map $J: W_o(\mathbb{R}^d) \mapsto W_o(M)$ maps a euclidean Brownian motion to a Riemannian Brownian motion on M, i.e., $\nu = \mu \circ J^{-1}$, where μ is the Wiener measure on $W_o(\mathbb{R}^d)$. The image of the vector field D_h under J^{-1}, which we denote by $J_*^{-1} D_h$, can be computed explicitly. The vector field $J_*^{-1} D_h$ on $W_o(\mathbb{R}^d)$ can be identified with an $\mathbb{R}^d$-valued semimartingale denoted by p_h. Let $\xi_h^t = J^{-1} \circ \zeta_h^t \circ J$. Then $\{\xi_h^t, t \in \mathbb{R}^1\}$ is the image of the flow $\{\zeta_h^t, t \in \mathbb{R}^1\}$ on the path space $W_o(\mathbb{R}^d)$ and should be the flow generated by the p_h as a vector field on $W_o(\mathbb{R}^d)$, namely, it satisfies the integral equation

$$\xi_h^t \omega = \omega + \int_0^t p_h(\xi_h^\lambda \omega)\, d\lambda. \tag{1.4}$$

This should be regarded as an equation in the space of semimartingales on the probability space $(W_o(\mathbb{R}^d), \mathscr{B}, \mu)$. We will single out a class of semimartingales (denoted by $SM(h)$ in the paper) so that under a suitably defined norm on this class, (1.4) can be solved by Picard's iteration method.

The existence of the flow generated by D_h and the quasi-invariance of the Wiener measure under the flow can also be proved by Euler's polygonal method and the infinitesimal quasi-invariance of the Wiener measure. This is done in Hsu [7].

The paper is organized as follows. In Section 2 we compute $p_h = J_*^{-1} D_h$, the image of the vector field D_h in $W_o(\mathbb{R}^d)$ under the development map J. In Section 3 we prove the existence of the flow $\{\xi_h^t, t \in \mathbb{R}^1\}$ generated by p_h and show that the usual euclidean Wiener measure μ on $W_o(\mathbb{R}^d)$ is quasi-invariant under this flow. In Section 4, we show how to transfer the euclidean flow to the flow on the path space $W_o(M)$ generated by D_h. In Section 5 we discuss the operators D_h and D, compute their adjoints, and prove the corresponding integration by parts formulas. In Section 6, we apply the results in Section 5 to give an explicit formula for the Ornstein–Uhlenbeck operator L on $W_o(M)$ and show that the set of cylindrical functions lies in the domain of definition of L.

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Added in the final version. O. Enchev and D. Stroock have worked out another approach to the problems treated in this paper. Their results will be published in an article now in preparation.

2. A Geometric Computation

The purpose of this section is to motivate the flow equation (3.2) we will solve in the next section. The computations are therefore carried out on smooth paths.

We continue to use the notations introduced in the preceding section. We assume that the compact Riemannian manifold M is equipped with a connection compatible with the Riemannian metric but not necessarily torsion-free. Let $\pi: O(M) \to M$ be the canonical projection. Each frame

$u \in O(M)$ can be regarded as a linear isometry $u: \mathbb{R}^d \to T_{\pi(u)}M$, the tangent space at $\pi(u)$. We will use the same letter π for the canonical projection from $W_o(O(M))$ to $W_o(M)$. The $\mathbb{R}^d$-valued 1-form θ on $O(M)$ defined by $\theta(X) = u^{-1}\pi_*(X)$ is called the canonical 1-form on $O(M)$. The connection we fixed on M gives rise to a decomposition of each tangent space $T_u O(M)$ into the direct sum of a horizontal subspace and a vertical subspace. Let $\{H_i, 1 \leqslant i \leqslant d\}$ be the canonical horizontal vector fields on $O(M)$. By definition, H_i is the unique horizontal vector field such that $\pi_* H_i = ue_i$ at each $u \in O(M)$, where e_i is the ith unit coordinate vector in $\mathbb{R}^d$. If $h \in \mathbb{R}^d$ we set $Hh = \sum_{i=1}^d H_i h^i$ and the value of the vector field at $u \in O(M)$ is denoted by $H_u h$; in other words, $H_u h$ is the unique horizontal vector field such that $\pi_*(H_u h) = uh$. The connection on M also gives rise to an $o(d)$-valued connection form ω on $O(M)$, where $o(d)$ is the set of $d \times d$ antisymmetric matrices, i.e., the Lie algebra of the Lie group $O(d)$ of $d \times d$ orthogonal matrices. The torsion form Θ, defined by the first structure equation (see below), is a $\mathbb{R}^d$-valued 2-form on $O(M)$. The curvature tensor Ω, defined by the second structure equation (see below), is a $o(d)$-valued 2-form on $O(M)$. See Bishop and Crittenden [1] or Kobayashi and Nomizu [9] for differential geometrical details.

Let $\omega \in W_o^\infty(\mathbb{R}^d)$. The development $I = I(\omega)$ of ω in $O(M)$ is a horizontal path in $W_o^\infty(O(M))$ satisfying the ordinary differential equation

$$\frac{dI_s}{ds} = H_{I_s} \frac{d\omega_s}{ds}, \qquad I_0 = u_o.$$

The projection $J\omega = \pi I(\omega)$ is a path on M. The map

$$J: W_o^\infty(\mathbb{R}^d) \to W_o^\infty(M)$$

is in fact invertible. This can be seen from the following argument. Suppose that $\gamma \in W_o^\infty(M)$, let $U = U(\gamma)$ be the horizontal lift of γ, i.e.,

$$\frac{dU_s}{ds} = H_{U_s} U_s^{-1} \frac{d\gamma_s}{ds}, \qquad U_0 = u_o.$$

It is the unique horizontal path in $O(M)$ starting from u_o such that $\pi U = \gamma$. Then $\omega = J^{-1}\gamma \in W_o^\infty(\mathbb{R}^d)$ is given by the line integral.

$$\omega_s = \int_{U[0,s]} \theta = \int_0^s \theta(dU_\tau)$$

and is called the development of γ in $\mathbb{R}^d$. Note that $I(\omega) = U(\gamma)$, i.e., $I = U \circ J$ if we use $U: W_o^\infty(M) \to W_o^\infty(O(M))$ to denote the operation of horizontal lift.

Suppose that $\gamma \in W_o^\infty(M)$ and $\omega = J^{-1}\gamma$ its parallel development in $\mathbb{R}^d$. Let $h \in W_o^\infty(\mathbb{R}^d)$ and consider the vector field on $W_o^\infty(M)$ defined by

$$D_h(\gamma)_s = U(\gamma)_s h_s.$$

Let $\{\zeta_h^t \gamma, t \geqslant 0\}$ be the flow generated by D_h, i.e.,

$$\frac{\partial(\zeta_h^t \gamma)_s}{\partial t} = D_h(\zeta^t \gamma)_s, \qquad \zeta_h^0 = \gamma. \tag{2.1}$$

Clearly

$$\xi_h^t \omega = J^{-1} \zeta_h^t J_\omega, \qquad t \in \mathbb{R}^1,$$

is a flow on $W_o^\infty(\mathbb{R}^d)$. We want to compute $J_*^{-1} D_h$, the pullback of the vector field D_h to $W_o^\infty(\mathbb{R}^d)$, namely

$$[J_*^{-1} D_h(\omega)]_s = \left.\frac{\partial(\xi_h^t \omega)_s}{\partial t}\right|_{t=0}.$$

For each $\omega \in W_o^\infty(\mathbb{R}^d)$, the above relation defines a vector field along ω, which we identified with an $\mathbb{R}^d$-valued function on $[0, 1]$ denoted by $p_h(\omega)$.

THEOREM 2.1. *Suppose that $\gamma \in W_o^\infty(M)$ and $h \in W_o^\infty(\mathbb{R}^d)$. The pullback $J_*^{-1} D_h$ of the vector field D_h under the development map J^{-1}: $W_o^\infty(M) \to W_o^\infty(\mathbb{R}^d)$ is given at $\omega = J^{-1}\gamma$ by the following $\mathbb{R}^d$-valued function on $[0, 1]$:*

$$p_h(\omega)_s = h_s - \int_0^s \Theta_{U_\tau}(H d\omega_\tau, H h_\tau) - \int_0^s K_h(\omega)_\tau \, d\omega_\tau, \tag{2.2}$$

where $U = U(\gamma)$ is the horizontal lift of γ in $O(M)$ and

$$K_h(\omega)_s = \int_0^s \Omega_{U_\tau}(H d\omega_\tau, H h_\tau).$$

The rest of this section is devoted to the proof of this theorem. We will use the following three facts:

- *Exterior differentiation formula.* If ϕ is a 1-form then the exterior differentiation $d\phi$ is a 2-form defined by

$$d\phi(S, T) = S\phi(T) - T\phi(S) - \phi([S, T]),$$

where $[S, T]$ is the Lie bracket of the vector fields S and T.

• *First structural equation.* The differential of the canonical horizontal 1-form θ is given by

$$d\theta = -\omega \wedge \theta + \Theta,$$

where Θ is the torsion form.

• *Second structural equation.* The differential of the connection 1-form ω is given by

$$d\omega = -\omega \wedge \omega + \Omega,$$

where Ω is the curvature form.

For discussions on these facts see Bishop and Crittenden [1] or Kobayashi and Nomizu [9].

