

## BOOK REVIEW

RICHARD F. BASS, *Probabilistic Techniques in Analysis*. Springer, New York, 1995.

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A significant portion of mathematical analysis, whether classical or modern, centers around the Laplace operator  $\Delta$  on Euclidean space and its generalizations to various other settings. On the other hand, the most important stochastic process in probability theory is Brownian motion  $B_t$ ,  $t \geq 0$ , because it lies at the intersection of several important classes of stochastic processes. The relation between these two objects is reflected in the classical formula of Dynkin

$$E_x f(B_t) = f(x) + \frac{1}{2} E_x \int_0^t \Delta f(B_s) ds,$$

which says that  $\Delta/2$  is the infinitesimal generator of the diffusion process  $B$ . It is this relation that provides the crucial link between probability theory and analysis. Analytically, Dynkin's formula says that the function  $u(t, x) = E_x f(B_t)$  is the solution of the heat equation  $(\partial/\partial t - \Delta/2)u = 0$  with the initial function  $f$ . In order to take full advantage of probability theory, we need to add two more ingredients to Dynkin's formula: Itô's formula and the use of stopping times. The result is a pathwise statement,

$$f(B_\tau) = f(B_0) + \int_0^\tau \nabla f(B_s) \cdot dB_s + \frac{1}{2} \int_0^\tau \Delta f(B_s) ds,$$

from which the original Dynkin formula can be easily recovered. The power of this formula lies in the fact that the added stochastic integral term is a martingale, which at the same time brings the gradient of the function into play. This formula, justifiably called the fundamental theorem of stochastic calculus, is the beginning of many applications of probability theory to analysis. The reader will find many examples of such applications in the book under review.

The book starts with a quick course on basic stochastic calculus in Chapter I. The chapter is self-contained to a certain extent, because proofs are generously supplied. Here the reader can find the author's own proof of the Doob–Meyer decomposition.

Chapter II covers applications of Brownian motion to classical potential theory. Topics treated in this chapter include the Dirichlet problem, Newtonian capacity, excessive functions and the Martin boundary. Most of the results in this chapter have appeared in book form [notably Port and Stone (1978), Doob (1984), and Blumenthal and Gettoor (1968)]. The author has worked hard

through the vast literature and the end result is an updated and very readable treatment of these topics.

Chapter III is devoted to topics related to Lipschitz Euclidean domains, including the boundary Harnack principle, the Martin boundary, Fatou's theorem on nontangential limits of harmonic functions and harmonic measures. It also includes a discussion of the conditional lifetime problem. Many of the results in this chapter appear in book form for the first time.

Chapter IV gives a probabilistic treatment of the theory of singular integrals. Here Brownian motion in a half Euclidean space stopped at the boundary plays a prominent role. Among the theorems proved in this chapter are the weak  $(1, 1)$  and/or  $L^p$  inequalities for maximal functions, Riesz transforms and the Littlewood–Paley functions. These results are applied to the Fourier multiplier problem,  $H^1$  spaces and functions of bounded mean oscillation. Most of the results in this chapter can be found in Stein (1993), and the reader should benefit from comparing the two approaches.

Chapter V contains applications to complex analysis. Topics include Picard's little theorem, Koebe's distortion theorem and several results concerning the boundary behavior of analytic functions. This chapter (and the book) concludes with a discussion of Carleson's corona theorem.

This well written book is primarily intended for probabilists who want to learn analysis from a probabilistic viewpoint. It should also be suitable for analysis-oriented advanced graduate students in probability, either for self-study or for classroom instruction. The carefully selected exercises at the end of each chapter are an integral part of the book, and the reader is strongly encouraged at least to read through them.

During my student years, my adviser, a dyed-in-the-wool probabilist, used to exhort me to find a probabilistic proof (or a probabilistic interpretation) for every theorem that came our way. As many of the proofs in the book under review illustrate, the line between a probabilistic proof and an analytical one is not always easily drawn. It is also unnecessary to draw such a line. As often happens, a proof that comes naturally is a mixture of techniques from both sources. Probability theory, as a field of research, is currently undergoing a transition not unlike what functional analysis underwent 20 years ago; namely, it is transforming itself into a collection of tools to be used in other fields of mathematics. It becomes more and more a way of thinking rather than a well defined branch of mathematics. In 20 years time or less, "pure" probabilists will be as rare as functional analysts are these days—they will have metamorphosed into hyphenated probabilists. This book is highly recommended both for current and for such future analyst–probabilists.

## REFERENCES

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