In the following computation we will omit the subscripts h. Let $U^t = U(\zeta^t\gamma)$ be the horizontal lift of $\zeta^t\gamma$ and define

$$S = \frac{\partial U_s^t}{\partial s}, \qquad T = \frac{\partial U_s^t}{\partial t}, \qquad N = \frac{\partial(\xi^t\omega)_s}{\partial s}.$$

Then we can write

$$p_h(\omega)_s = \int_0^s \left\{ \frac{\partial N_\tau^t}{\partial t} \bigg|_{t=0} \right\} d\tau. \tag{2.3}$$

By $\xi^t = J \circ \zeta^t \circ J^{-1}$ we see that $U^t = I(\xi^t\omega)$ is the development of $\xi^t\omega$ in $O(M)$, hence $S = HN$, which is equivalent to $N = \theta(S)$. Differentiating with respect to t, we have

$$\frac{\partial N}{\partial t} = T\theta(S).$$

By the exterior differentiation formula,

$$T\theta(S) = S\theta(T) - \theta([S, T]) - d\theta(S, T).$$

Clearly, $[S, T] = 0$. On the other hand, since $\pi(U^t) = \zeta^t\gamma$, at $t = 0$ we have $\pi_*(T) = D_h(\gamma) = U(\gamma)h$ by (2.1). This means

$$\text{the horizontal component of } T = Hh, \quad \text{or} \quad \theta(T) = h \tag{2.4}$$

Hence

$$\frac{\partial N_s^t}{\partial t}\bigg|_{t=0} = \dot{h}_s - d\theta(S, T). \tag{2.5}$$

We now compute $d\theta(S, T)$ by the first structural equation. We have $\omega(S) = 0$ because S is horizontal and the connection form ω is a vertical form. We also have $\theta(S) = N$ as before. Hence the first structural equation gives

$$d\theta(S, T) = \Theta(S, T) + \omega(T)N. \tag{2.6}$$

We now compute $\omega(T)$. Using $[S, T] = 0$ and $\omega(S) = 0$ we have by the exterior differentiation formula and the second structural equation

$$S\omega(T) = d\omega(S, T) = \Omega(S, T).$$

Integration with respect to s, we have

$$\omega_{U_s^t}(T) = \int_0^s \Omega_{U_\tau^t}(S, T)\, d\tau. \tag{2.7}$$

Let $t = 0$ in this relation. We have seen that the horizontal component of T is Hh. We also have $S = H\dot{\omega}_s$ because $U(\gamma) = I(\omega)$. Hence at $t = 0$ the $\Omega_{U_\tau^t}(S, T)$ in (2.7) can be replaced by $\Omega_{U_\tau}(H\dot{\omega}_\tau, Hh_\tau)$ because Ω is a horizontal form. Therefore we have

$$\omega(T) = K_h. \tag{2.8}$$

From (2.5)–(2.8), and the fact that $N = \dot{\omega}_s$ at $t = 0$ we have

$$\left.\frac{\partial N_s^t}{\partial t}\right|_{t=0} = \dot{h}_s - \Theta(H\dot{\omega}_s, Hh_s) - K_h(\omega)\dot{\omega}_s.$$

Integrating with respect to s and using (2.3) we obtain the theorem.

3. Flows on Euclidean Path Space

In this section, we will work in the probability space $(W_o(\mathbb{R}^d), \mathscr{B}, \mu)$, where $\mathscr{B}$ is the Borel σ-field on the path space $W_o(\mathbb{R}^d)$ and μ is the Wiener measure. The canonical filtration of σ-fields on $W_o(\mathbb{R}^d)$ will be denoted by $\{\mathscr{B}_s, 0 \leqslant s \leqslant 1\}$. The coordinate process $\{\omega_s, 0 \leqslant s \leqslant 1\}$ is a $\mathscr{B}_s$-adapted Brownian motion.

In Section 2 we have introduced the function $p_h(\omega)$ in Theorem 2.1. We regard p_h as a vector field on the path space $W_o(M)$ whose value at ω is $p_h(\omega)$. The purpose of this section is to prove the existence of a flow on $W_o(M)$ generated by the vector field p_h (in a sense to be made precise later) under an antisymmetry assumption on the torsion form (see below) and the assumption that $h \in \mathbb{H}$.

Throughout this section we will fix an element $h \in \mathbb{H}$, the $\mathbb{R}^d$-valued Cameron–Martin space. We use $SM(h)$ to denote the space of $\mathbb{R}^d$-valued and $\mathscr{B}_s$-adapted continuous semimartingales z of the special form

$$z_s = \int_0^s A_\tau \, d\tau + \int_0^s O_\tau \, d\omega_\tau, \tag{3.1}$$

where O is an $O(d)$-valued, $\mathscr{B}_s$-adapted process and A is a $\mathbb{R}^d$-valued, $\mathscr{B}_s$-adapted process such that

$$|A_s| \leqslant K\{|\dot{h}_s| + 1\}$$

for some (nonrandom constant) K. Note that since A is bounded by a deterministic function in $L^2[0, 1]$, the law of z in $W_o(\mathbb{R}^d)$ is mutually absolutely continuous with respect to the Wiener measure μ by the usual Cameron–Martin theorem for $\mathbb{R}^d$.

Recall that we assume that the connection is compatible with the Riemannian metric, but not necessarily torsion-free. From now on, the following assumption introduced by Driver [2] will be in force:

The torsion of the connection is antisymmetric, i.e., for all $Z \in TM$, the matrix

$$\Theta(H, Z) = \{\Theta^i(H_j, Z)\} \in o(d).$$

We define the semimartingale p_h on $(W_o(\mathbb{R}^d), \mathscr{B}, \mu)$ simply by replacing the integrals in (2.2) by Stratonovich stochastic integrals

$$\begin{cases} p_h(\omega)_s = h_s - \int_0^s \Theta_{U_\tau}(H \circ d\omega_\tau, Hh_\tau) - \int_0^s K_h(\omega)_\tau \circ d\omega_\tau, \\ K_h(\omega)_s = \int_0^s \Omega_{U_\tau}(H \circ d\omega_\tau, Hh_\tau), \end{cases} \tag{3.2}$$

where $U = I(\omega)$ is the stochastic development of z in $O(M)$ determined by

$$dU_s = H_{U_s}, \qquad U_0 = u_o. \tag{3.3}$$

Since p_h is defined μ-a.s. and the law of a semimartingale $z \in SM(h)$ is equivalent to μ, the composition $p_h \circ z = p_h(z)$ is a well defined semimartingale.

For the rest of this section, $h \in \mathbb{H}$ is fixed and we will drop the subscripts h if doing so causes no confusion.

Let us rewrite $p(z)$ in the Itô form. Let z be given by (3.1). After a straightforward computation, we obtain

$$\begin{cases} p(z)_s = h_s - \int_0^s a(z)_\tau \, d\tau - \int_0^s \langle b(z)_\tau, dz_\tau \rangle, \\ a(z)_s = \frac{1}{2} H_i \Theta_{U_s}(H_i, Hh_s) + \frac{1}{2} \mathrm{Ric}_{U_s}(Hh_s), \\ b(z)_s = \Theta_{U_s}(H, Hh_s) + K(z)_s. \end{cases} \tag{3.4}$$

b can be further written as

$$b(z)s = \Theta_{U_s}(H, Hh_s) + \tfrac{1}{2}\int_0^s H_i \Omega_{U_\tau}(H_i, Hh_\tau)\, d\tau$$
$$+ \int_0^s \Omega_{U_\tau}(HA_\tau d\tau, Hh_\tau) + \int_0^s \Omega_{U_\tau}(HO_\tau d\omega_\tau, Hh_\tau). \qquad (3.5)$$

Here $\mathrm{Ric}_U(\cdot)$ is the Ricci curvature tensor and is regarded as an $\mathbb{R}^d$-valued horizontal 1-form on $O(M)$ defined by

$$\mathrm{Ric}_U(Z)^i = \sum_{j=1}^n \Omega_U^{ij}(H_j, Z).$$

Note that our basic antisymmetry assumption on the torsion form Θ implies that $b(z)_\cdot$ is antisymmetric.

Theorem 3.1. *Suppose that $h \in \mathbb{H}$. There exists a unique family of semimartingales $\{\xi_h^t, t \in \mathbb{R}^1\}$ such that*

(i) *$\xi_h^t \in SM(h)$ for all $t \in \mathbb{R}^1$ and $\xi^0\omega = \omega$; hence the law of ξ_h^t is equivalent to μ;*

(ii) *For μ-almost all ω, the function $t \mapsto \xi_h^t(\omega)$ is a $W_o(\mathbb{R}^d)$-valued continuous function;*

(iii) *There exists a continuous version of $\{p_h(\xi_h^t), t \in \mathbb{R}^1\}$ such that μ-almost surely, $\{\xi_h^t, t \in \mathbb{R}^1\}$ satisfies the equation*

$$\xi_h^t\omega = \omega + \int_0^t p_h(\xi_h^\lambda\omega)\, d\lambda. \qquad (3.6)$$

Proof. The basic strategy is to solve (3.6) by Picard's iteration. We divide the proof into several steps.

(a) *An equivalent formulation.* Consider the equations

$$\begin{cases} O^t = I - \int_0^t b(\xi^\lambda)\, O^\lambda\, d\lambda, \\ A^t = \dot{h}t - \int_0^t a(\xi^\lambda)\, d\lambda - \int_0^t b(\xi^\lambda)\, A^\lambda d\lambda, \\ \xi^t = \int_0^s A_\tau^t d\tau + \int_0^s O_\tau^t d\omega_\tau. \end{cases} \qquad (3.7)$$

I is the identity matrix. Suppose that $\{\xi^t, t \in \mathbb{R}^1\}$ is given by the third equation, then $\{A^t, O^t, t \in \mathbb{R}^1\}$ satisfy the first and second equations, respectively. This fact follows from (3.4) by a simple computation. Note that we can write A^t in terms of O^t:

$$A^t = O^t \int_0^t [O^\lambda]^{-1} \{\dot{h} - a(x^\lambda)\}\, d\lambda. \qquad (3.8)$$

Thus it is enough to solve for O^t.

(b) *Picard's iteration.* We may assume that $|t| \leqslant T$ for some fixed positive T. The letter C will denote constants whose actual values may vary from one appearance to another.

Let $A^{t,0}=0$, $O^{t,0}=I$. Consider the iterative equations

$$\begin{cases} O^{t,n} = I - \int_0^t b(\xi^{\lambda,n-1})\, O^{\lambda,n}\, d\lambda, \\ A^{t,n} = O^{t,n} \int_0^t [O^{\lambda,n}]^{-1} \{\dot{h} - a(\xi^{\lambda,n-1})\}\, d\lambda, \\ \xi_s^{t,n} = \int_0^s A_\tau^{t,n}\, d\tau + \int_0^s O_\tau^{t,n}\, d\omega_\tau. \end{cases} \tag{3.9}$$

We prove that the above iteration process converges in a judiciously chosen norm on $SM(h)$. If $\xi \in SM(h)$ is given by

$$\xi_s = \int_0^s A_\tau d\tau + \int_0^s O_\tau d\omega_\tau,$$

we define

$$\|A\|^2 = E\left[\int_0^1 |A_s|^2\, ds\right],$$

$$|O|^2 = E[\sup_{0 \leqslant s \leqslant 1} |O_s|^2],$$

$$\langle \xi \rangle^2 = \|A\|^2 + |O|^2.$$

The norm in which we show the convergence is $\langle \cdot \rangle$ defined above. We will prove for $n \geqslant 2$ the inequality

$$\langle \xi^{t,n} - \xi^{t,n-1} \rangle \leqslant C \int_0^t \langle \xi^{\lambda,n-1} - \xi^{\lambda,n-2} \rangle\, d\lambda. \tag{3.10}$$

This together with easily proved inequality

$$\langle \xi^{t,1} - \xi^0 \rangle \leqslant Ct$$

implies that

$$\langle \xi^{t,n} - \xi^{t,n-1} \rangle \leqslant \frac{(Ct)^n}{n!}.$$

Therefore the limits

$$\lim_{n\to\infty} A^{t,n} = A^t \qquad \text{and} \qquad \lim_{n\to\infty} O^{t,n} = O^t$$

exist in the $\|\cdot\|$-norm and the $|\cdot|$-norm, respectively, and uniformly for $|t| \leqslant T$. Letting $n \to \infty$ in (3.9) we see that $\{A^t, O^t, \xi^t, t \in \mathbb{R}^1\}$ satisfy (3.7).

We now prove (3.10). We first show that

$$|O^{t,n} - O^{t,n-1}| \leqslant C \int_0^t \langle \xi^{\lambda,n-1} - \xi^{\lambda,n-2} \rangle \, d\lambda. \tag{3.11}$$

Since h is $o(d)$-valued, we have $O_s^{t,n} \in O(d)$ and by the first equation in (3.9)

$$\frac{d}{dt}[O^{t,n}]^{-1} O^{t,n-1} = [O^{t,n}]^{-1} \{b(\xi^{t,n-1}) - b(\xi^{t,n-2})\} O^{t,n-1}.$$

Integrating with respect to t and using the equality $O_1 - O_2 = O_1(I - O_1^{-1} O_2)$ we obtain

$$|O^{t,n} - O^{t,n-1}| \leqslant C \int_0^t |b(\xi^{\lambda,n-1}) - b(\xi^{\lambda,n-2})| \, d\lambda.$$

[I am indebted to Bruce Driver, whose suggestion of using the above inequality greatly simplifies an early version of this proof.] Hence it is sufficient to show that

$$|b(\xi^{\lambda,n-1}) - b(\xi^{\lambda,n-2})| \leqslant C \langle \xi^{\lambda,n-1} - \xi^{\lambda,n-2} \rangle. \tag{3.12}$$

The expression of $b(\xi^{\lambda,n-1})$ in (3.5) has four terms. Correspondingly the difference $b(\xi^{\lambda,n-1}) - b(\xi^{\lambda,n-2})$ is the sum of four differences $D_1, ..., D_4$. It is easily to see that

$$|D_i| \leqslant C\,|U^{\lambda,n-1} - U^{\lambda,n-2}|, \qquad i = 1, 2.$$

The distance function on $O(M)$ can be understood by embedding $O(M)$ in some euclidean space $\mathbb{R}^L$ for a large integer L. From the second equation in (3.9) and the fact that a is uniformly bounded we have

$$\forall n: \quad |A_s^{t,n}| \leqslant K\{|\dot{h}_s| + 1\}.$$

Hence for a constant C_1 independent of h,

$$|D_3| \leqslant C_1\{|h|_{0t} + 1\}\, |h|_\infty\, |U^{\lambda,n-1} - U^{\lambda,n-2}| + C_1\, |h|_\infty\, \|A^{\lambda,n-1} - A^{\lambda,n-2}\|.$$

Using standard L^2-estimates for stochastic integrals we have

$$|D_4| \leqslant C\,|U^{\lambda,n-1} - U^{\lambda,n-2}| + C\,|O^{\lambda,n-1} - O^{\lambda,n-2}|.$$

Combining the above estimates for D_i we have

$$|b(\xi^{\lambda,n-1})-b(\xi^{\lambda,n-2})| \leqslant C\,|U^{\lambda,n-1}-U^{\lambda,n-2}|+C\langle \xi^{\lambda,n-1}-\xi^{\lambda,n-2}\rangle \tag{3.13}$$

for a constant C dependent on h. Here $U^{\lambda,n}=I(\xi^{\lambda,n})$ is the solution of the SDE

$$dU_s = H_{U_s}\, d\xi^{\lambda,n}, \qquad U_0 = u_0.$$

If we embed $O(M)$ in some $\mathbb{R}^L$ and extend H_i to vector fields vanishing outside a compact set, then the above equation can be regarded as an SDE on $\mathbb{R}^L$ with coefficients of compact support. Hence simple estimates on solutions of SDEs gives the estimate

$$|U^{\lambda,n-1}-U^{\lambda,n-2}| \leqslant C\langle \xi^{\lambda,n-1}-\xi^{\lambda,n-2}\rangle. \tag{3.14}$$

This together with (3.13) implies (3.12) and (3.11) is proved.

Next we prove the inequality

$$\|A^{t,n}-A^{t,n-1}\| \leqslant C\int_0^t \langle \xi^{\lambda,n-1}-\xi^{\lambda,n-2}\rangle\, d\lambda. \tag{3.15}$$

We use the second equation in (3.9). Using the identity

$$O_1^{-1}-O_2^{-1}=O_1^{-1}\{O_2-O_1\}\,O_2^{-1}$$

and the fact that a, $O^{t,n}$, and $[O^{t,n}]^{-1}$ are uniformly bounded we have

$$\begin{aligned}\|A^{t,n}-A^{t,n-1}\| &\leqslant C_1\int_0^t \|a(\xi^{\lambda,n-1})-a(\xi^{\lambda,n-2})\|\, d\lambda\\ &\quad + C_1\{|h|_{\mathbb{H}}+1\}\,|O^{t,n-1}-O^{t,n-2}|\\ &\quad + C_1\{|h|_{\mathbb{H}}+1\}\int_0^t |O^{\lambda,n-1}-O^{\lambda,n-2}|\, d\lambda. \end{aligned} \tag{3.16}$$

From the second equation in (3.4) and (3.14) it is clear that

$$\|a(\xi^{\lambda,n-1})-a(\xi^{\lambda,n-2})\| \leqslant C\,|U^{\lambda,n-1}-U^{\lambda,n-2}| \leqslant C\langle \xi^{\lambda,n-1}-\xi^{\lambda,n-2}\rangle. \tag{3.17}$$

Now (3.15) follows from (3.11), (3.14), (3.16), and (3.17).

(c) *End of the proof.* So far we have shown that there exists a family of meaurable random variables $\{A^t, O^t, \xi^t, t\in\mathbb{R}^1\}$ such that (3.7) is satisfied a.s. for each fixed t. From the first equation in (3.7) we have O^t

is $O(d)$-valued; from the second equation there $|A^t_s| \leq K\{|\dot{h}_s| + 1\}$ for some K. Hence $\xi^t \in SM(h)$, which proves (i). We see from (3.7) (using Kolmogorov's criterion, for example) that there exists a continuous version of $\{A^t, O^t, \xi^t, t \in \mathbb{R}\}$, which proves (ii). Finally from (3.4), $\{p(\xi^t), t \in \mathbb{R}^1\}$ has a continuous version. From (3.7) and (3.4) we see that (3.6) holds for each fixed t. It therefore holds for all t by continuity. This proves (iii). ∎

Remark 3.2. It can be shown that $\{\xi^t, t \in \mathbb{R}^1\}$ has a smooth version.

For each fixed t, the random variable ξ^t_h: $W_o(\mathbb{R}^d) \to W_o(\mathbb{R}^d)$ can be regarded as a μ-almost surely defined map from the path space to itself. In Proposition 3.4 below we will prove the $\{\xi^t, t \in \mathbb{R}^1\}$ has the group property. Thus we can regard $\{\xi^t_h, t \in \mathbb{R}^1\}$ as the flow on $W_o(\mathbb{R}^d)$ generated by $p_h = J_*^{-1} D_h$. If $z \in SM(h)$, then the law of z is equivalent to μ, hence the composition $\xi^t_h \circ z = \xi_h(z)$ is well defined. Furthermore if

$$z_s = \int_0^s A_\tau d\tau + \int_0^s O_\tau \, d\omega_\tau,$$

then

$$\xi^t(z)_s = \int_0^s \{A^t_\tau(z) + O^t_\tau(z) A_\tau\} \, d\tau + \int_0^s O^t_\tau(z) O_\tau \, d\omega_\tau,$$

which implies that $\xi^t \circ z \in SM(h)$. Thus the space $SM(h)$ is invariant under ξ^t_h.

Because $\xi^t_h \circ z \in SM(h)$, its law is equivalent to μ. Hence we may replace the ω in (3.6) by $z(\omega)$. The resulting equation shows that $x^t = \xi^t \circ z$ is the solution of the integral equation

$$x^t = z + \int_0^t p(x^\lambda) \, d\lambda. \tag{3.18}$$

We prove a uniqueness result for the above equation.

PROPOSITION 3.3. *The solution to* (3.18) *in the class* $SM(h)$ *is unique.*

Proof. Let $x^{t,i}$, $i = 1, 2$, be two solutions in $SM(h)$. Then we have as in the proof of Theorem 3.1,

$$\begin{cases} O^{t,i} = O - \int_0^t b(x^{\lambda,i}) O^{\lambda,i} \, d\lambda, \\ A^{t,i} = A + \dot{h}t - \int_0^t a(x^{\lambda,i}) \, d\lambda - \int_0^t b(x^{\lambda,i}) A^{\lambda,i} \, d\lambda, \\ x^{t,i} = \int_0^s A^{t,i}_\tau d\tau + \int_0^s O^{t,i}_\tau d\omega_\tau. \end{cases} \tag{3.19}$$

Following the proof of 3.1 we can show from the above relations that there is a constant C such that

$$\langle x^{t,1} - x^{t,2} \rangle \leqslant C \int_0^t \langle x^{\lambda,1} - x^{\lambda,2} \rangle \, d\lambda,$$

which implies $x^{t,1} = x^{t,2}$ by Gronwall's lemma. ∎

PROPOSITION 3.4. *Suppose that $h \in \mathbb{H}$ and $\{\xi_h^t, t \in \mathbb{R}^1\}$ is the unique flow on $W_o(\mathbb{R}^d)$ generated by p_h. Then μ-almost surely,*

$$\xi_h^{t_1} \circ \xi_h^{t_2} = \xi_h^{t_1 + t_2}, \qquad \textit{for all} \quad (t_1, t_2) \in \mathbb{R}^1 \times \mathbb{R}^1.$$

Proof. The composition $\xi^{t_1} \circ \xi_2^t$ makes sense because the law of ξ^t is equivalent to μ for all t. It can be shown (by Kolmogorov's criterion, for example) that

$$\{\xi^{t_1} \circ \xi^{t_2}, (t_1, t_2) \in \mathbb{R}^1 \times \mathbb{R}^1\}$$

has a continuous version. Now both $\{\xi^t \circ \xi^{t_2}, t \in \mathbb{R}^1\}$ and $\{\xi^{t+t_2}, t \in \mathbb{R}^1\}$ are solutions of (3.18) with initial value ξ^{t_2}. Therefore by the uniqueness (Proposition 3.3), for each fixed t_2, we have μ-a.s., $\xi^{t_1} \circ \xi^{t_2} = \xi^{t_1 + t_2}$ for all t_1. Therefore μ-a.s., it holds for all t_1 and all rational t_2. Since both sides are continuous in (t_1, t_2), the equality automatically holds for all t_1 and t_2. ∎

Let μ_h^t be the law of ξ_h^t. We give an explicit formula for the Radon–Nikodym derivative $d\mu_h^t/d\mu$.

THEOREM 3.5. *Let $h \in \mathbb{H}$ and $\{\xi_h^t, t \in \mathbb{R}\}$ the flow on on $W_o(\mathbb{R}^d)$ generated by $p_h = J_*{}^{-1} D_h$. Let $\{A^t, t \in \mathbb{R}\}$ be the solution of the equation*

$$A^t = \dot{h}t - \int_0^t a(\xi_h^\lambda) \, d\lambda - \int_0^t b(\xi_h^\lambda) A^\lambda \, d\lambda. \tag{3.20}$$

Then the Radon–Nikydom derivative of the law μ_h^t of ξ_h^t with respect to the Wiener measure μ is given by

$$\frac{d\mu_h^t}{d\mu}(\omega) = \exp\left[\int_0^1 A_s^t(\xi^{-t}\omega)^* \, d\omega_s - \tfrac{1}{2} \int_0^1 |A_s^t(\xi^{-t}\omega)|^2 \, ds\right]. \tag{3.21}$$

Proof. By Theorem 3.1 ξ^t is given by

$$\xi_s^t = \int_0^s A_\tau^t \, d\tau + \int_0^s O_\tau^t \, d\omega_\tau, \tag{3.22}$$

Define the exponential martingale

$$e_s(\omega)=\exp\left[-\int_0^s (A')^*_\tau O'_\tau \, d\omega_\tau - \tfrac{1}{2}\int_0^s |A'_\tau|^2 \, d\tau\right]. \tag{3.23}$$

We have $Ee_1 = 1$. Define a new probability measure χ on $W_o(\mathbb{R}^d)$ by

$$\frac{d\chi}{d\mu} = e_1.$$

By Girsanov's theorem , ξ' is a Brownian motion under the measure χ. Hence for any measurable set $C \subset W_o(\mathbb{R}^d)$,

$$\begin{aligned}\mu[\omega \in C] &= \chi[\xi'\omega \in C]\\ &= \mu[e_1(\omega);\ \xi'\omega \in C]\\ &= \mu'[e_1(\xi^{-1}\omega);\ \omega \in C].\end{aligned}$$

This implies immediately that μ' is equivalent to μ and

$$\frac{d\mu'}{d\mu}(\omega) = \frac{1}{e_1(\xi^{-1}\omega)}. \tag{3.24}$$

Now using (3.22) (with t replaced by $-t$) and (3.23) we have

$$\begin{aligned}\log e_1(\xi^{-1}\omega) = &-\int_0^1 A'_s(\xi^{-1}\omega)^* \, O'_s(\xi^{-1}\omega)\, O_s^{-1}(\omega)\, d\omega_s\\ &-\int_0^1 A'_s(\xi^{-1}\omega)^* \, O'_s(\xi^{-1}\omega)\, A_s^{-1}(\omega)\, ds\\ &-\tfrac{1}{2}\int_0^1 |A'_s(\xi^{-1}\omega)|^2\, ds\end{aligned} \tag{3.25}$$

Replacing ω in (3.22) by $\xi^{-1}\omega$ and using $\xi' \circ \xi^{-1}\omega = \omega$ (from Proposition 3.4) we have

$$\omega_s = \int_0^s [A'_\tau(\xi^{-1}\omega) + O'_\tau(\xi^{-1}\omega)\, A_\tau^{-1}(\omega)]\, d\tau + \int_0^s O'_\tau(\xi^{-1}\omega)\, O_\tau^{-1}(\omega)\, d\omega_\tau.$$

By the uniqueness of the Doob–Meyer decomposition we have

$$A'_s(\xi^{-1}\omega) + O'_s(\xi^{-1}\omega)\, A_s^{-1}(\omega) = 0, \quad O'_s(\xi^{-1}\omega)\, O_s^{-1}(\omega) = I.$$

Using these two identities in (3.25) we obtain immediately (3.21) from (3.24). ∎

4. Flows on Riemannian Path Space

The purpose of this section is to transfer the flow $\{\xi_h^t, t \in \mathbb{R}^1\}$ on $W_o(\mathbb{R}^d)$ constructed in the last section to a flow on the path space $W_o(M)$ by using the Itô map $J: W_o(\mathbb{R}^d) \to W_o(M)$.

On the probability space $(W_o(\mathbb{R}^d), \mathscr{B}, \mu)$ consider the following SDE for a process $I = I(\omega)$ on $O(M)$:

$$dI_s = H_{I_s} \circ d\omega_s, \qquad I_0 = u_0.$$

By the pathwise uniqueness for this SDE, the solution gives a progressively measurable map I: $W_o(\mathbb{R}^d) \to W_o(O(M))$ defined μ-a.s. Thus if z is an $\mathbb{R}^d$-valued continuous semimartingale whose law is absolutely continuous with respect μ, then the composition $I \circ z$ is a well defined, $O(M)$-valued semimartingale and is the unique solution of the SDE with the driving process ω replaced by z. This holds in particular if $z \in SM(h)$.

Let $J = \pi \circ I$, where π: $W(O(M)) \to W(M)$ is the canonical projection. Then the Itô map J: $W_o(\mathbb{R}^d) \to W_o(M)$ is a progressively measurable map defined μ-a.s. As a $W_o(M)$-valued random variable, J is a Riemannian Brownian motion on M, whose law on $W_o(M)$ is the Wiener measure ν on $W_o(M)$.

We now define an inverse of J. We will work in the probability space $(W_o(M), \mathscr{F}, \nu)$, where $\mathscr{F}$ is the Borel σ-field on $W_o(M)$ and ν is the Wiener measure on $W_o(M)$. Let γ the coordinate process on $W_o(M)$. The horizontal lift $U = U(\gamma)$ of the Riemannian Brownian motion $\{\gamma_s, 0 \leqslant s \leqslant 1\}$ is the solution of the SDE

$$dU_s = H_{U_s} U_s^{-1} \circ d\gamma_s, \qquad U_0 = u_o.$$

Let θ be the canonical 1-form on $O(M)$. The stochastic line integral

$$L(\gamma)_s = \int_{U[0,s]} \theta = \int_0^s \theta \circ dU_\tau$$

is called the stochastic parallel development of γ and as a $W_o(\mathbb{R}^d)$-valued random variable the law of L is the Wiener measure μ. We therefore have a progressively measurable map L: $W_o(M) \to W_o(\mathbb{R}^d)$ defined ν-a.s.

From the above discussion, we see that the compositions $L \circ J: W_o(\mathbb{R}^d) \to W_o(\mathbb{R}^d)$ and $J \circ L: W_o(M) \to W_o(M)$ are well defined μ-a.s. and ν-a.s., respectively. The map L is the inverse map of the map J in the sense that $L \circ J(\omega) = \omega$, μ-a.s. and $J \circ L(\gamma) = \gamma$, ν-a.s. For this reason we denote L by J^{-1}.

From now on we work in the probability space $(W_o(M), \mathscr{F}, \nu)$. Define the flow $\zeta_h^t\colon W_o(M) \to W_o(M)$ by

$$\zeta_h^t = J \circ \xi_h^t \circ J^{-1}. \tag{4.1}$$

The composition is well defined ν-a.s. and ζ_h^t is an M-valued semimartingale. The problem remains to show that there is a nice version of $\{\zeta_h^t, t \in \mathbb{R}^1\}$ which is the flow generated by D_h.

THEOREM 4.1. *Let $h \in \mathbb{H}$. There is a family of measurable maps ($W_o(M)$-valued random variables)*

$$\zeta_h^t\colon W_o(M) \to W_o(M), \qquad t \in \mathbb{R}^1$$

with the following properties:

(i) *For each fixed $t \in \mathbb{R}^1$, the law ν_h^t of ζ_h^t is equivalent to the Wiener measure ν (quasi-invariance of the Wiener measure) and the Radon–Nikodym derivative is given by*

$$\frac{d\nu_h^t}{d\nu}(\gamma) = \frac{d\mu^t}{d\mu}(J^{-1}\gamma), \tag{4.2}$$

where $J^{-1}\colon W_o(M) \to W_o(\mathbb{R}^d)$ is the inverse Itô map and $d\mu_h^t/d\mu$ is given by (3.21);

(ii) *ν-almost surely, the function $t \mapsto \zeta^t\gamma$ is a $W_o(M)$-valued continuously differentiable function;*

(iii) *There is a continuous version of $t \mapsto U(\zeta_h^t\gamma)\, h = D_h(\zeta_h^t\gamma)$ such that ν-almost surely, $\zeta_h^t\gamma$ satisfies the differential equation*

$$\frac{d\zeta_h^t\gamma}{dt} = D_h(\zeta_h^t\gamma); \tag{4.3}$$

(iv) *ν-almost surely,*

$$\zeta_h^{t_1} \circ \zeta_h^{t_2} = \zeta_h^{t_1+t_2}\gamma \qquad \textit{for all} \quad (t_1, t_2) \in \mathbb{R}^1 \times \mathbb{R}^1.$$

Proof. Define a new probability measure η on $W_o(M)$ by

$$\frac{d\eta}{d\nu}(\gamma) = e_1(J^{-1}\gamma),$$

where e_1 is defined on $W_o(\mathbb{R}^d)$ by (3.23). Then by Girsanov's theorem, the law of $\xi^t \circ J^{-1}$ under η is the Wiener measure μ. Therefore the law of

$\zeta^t = J \circ \xi^t \circ J^{-1}$ under η is the measure ν. Now for any measurable set $C \subset W_o(M)$,

$$\nu[\gamma \in C] = \eta[\zeta^t \gamma \in C]$$
$$= \nu[e_1(J^{-1}\gamma); \zeta^t \gamma \in C]$$
$$= \nu^t[e_1(J^{-1}\zeta^{-t}\gamma); \gamma \in C].$$

This implies immediately ν^t is equivalent to ν, and by (3.24)

$$\frac{d\nu^t}{d\nu}(\gamma) = \frac{1}{e_1(J^{-1}\zeta^{-t}\gamma)} = \frac{d\mu^t}{d\mu}(J^{-1}\gamma).$$

This proves (i).

From $J = \pi \circ I$ and the SDE for I: $W_o(\mathbb{R}^d) \to W_o(O(M))$ it is not difficult to prove that for $\xi^1, \xi^2 \in SM(h)$,

$$|J \circ \xi^1 - J \circ \xi^2| \leqslant C\langle \xi^1 - \xi^2 \rangle.$$

From (3.6) and (3.4) we have the estimate

$$\langle \xi^{t_1} - \xi^{t_2} \rangle \leqslant C\,|t_1 - t_2|.$$

It follows that

$$|\zeta^{t_1} - \zeta^{t_2}| = |J \circ \xi^{t_1} \circ J^{-1} - J \circ \xi^{t_2} \circ J^{-1}|$$
$$= |J \circ \xi^{t_1} - J \circ \xi^{t_2}|$$
$$\leqslant C\langle \xi^{t_1} - \xi^{t_2} \rangle$$
$$\leqslant C\,|t_1 - t_2|,$$

from which we conclude that $\{\zeta^t, t \in \mathbb{R}^1\}$ has a continuous version. Using the same argument and the fact that $\{\xi^t, t \in \mathbb{R}^1\}$ has a continuously differentiable version, we can show that $\{\zeta^t, t \in \mathbb{R}^1\}$ has a continuously differentiable version. This proves (ii).

To prove (iii), we first show that $\{U(\zeta^t), t \in \mathbb{R}^1\}$ has a continuous version. From $U(\zeta^t) = I(\xi^t)$ we have

$$|U(\zeta^{t_1}) - U(\zeta^{t_2})| \leqslant C\langle \xi^{t_1} - \xi^{t_2} \rangle \leqslant C\,|t^1 - t^2|.$$

See the proof of (3.14). It follows that $\{U(\zeta^t), t \in \mathbb{R}^1\}$ has a continuous version. Differenting the equation

$$dU_s^t = H_{U_s^t}(U_s^t)^{-1} \circ d\zeta_s^t$$

with respect to t, we obtain a linear SDE for dU^t/dt. Hence it is not difficult to show as before that $\{dU(\zeta^t)/dt, t \in \mathbb{R}^1\}$ exists and has a continuous version.

Let

$$\tilde{h} = \theta\left[\frac{dU(\zeta^t)}{dt}\right].$$

$\tilde{h}$ is an $\mathbb{R}^d$-valued semimartingale. Since $\zeta^t = \pi(U^t)$, the assertion in (iii) is equivalent to $\tilde{h} = h$. To prove this we have to essentially repeat the computation in Section 2 with stochastic calculus. Let $T = \partial U^t_s/\partial t$ as before. Using the exterior differentiation formula, we have

$$d\tilde{h}_s = \frac{\partial}{\partial t}\,\theta(\circ\, dU^t_s) + d\theta(\circ\, dU^t_s, T),$$

Let $x^t = \xi^t \circ J^{-1}$. We have $U^t = I(x^t)$. Hence by the SDE for $I(x^t)$ we have we $\theta(\circ\, dU^t_s) = dx^t_s$, whose derivative with respect to t is $dp(x^t)_s$. We use the first structural equation on the second term on the right-hand side and obtain

$$d\tilde{h}_s = dp_h(x^t)_s + \Theta(\circ\, dU^t_s, T) + \omega(T) \circ \theta(\circ\, dU^t_s).$$

We have $dU^t_s = H_{U^t_s} \circ dx^t_s$ by the SDE for U^t. By the definition of $\tilde{h}_s$, the horizontal component of T is just $H\tilde{h}_s$. Thus the second term on the right-hand side can be written as $\Theta(H_{U^t_s} \circ dx^t_s, H\tilde{h}_s)$. Using the second structural equation on the third term on the right-hand side we have

$$\omega_{U^t_s}(T) = \int_0^s \Omega_{U^t_\tau}(\circ\, dU^t_\tau, T) = \int_0^s \Omega_{U^t_\tau}(H \circ dx^t_\tau, H\tilde{h}_\tau).$$

It follows that

$$d\tilde{h}_s = dp_h(x^t)_s + \Theta(H \circ dx^t_s, H\tilde{h}_s) + \left\{\int_0^s \Omega_{U^t_\tau}(H \circ dx^t_\tau, H\tilde{h}_\tau)\right\} \circ dx^t_s.$$

From the above equation and the definition of p_h in (3.2), we see that the above equation is just $dp_h(x^t)_s = dp_{\tilde{h}}(x^t)_s$, or equivalently $p_h(x^t) = p_{\tilde{h}}(x^t)$. Because the law of x^t is equivalent to the Wiener measure μ and because p_h is linear in h, we conclude that μ-a.s., $p_{h-\tilde{h}} = 0$. We have to show from this that $\tilde{h} = h$.

Let $\phi = h - \tilde{h}$. Using simple L^2-estimates on stochastic integrals, we see from (3.4) that there exists a constant C such that

$$E\,|(p_\phi)_s - \phi_s|^2 \leqslant C \int_0^s E\,|\phi_s|^2\, ds$$

Since $p_\phi = 0$, we can write

$$E\,|\phi_s|^2 \leqslant C \int_0^s E\,|\phi_\tau|^2 \, d\tau,$$

from which we have immediately $\phi = 0$. This completes the proof of (iii).

Part (iv) follows from Proposition 3.4. ∎

5. Gradient Operator and Integration by Parts

Let $\mathbb{E}$ be a Hilbert space. An $\mathbb{E}$-valued function F on $W_o(M)$ is called cylindrical if there is a positive integer n, a set of n points $0 \leqslant s_1 < \cdots < s_n \leqslant 1$ and a smooth function $\tilde{F}\colon M \times \cdots \times M \to \mathbb{E}$ such that

$$F(\gamma) = \tilde{F}(\gamma_{s_1}, ..., \gamma_{s_n}). \tag{5.1}$$

The set of $\mathbb{E}$-valued cylindrical functions on $W_o(M)$ is denoted by $\mathscr{C}(\mathbb{E})$. Typically $\mathbb{E} = \mathbb{R}$, $\mathbb{R}^d$, $\mathbb{H}$, $\mathbb{H} \otimes \mathbb{H}$ (with the usual Hilbert–Schmidt norm). We denote $\mathscr{C}(\mathbb{R})$ simply by $\mathscr{C}$.

We will use $L^2(\mathbb{E}; \nu)$ to denote the Hilbert space of $\mathbb{E}$-valued measurable functions F on $W_o(M)$ such that

$$\|F\|^2_{L^2(\mathbb{E};\nu)} = \int_{W_o(M)} |F(\gamma)|^2_{\mathbb{E}}\, \nu(d\gamma) < \infty.$$

The inner production on $L^2(\mathbb{E}; \nu)$ is denoted by $(\cdot, \cdot)_{L^2(\mathbb{E};\nu)}$. We write $L^2(\nu)$ instead of $L^2(\mathbb{R}^1; \nu)$ and the inner product $(\cdot, \cdot)_{L^2(\mathbb{R}^1;\nu)}$ is simply written as $(\cdot, \cdot)$.

Let $F \in \mathscr{C}(\mathbb{E})$. It is natural to define the directional derivative

$$D_h F = \lim_{t \to 0} \frac{F \cdot \zeta_h^t - F}{t}.$$

The limit takes place in $L^2(\mathbb{E}; \nu)$. If F is given by (5.1), then by Theorem 4.1,

$$D_h F(\gamma) = \sum_{p=1}^{n} \langle \nabla^{(p)} \tilde{F}(\gamma), U(\gamma)_{s_p} h_{s_p} \rangle, \tag{5.2}$$

where $\nabla^{(p)} \tilde{F}$ denotes the gradient of $\tilde{F}$ with respect to the pth variable. There exists an element $DF \in L^2(\mathbb{H} \otimes \mathbb{E}, \nu)$ such that for all $h \in \mathbb{H}$

$$\langle DF, h \rangle_{\mathbb{H}} = D_h F.$$

DF is called the gradient of F and is given by

$$DF(\gamma)=\sum_{i=1}^{d}\sum_{p=1}^{n}(s\wedge s_p)\,e^i\otimes\langle\nabla^{(p)}\tilde{F}(\gamma),\,U(\gamma)_{s_p}e^i\rangle. \tag{5.3}$$

We want to show that D_h and D defined on $\mathscr{C}$ as above are closable and to describe their adjoints D_h^* and D^*.

For an $h\in\mathbb{H}$, define a martingale

$$\begin{aligned} l_h(\gamma)&=\int_0^1\langle\dot{h}_s-a(\omega)_s,\,d\omega_s\rangle\\ &=\int_0^1\langle\dot{h}_s-\tfrac{1}{2}H_i\Theta_{U_s}(H_i,\,Hh_s)-\tfrac{1}{2}\operatorname{Ric}_{U_s}(Hh_s),\,d\omega_s\rangle, \end{aligned} \tag{5.4}$$

where $\omega=J^{-1}\gamma$ and $U=U(\gamma)$ is the horizontal lift of γ to $O(M)$. The following theorem gives the formal adjoint of D_h on $\mathscr{C}$.

THEOREM 5.1 (Integration by Parts Formula). *Let F, G be two cylindrical functions. Then*

$$(D_hF,\,G)=(F,\,D_h^*G), \tag{5.5}$$

where

$$D_h^*=-D_h+l_h. \tag{5.6}$$

Proof. Since $F\in\mathscr{C}$, we have

$$\frac{d}{dt}F\circ\zeta_h^t\bigg|_{t=0}=D_hF.$$

Hence we can write

$$(D_hF,\,G)=\frac{d}{dt}(F\circ h^t,\,G). \tag{5.7}$$

The derivative is evaluated at $t=0$. We have $\zeta_h^{-t}\circ\zeta_h^t\gamma=\gamma$. The law v_h^t of ζ_h^t being equivalent to v by Theorem 4.1 we have by the change of variables $\zeta_h^t\gamma\mapsto\gamma$,

$$(F\circ\zeta_h^t,\,G)=(F,\,G\circ\zeta_h^{-t})_{L^2(v_h^t)}=\left(F,\,G\circ\zeta_h^{-t}\left\{\frac{dv_h^t}{dv}\right\}\right). \tag{5.8}$$

The Radon–Nikodym derivative is given by (4.2) and (3.21). From (3.20) we find that at $t = 0$

$$\frac{dA^t}{dt}(\omega) = \dot{h} - a(\omega).$$

Hence from (3.21) and (4.2),

$$\frac{d}{dt}\left\{\frac{d\nu_h^t}{d\nu}(\gamma)\right\} = \frac{d}{dt}\left\{\frac{d\mu^t}{d\mu}(\omega)\right\} = \int_0^1 \left\{\frac{dA^t(\omega)^*}{dt}\right\}_s d\omega_s = l_h(\omega).$$

It follows that in $L^2(\nu)$,

$$\frac{d}{dt}\left\{G(\zeta^{-t}\gamma)\frac{d\nu_h^t}{d\nu}(\gamma)\right\} = -D_h G(\gamma) + l_h(\gamma)\, G(\gamma) = D_h^* G(\gamma).$$

From (5.7), (5.8), and the above identity we have immediately the integration by parts formula (5.5). ∎

Having computed the formal adjoint D_h^* on cylindrical functions, we can extend the derivative operator and the gradient operator by the usual method in functional analysis. We will use Dom(A) to denote the domain of a linear operator A. Let $L^{2+}(\nu) = \bigcup_{p>2} L^p(\nu)$.

THEOREM 5.2. *Let $h \in \mathbb{H}$. The directional derivative operator $D_h: \mathscr{C} \to L^2(\nu)$ is closable in $L^2(\nu)$. Denote its closure by D_h again. Let D_h^* be its adjoint. Then*

$$\mathrm{Dom}(D_h) \cap L^{2+}(\nu) \subset \mathrm{Dom}(D_h^*)$$

and for all $G \in \mathrm{Dom}(D_h) \cap L^{2+}(\nu)$ we have

$$D_h^* G = -D_h G + l_h G.$$

Proof. By definition we have $\mathrm{Dom}(D_h) \supset \mathscr{C}$ and $\mathscr{C}$ is dense in $L^2(\nu)$, the operator D_h is densely defined. The closability of D_h follows from the existence of a formal adjoint D_h^* on $\mathscr{C}$.

If $h \in \mathbb{H}$, then l_h is a continuous martingale with uniformly bounded quadratic variation. Hence $l_h \in L^q(\nu)$ for all $q > 0$ by moment estimates for continuous martingales (see Ikeda and Watanabe [8], 110–113.) Suppose that $G \in \mathrm{Dom}(D_h) \cap L^p(\nu)$ for some $p > 2$. To show that $G \in \mathrm{Dom}(D_h^*)$ it is enough to show that there is a $D_h^* G \in L^2(\nu)$ such that

$$(D_h F, G) = (F, D_h^* G)$$

for all $F \in \mathscr{C}$. From the closability of D_h there is a sequence $\{G_n\}$ of cylindrical functions such that

$$G_n \to G \qquad \text{and} \qquad D_h G_n \to D_h G$$

in $L^2(\nu)$. We have

$$(D_h F, G_n) = -(F, D_h G_n) + (F, l_h G_n). \tag{5.9}$$

We want to let $n \to \infty$ in the above relation, but we do not know if $l_h G_n$ converges to $l_h G$ in $L^2(\nu)$. We overcome this difficulty by a truncation argument. Let $\phi: \mathbb{R}^1 \to \mathbb{R}^1$ be a bounded smooth function with bounded first derivative ϕ'. Then $\phi(G_n) \in \mathscr{C}$ and we have $D_h \phi(G_n) = \phi'(G_n) D_h G_n$. Now we write down (5.9) with G_n replaced by $\phi(G_n)$ and obtain

$$(D_h F, \phi(G_n)) = -(F, \phi'(G_n) D_h G_n) + (F, l_h \phi(G_n)).$$

Letting $n \to \infty$ we have

$$(D_h F, \phi(G)) = -(F, \phi'(G) D_h G) + (F, l_h \phi(G)). \tag{5.10}$$

Now let ϕ go through a sequence of functions $\{\phi_N\}$ such that (i) $\phi_N(t) = t$ for $|t| \leqslant N$; (ii) $|\phi_N(t)| \leqslant 2|t|$ for all $t \in \mathbb{R}^1$; (iii) $|\phi'_N(t)| \leqslant 1$ for all $t \in \mathbb{R}^1$. Recall that $G \in L^p(\nu)$ for some $p > 2$. Choose q such that $1/p + 1/q = 1/2$. We have as $N \to \infty$,

$$\|l_h \phi_N(G) - l_h G\| \leqslant \|l_h\|_{L^q(\nu)} \|\phi_N(G) - G\|_{L^p(\nu)} \to 0. \tag{5.11}$$

In the last step we have used the dominated convergence theorem, which is permissible because $\phi_N(G) \to G$ and $|\phi_N(G) - G| \leqslant 3|G| \in L^p(\nu)$ by the choice of ϕ_N. Now replace ϕ by ϕ_N in (5.10) and let $N \to \infty$. Using (5.11) and the fact that $\phi'_N(G)$ is bounded by 1 and converges to 1 we have

$$(D_h F, G) = -(F, D_h G) + (F, l_h G).$$

This shows immediately that $G \in \mathrm{Dom}(D_h^*)$ and $D_h^* G = -D_h + l_h G$. ∎

From now on we fix an orthonormal basis $\{h^\alpha\}$ for the Cameron–Martin space $\mathbb{H}$. The orthonormal basis satisfies the following relation:

$$\sum_\alpha h_{s_1}^{\alpha, i} h_{s_2}^{\alpha, j} = (s_1 \wedge s_2) \delta^{ij}. \tag{5.12}$$

From (5.2), (5.3), and (5.12) we have for any $F \in \mathscr{C}(\mathbb{E})$,

$$DF = h^\alpha D_{h^\alpha} F.$$

PROPOSITION 5.3. *The following assertions hold.*

(i) *The gradient operator* $D: \mathscr{C}(\mathbb{E}) \to L^2(\mathbb{E} \otimes \mathbb{H}; \nu)$ *is closable on* $L^2(\mathbb{E}; \nu)$. *Denote its closure by D again;*

(ii) $\mathrm{Dom}(D) \subset \mathrm{Dom}(D_h)$ *for all* $h \in \mathbb{H}$; *if* $F \in \mathrm{Dom}(D)$ *and* $h \in \mathbb{H}$, *then* $D_h F = \langle DF, h \rangle_{\mathbb{H}}$;

(iii) *If* $F \in \mathrm{Dom}(D)$ *then*

$$\sum_\alpha \|D_{h^\alpha} F\|^2_{L^2(\mathbb{E};\, \nu)} < \infty$$

and $DF = h^\alpha D_{h^\alpha} F$; *the convergence takes place* ν*-almost surely as well as in* $L^2(\mathbb{E} \otimes \mathbb{H}; \nu)$.

Proof. The proof for a general $\mathbb{E}$ having no particular difficulty, we assume for simplicity that $\mathbb{E} = \mathbb{R}^1$

(i) Let $\mathscr{C}_0(\mathbb{H})$ be the cylindrical functions of the form $G = \sum_{\alpha=1}^N h^\alpha G_\alpha$, with $G_\alpha \in \mathscr{C}$. We have

$$(DF, G)_{L^2(\mathbb{H};\, \nu)} = (D_{h^\alpha} F, G_\alpha) = (F, D^*_{h^\alpha} G_\alpha). \tag{5.13}$$

The above equality shows that D has a formal adjoint on $\mathscr{C}_0(\mathbb{H})$ and is given by

$$D^* G = D^*_{h^\alpha} G_\alpha.$$

Note that because $G \in \mathscr{C}_0(\mathbb{H})$, the sum is actually finite. The existence of a formal adjoint for D on $\mathscr{C}_0(\mathbb{H})$, which is dense in $L^2(\mathbb{H}; \nu)$, shows that D is closable as an operator from $L^2(\nu)$ to $L^2(\mathbb{H}; \nu)$.

(ii) Suppose that $F \in \mathrm{Dom}(D)$. Then there exists a sequence of cylindrical functions $\{F_n\}$ such that $F_n \to F$ in $L^2(\nu)$ and $DF_n \to DF$ in $L^2(\mathbb{H}; \nu)$. Let $h \in \mathbb{H}$. We have

$$|D_h(F_n - F_m)| = |\langle D(F_n - F_m), h \rangle_{\mathbb{H}}| \leqslant |D(F_n - F_m)|_{\mathbb{H}} \cdot |h|_{\mathbb{H}}.$$

Thus the sequence $D_h F_n$ converges in $L^2(\nu)$. It follows from the closedness of D_h that $F \in \mathrm{Dom}(D_h)$ and $D_h F_n \to D_h F$. From $DF_n = \langle DF_n, h \rangle_{\mathbb{H}}$ we have $D_h = \langle DF, h \rangle_{\mathbb{H}}$.

(iii) Suppose that $F \in \mathrm{Dom}(D)$. Then for ν-almost all γ we have $DF(\gamma) \in \mathbb{H}$. By the orthogonal expansion in the basis $\{h^\alpha\}$ and (ii) we have ν-almost surely

$$DF(\gamma) = h^\alpha \langle DF(\gamma), h^\alpha \rangle_{\mathbb{H}} = h^\alpha D_{h^\alpha} F(\gamma).$$

Taking $|\cdot|^2_{\mathbb{H}}$ on both sides and integrating, we have

$$\sum_\alpha \|D_{h^\alpha} F\|^2_{L^2(\nu)} = \|DF\|^2_{L^2(\mathbb{H};\, \nu)} < \infty.$$

This also shows that the series for DF converges in $L^2(\mathbb{H}; \nu)$. ∎

We define a symmetric quadratic form on $\mathscr{C}$ as follows:

$$\mathscr{E}(F, F) = \int_{W_o(M)} |DF(\gamma)|^2_{\mathbb{H}}\, \nu(d\gamma).$$

Proposition 5.3 gives immediately the following result (see Fukushima [6] or Ma and Röckner [12] for the definition of closed symmetric quadratic forms).

PROPOSITION 5.4. *The symmetric quadratic form $\mathscr{E}$ on $L^2(\nu)$ is closed. It is a Dirichlet form with* $\mathrm{Dom}(\mathscr{E}) = \mathrm{Dom}(D)$ *and $\mathscr{C}$ is dense in* $\mathrm{Dom}(\mathscr{E})$.

We now give a formula for the adjoint operator

$$D^*\colon L^2(\mathbb{H} \otimes \mathbb{E};\, \nu) \to L^2(\mathbb{E};\, \nu).$$

For simplicity we will assume that $\mathbb{E} = \mathbb{R}^1$. If $F \in \mathscr{C}(\mathbb{H})$, then $D_h F$ and DF are well defined and

$$DF = \sum_{\alpha, \beta} h^\beta \otimes h^\alpha D_{h^\beta} \langle F, h^\alpha \rangle_{\mathbb{H}}.$$

The convergence takes place ν-almost surely as well as in $L^2(\mathbb{H} \otimes \mathbb{H}; \nu)$.

DEFINITION 5.1. *We say that an element $K \in L^2(\mathbb{H} \otimes \mathbb{H}; \nu)$ has L^2-trace if the series*

$$\mathrm{Trace}\, K = \sum_\alpha \langle K, h^\alpha \otimes h^\alpha \rangle_{\mathbb{H} \otimes \mathbb{H}} \tag{5.14}$$

converges in $L^2(\nu)$.

If $G \in \mathscr{C}_0(\mathbb{H})$, then it is easy to verify that DG has L^2-trace and

$$\text{Trace } DG = \sum_\alpha \langle D_{h^\alpha} G, h^\alpha \rangle_{\mathbb{H}}.$$

In fact in this case (5.14) is a finite sum.

Define a martingale

$$A(\gamma)_s = \frac{1}{2} \int_0^s \langle H_i \Theta_{U_\tau}(H_i, H), d\omega_\tau \rangle + \frac{1}{2} \int_0^s \langle \text{Ric}_{U_\tau}(H), d\omega_\tau \rangle, \quad (5.15)$$

where $\omega = J^{-1}\gamma$ and $U = U(\gamma)$, the horizontal lift of γ.

Recall that $\mathscr{C}_0(\mathbb{H})$ is the set of $\mathbb{H}$-valued cylindrical functions of the form $G = \sum_{\alpha=1}^N h^\alpha G_\alpha$, with $G_\alpha \in \mathscr{C}$. We have the following integration by parts formula.

LEMMA 5.5. *If $F \in \text{Dom}(D)$ and $G \in \mathscr{C}_0(\mathbb{H})$, then*

$$(DF, G)_{L^2(\mathbb{H};\nu)} = (F, D^*G), \quad (5.16)$$

where D^*G is given by

$$D^*G = -\text{Trace } DG + \int_0^1 \langle \dot{G}_s, d\omega_s \rangle - \int_0^1 \langle G_s, dA_s \rangle.$$

Proof. Let $G \in \mathscr{C}_0(\mathbb{H})$ has the form $G = \sum_{\alpha=1}^N h^\alpha G_\alpha$, where $G_\alpha \in \mathscr{C}$. We have from (5.13) that (5.16) holds with

$$D^*G = D^*_{h^\alpha} G_\alpha = -D_{h^\alpha} G_\alpha + l_{h^\alpha} G_\alpha = -\text{Trace } DG + l_{h^\alpha} G_\alpha.$$

Thus it is enough to show that

$$l_{h^\alpha} G_\alpha = \int_0^1 \langle \dot{G}_s, d\omega_s \rangle - \int_0^1 \langle G_s, dA_s \rangle. \quad (5.17)$$

Note that all sums over α are finite sums from 1 to N. From (5.4) we see that l_{h^α} is a sum of three terms. Thus the left side of (5.17) is correspondingly a sum of three series, say S_1, $-S_2$ and $-S_3$. We have

$$S_1 = G_\alpha \int_0^1 \langle \dot{h}^\alpha_s, d\omega_s \rangle = \int_0^1 \langle \dot{G}_s, d\omega_s \rangle,$$

which coincides with the first term on the right side of (5.17). For S_2 and S_3 we have

$$S_2 = \tfrac{1}{2} G_\alpha \int_0^1 \langle H_i \Theta_{U_s}(H_i, Hh_s^\alpha), d\omega_s \rangle$$

$$= \tfrac{1}{2} \int_0^1 \langle H_i \Theta_{U_s}(H_i, HG_s), d\omega_s \rangle,$$

$$S_3 = \tfrac{1}{2} G_\alpha \int_0^1 \langle \mathrm{Ric}_{U_s}(Hh_s^\alpha), d\omega_s \rangle$$

$$= \tfrac{1}{2} \int_0^1 \langle \mathrm{Ric}_{U_s}(HG_s), d\omega_s \rangle.$$

It is clear now that the sum $S_1 - S_2 - S_3$ is equal to the right side of (5.17). ∎

Compare the following theorem with Theorem 5.2.

THEOREM 5.6. *If* $G \in \mathrm{Dom}(D) \cap L^2(\mathbb{H}; \nu)$ *and* DG *has* L^2*-trace, then* $G \in \mathrm{Dom}(D^*)$ *and*

$$D^*G = -\mathrm{Trace}\, DG + \int_0^1 \langle \dot{G}_s, d\omega_s \rangle - \int_0^1 \langle G_s, dA_s \rangle. \tag{5.18}$$

In particular, $\mathscr{C}(\mathbb{H}) \subset \mathrm{Dom}(D^*)$ *and*

$$(DF, G)_{L^2(\mathbb{H}; \nu)} = (F, D^*G)$$

for $F \in \mathrm{Dom}(D)$ *and* $G \in \mathscr{C}(\mathbb{H})$.

Proof. Let G^N be the N-truncation of G:

$$G^N = \sum_{\alpha=1}^N h^\alpha \langle G, h^\alpha \rangle.$$

There is a sequence $\{G_n\} \subset \mathscr{C}(\mathbb{H})$ such that $G_n \to G$ in $L^2(\mathbb{H}; \nu)$ and $DG_n \to DG$ in $L^2(\mathbb{H} \otimes \mathbb{H}; \nu)$. Fix a positive integer N and let G_n^N be the N-truncation of G_n. Then $G_n^N \in \mathscr{C}_0(\mathbb{H})$. By Lemma 5.5 we have for any $F \in \mathrm{Dom}(D)$,

$$(DF, G^N)_{L^2(\mathbb{H}; \nu)} = \lim_{n \to \infty} (DF, G_n^N)_{L^2(\mathbb{H}; \nu)} = \lim_{n \to \infty} (F, D^*G_n^N). \tag{5.19}$$

Now the convergence $DG_n \to DG$ in $L^2(\mathbb{H} \otimes \mathbb{H}; \nu)$ implies that for each fixed N,

$$\text{Trace } D^{\mathbb{H}} G_n^N \to \text{Trace } D^{\mathbb{H}} G^N \text{ in } L^2(\nu).$$

The convergence $G_n \to G$ in $L^2(\mathbb{H}; \nu)$ implies that

$$\int_0^1 \langle \dot{G}_{n,s}^N, d\omega_s \rangle \to \int_0^1 \langle \dot{G}_s^N, d\omega_s \rangle$$

and

$$\int_0^1 \langle G_{n,s}^N, dA_s \rangle \to \int_2^1 \langle G_s^N, dA_s \rangle.$$

Both convergence take place in $L^2(\nu)$. Thus we have $D^* G_n^N \to D^* G^N$ in $L^2(\nu)$ with $D^* G^N$ given as in (5.18). It follows from (5.19) that

$$(DF, G^N)_{L^2(\mathbb{H};\, \nu)} = (F, D^* G^N). \tag{5.20}$$

We take the limit in the above relation as $N \to \infty$. Since $G^N \to G$ in $L^2(\nu)$, the left-hand side goes to (DF, G). For the right-hand side we have

$$D^* G^N = -\text{Trace } DG^N + \int_0^1 \langle \dot{G}_s^N, d\omega_s \rangle - \int_0^1 \langle G_s^N, dA_s \rangle.$$

We have

$$\text{Trace } DG^N = \sum_{\alpha \leq N} \langle DG, h^\alpha \otimes h^\alpha \rangle.$$

Since we assume that DG has L^2-trace, Trace $DG^N \to$ Trace DG in $L^2(\nu)$. Now $G^N \to G$ in $L^2(\mathbb{H}; \nu)$, which implies both $\dot{G}^N \to \dot{G}$ and $G^N \to G$ in $L^2(\mathbb{H}^0; \nu)$, where $\mathbb{H}^0 = L^2[0, 1]$. This shows that

$$\int_0^1 \langle \dot{G}_s^N, d\omega_s \rangle \to \int_0^1 \langle \dot{G}_s, d\omega_s \rangle,$$

and

$$\int_0^1 \langle G_s^N, d\omega_s \rangle \to \int_0^1 \langle G_s, d\omega_s \rangle$$

in $L^2(\nu)$. It follows that $D^*G^N \to D^*G$, where D^*G is given by (5.18). Thus by letting $N \to \infty$ in (5.20) we obtain

$$(DF, G)_{L^2(\mathbb{H};\,\nu)} = (F, D^*G).$$

This implies that $G \in \mathrm{Dom}(D^*)$ with D^*G given by (5.18). ∎

6. Ornstein–Uhlenbeck Operator in Path Spaces

The usual Ornstein–Uhlenbeck operator in euclidean path spaces can be generalized to path spaces over Riemannian manifolds. We define the Ornstein–Uhlenbeck operator L on $W_o(M)$ to be the unique self-adjoint operator associated with the Dirichlet form

$$\mathscr{E}(F, F) = \|DF\|_{L^2(\mathbb{H};\,\nu)}.$$

By the general theory of Dirichlet forms (see Fukushima [6] or Ma and Röckner [12]), we have $\mathrm{Dom}(\mathscr{E}) = \mathrm{Dom}(\sqrt{-L})$ and $\mathscr{E}(F, F) = \|\sqrt{-L}F\|_{L^2(\nu)}$. The semigroup generated by the Dirichlet form is $P_t = e^{tL}$ and L is the L^2-infinitesimal generator of P_t.

Theorem 6.1. *We have $L = -D^*D$. If $F \in \mathrm{Dom}(D^2)$ and D^2F is of trace class, then $F \in \mathrm{Dom}(L)$ and at $\gamma \in W_o(M)$,*

$$LF = \mathrm{Trace}\ D^2F - \int_0^1 \langle (DF)_s^{\bullet}, d\omega_s \rangle + \int_0^1 \langle (DF)_s, dA_s \rangle, \qquad (6.1)$$

where $\omega = J^{-1}\gamma$ is the stochastic development of γ in $\mathbb{R}^d$ and A is defined in (5.15).

Proof. Assume that $F \in \mathrm{Dom}(D^*D)$ and $G \in \mathrm{Dom}(\sqrt{-L})$. Then they are both in $\mathrm{Dom}(D)$. We have

$$(\sqrt{-L}F, \sqrt{-L}G) = \mathscr{E}(F, G) = (DF, DG)_{L^2(\mathbb{H};\,\nu)} = (D^*DF, G).$$

Hence $\sqrt{-L}F \in \mathrm{Dom}(\sqrt{-L})$ and $-LF = D^*DF$, that is, $-D^*D \subset L$.

If $F \in \mathrm{Dom}(L)$, then $F \in \mathrm{Dom}(\sqrt{-L}) = \mathrm{Dom}(D)$. For any $G \in \mathrm{Dom}(D)$, we have

$$(DF, DG)_{L^2(\mathbb{H};\,\nu)} = \mathscr{E}(F, G) = (\sqrt{-L}F, \sqrt{-L}G) = (-LF, G).$$

Thus we have $DF \in \mathrm{Dom}(D^*)$ and $D^*DF = -LF$. Therefore $L \subset -D^*D$. It follows that $L = -D^*D$. The formula for L then follows from Theorem 5.6. ∎

We give an equivalent but more instructive formula for L. If Q is a continuous semimartingale and $F \in \mathrm{Dom}(D)$, we write

$$D_Q F = \int_0^1 \langle (DF)_s^{\bullet}, dQ_s \rangle.$$

PROPOSITION 6.2. *If $F \in \mathrm{Dom}(D^2)$ and D^2F is of trace class, then*

$$LF = \mathrm{Trace}\, D^2F - D_Q F,$$

where the "drift vector field" is given at $\gamma \in W_o(M)$ by

$$Q_s = \omega_s - \tfrac{1}{2} \int_0^1 (s \wedge \tau) \langle H_i \Theta_{U_\tau}(H_i, H) + \mathrm{Ric}_{U_\tau}(H), d\omega_\tau \rangle. \tag{6.2}$$

Here $\omega = J^{-1}\gamma$ is the stochastic development of γ in $\mathbb{R}^d$ and $U = U(\gamma)$ is the horizontal lift of γ in $O(M)$.

Proof. The second term on the right-hand side of (6.1) corresponds to the first term in the formula (6.2) for Q. For the third term we have

$$\begin{aligned}\int_0^1 \langle (DF)_s, dA_s \rangle &= \sum_\alpha \langle DF, h^\alpha \rangle_{\mathbb{H}} \int_0^1 \langle h_s^\alpha, dA_s \rangle \\ &= \left\langle DF, \sum_\alpha h^\alpha \int_0^1 \langle h_s^\alpha, dA_s \rangle \right\rangle_{\mathbb{H}}\end{aligned}$$

and by (5.12)

$$\sum_\alpha h_\tau^\alpha \int_0^1 \langle h_s^\alpha, dA_s \rangle = \int_0^1 (s \wedge \tau)\, dA_s.$$

It follows that the drift vector field is given by

$$Q_s = \omega_s - \int_0^1 (s \wedge \tau)\, dA_\tau. \quad \blacksquare$$

Finally we show that the above formula can be applied to cylindrical functions. The result is stated in Theorem 6.6 below. We divide the proof of this technical result into several steps.

LEMMA 6.3. *If $h \in \mathbb{H}$ and $K \in \mathrm{Dom}(D)$, then $hK \in \mathrm{Dom}(D)$ and $D(hK)$ is of trace class.*

Proof. There is a sequence $\{K_n\}$ of cylindrical functions such that $K_n \to K$ in $L^2(\nu)$ and $DK_n \to DK$ in $L^2(\mathbb{H}; \nu)$. We have $hK_n \to hK$ in

$L^2(\mathbb{H}; \nu)$ and $D(hK_n) = h \otimes DK_n \to h \otimes DK$ in $L^2(\mathbb{H} \otimes \mathbb{H}; \nu)$. Since DF is closed we have $hK \in \mathrm{Dom}(D)$ and $D(hK) = h \otimes DK$. We have

$$D(hK) = h^\beta \otimes h^\alpha \langle h, h^\beta \rangle_{\mathbb{H}} D_{h^\alpha} K.$$

Hence $D(hK)$ is of trace class and

$$\mathrm{Trace}\, D(hK) = \langle h, h^\alpha \rangle_{\mathbb{H}} D_{h^\alpha} K = D_h K. \quad \blacksquare$$

We use e_i to denote the ith unit coordinate vector in $\mathbb{R}^d$.

LEMMA 6.3. *Let* $F \in \mathscr{C}$ *and* $g_{i,s}(\tau) = (\tau \wedge s)\, e_i \in \mathbb{H}$. *Then*

$$(DF)_{i,s} = D_{g_{i,s}} F.$$

Proof. By (5.12) we have $g_{i,s}(\tau) = h_s^{\alpha, i} h_\tau^\alpha$. Hence

$$(DF)_{i,s} = h_s^{\alpha, i} D_{h^\alpha} F = D_{h_s^{\alpha, i} h^\alpha} F = D_{g_{i,s}} F. \quad \blacksquare$$

LEMMA 6.5. *Let F be a smooth function on $O(M)$ and $G(\gamma) = F(U(\gamma)_\tau)$ for some $0 \leqslant \tau \leqslant 1$. Then for any $h \in \mathbb{H}$ we have at $t = 0$,*

$$\frac{d}{dt} G(\zeta_h^t \gamma) = \{ H h_\tau + (K_h)_\tau^* \}\, F(U(\gamma)_\tau),$$

where K^* denotes the canonical vertical vector field on $O(M)$ corresponding to K, i.e., $\omega(K^*) = K$.

Proof. This follows directly from (2.4) and (2.8). $\blacksquare$

THEOREM 6.6. *If $F \in \mathscr{C}$, then $F \in \mathrm{Dom}(D^2)$ and $D^2 F$ is of trace class. Hence $\mathscr{C} \subset \mathrm{Dom}(L)$ and (6.1) holds for all $F \in \mathscr{C}$.*

Proof. Suppose that $F \in \mathscr{C}$. From (5.3) we have

$$DF = \sum_{p=1}^{n} (s \wedge s_p)(H^{(p)} \hat{F}) \circ U,$$

where $\hat{F} = F \circ \pi$ and U is the horizontal lift operator. Note that $H^{(p)} \hat{F}$ is a cylindrical function on $O(M)$. Thus to prove the theorem, it is enough to show the following assertion: If $h \in \mathbb{H}$ and $K = \tilde{F} \circ U$ with a cylindrical function $\tilde{F}$ on $O(M)$, then $hK \in \mathrm{Dom}(D)$ and $D(hK)$ is of trace class. By Lemma 6.3 all we need to show is $K \in \mathrm{Dom}(D)$. For the sake of simplicity, we assume that $\tilde{F}$ depends only on one time parameter, i.e., $\tilde{F}(u) = F(u_\tau)$ for a smooth function F on $O(M)$ and some $0 \leqslant \tau \leqslant 1$.

The basic idea of the proof is to approximate K by cylindrical functions on $W_o(M)$. Let $\psi^n \colon W_o(M) \to W_o(M)$ be the following piecewise geodesic approximation. Denote the injectivity radius of M by r. If there is a k such that $d(\gamma_{k/n}, \gamma_{(k+1)/n}) \geqslant r/2$, then $\psi^n(\gamma)_s = o$ for all $0 \leqslant s \leqslant 1$; otherwise $\psi^n(\gamma)_s = q_{\gamma_{k/n}, \gamma_{(k+1)/n}}(t - k/n)$, where $q_{x,y}$ denotes the unique geodesic joining x and y in time $1/n$. Now $\psi^n(\gamma)$ depends only $\gamma_{k/n}$, $0 \leqslant k \leqslant n$. Let $K_n = K \circ \psi^n$. Then $K_n \in \mathscr{C}$. By approximation theory of stochastic integrals and SDEs (see Ikeda and Watanabe [8], Chapter 6), we can show that $K_n \to K$ in $L^2(\nu)$. Hence it is enough to show that DK_n converges in $L^2(\mathbb{H}; \nu)$.

For $h \in \mathbb{H}$, let $\psi^n_* h \in \mathbb{H}$ be defined by the equation $D_{\psi^n_* h} = \psi^n_* D_h$. We have by Lemma 6.4

$$(DK_n)_{i,s} = D_{g_{i,s}}(K \circ \psi^n).$$

Let $f^n_{i,s} = \psi^n_* g_{i,s}$. Then we have from Lemma 6.5,

$$(DK_n)_{i,s} = D_{f_{i,s}} K = \{ Hf^n_{i,s}(\tau) + (K_{f^n_{i,s}})^*_\tau \} F. \tag{6.3}$$

Let

$$j^n_{i,s} = \psi^n_* \left\{ \frac{dg_{i,s}}{ds} \right\} = \psi^n_* h_{i,s},$$

where $h_{i,s}(\tau) = \chi_{[0,s]}(\tau) e_i$. Then

$$\frac{d}{ds}(DK_n)_{i,s} = \{ Hj^n_{i,s}(\tau) + (K_{j^n_{i,s}})^*_\tau \} F. \tag{6.4}$$

By approximation theory it can be shown that as $n \to \infty$,

$$E \int_0^1 |j^n_{i,s} - h_{i,s}|^2 \, ds \to 0, \qquad E \int_0^1 |K_{j^n_{i,s}} - K_{h_{i,s}}|^s \, ds \to 0. \tag{6.5}$$

Define DK by

$$(DK)_{i,s} = \{ Hg_{i,s}(\tau) + (K_{g_{i,s}})^*_\tau \} F.$$

Then

$$\frac{d}{ds}(DK)_{i,s} = \{ Hh_{i,s}(\tau) + (K_{h_{i,s}})^*_\tau \} K. \tag{6.6}$$

It follows from (6.4)–(6.6) that as $n \to \infty$,

$$\|DK_n - DK\|^2_{L^2(\mathbb{H};\nu)} = \sum_{i=1}^d E \int_0^1 \left\{ \frac{d}{ds}(DK_n)_{i,s} - \frac{d}{ds}(DK)_{i,s} \right\}^2 ds \to 0.$$

This shows that $K \in \mathrm{Dom}(D)$ and the proof is completed. ∎

Remark 6.7. It is proved in Driver and Röckner [4] that there exists a $W_o(M)$-valued diffusion process $\{X_\sigma, \sigma \in \mathbb{R}^1_+\}$ generated by $L/2$. Using the explicit formula (6.1) it is easy to see that the one-point motion $\{X_{\sigma,s}, \sigma \in \mathbb{R}^1_+\}$ is a Brownian motion on M with a drift, which is given at $\gamma \in W_o(M)$ by

$$-\tfrac{1}{2} U(\gamma)_s \left\{ (J^{-1}\gamma)_s - \tfrac{1}{2} \int_0^1 (s \wedge \tau)\, dA_\tau \right\},$$

where $\omega = J^{-1}\gamma$ and A is given by (5.15).

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