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★**An introduction to the analysis of paths on a Riemannian manifold. (English summary)**
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FEATURED REVIEW.

This latest book on stochastic analysis on manifolds from one of the major players in modern probability theory is a welcome addition to the existing literature on the subject. With this volume Stroock has brought into the field his unique approach as well as his highly precise expository style.

Unlike with many other mathematical books, I never skip the preface of a book by Stroock because I know there are always interesting stories told and at times confessions made. The dedication page reveals the names of the respective gurus of the two disciplines—probability and differential geometry—which the subject of the book bestrides. The preface tells the history of the author's marriage with the subject, a history not uncommon among many probabilists in the field. Coming to the point where the author confesses that the book is intended for his differential geometry friends and, by implication, not for me, I was seized by the desire to file the book into my well-arranged bookshelves and wondered whether it belonged to the geometry section or the probability section. Realizing that the book did not belong to me yet, I rationalized that the author didn't mean that literally, and with every book of his, even on a subject I know well, I have learned many useful and interesting things. With this conviction and the prospect of owning a volume without shedding any shekels, I decided to delve into the book.

I started with Chapter 1. Being familiar with the topics of the chapter, I soon found myself skipping to whichever sections caught my fancy, using the table of contents as a guide. It turned out to be a big mistake, for no matter how hard I tried to second guess the meanings of many symbols in the beautifully typeset and tantalizing displays, I was unable to resolve them locally. I rushed back to the table of contents and looked in vain for a list of notations. But once started on the book, I was not prepared to give it up. At this point I made the fateful decision to read the book in the way most mathematical books either are not meant to or do not deserve to be read, namely page by page. It turned out to be a decision, albeit costly, that paid off handsomely; otherwise I would have certainly missed many interesting facts and perspicacious observations scattered throughout the whole book. Having finished it in this fashion, I can now confidently share with the general public my appreciation of this unique work.

Chapter 1 deals with Brownian motion in Euclidean space. It shows what the subject of the book is like in the simplest setting. I paid special attention to the sections on the infinite-dimensional sphere and Feynman's picture of the Wiener measure.

Chapter 2 discusses diffusions in Euclidean space and reveals the author's unique treatment of the subject, which I remembered him once to have described as "Itô's calculus without Itô". That such an approach is possible is based on two fundamental ideas: the possibility of smooth approximation of diffusion paths and the concept of martingale problems, of which the author is one of the two creators. The chapter starts with a simple generator of the form $L = X^2/2$ and shows that the associated diffusion can be constructed easily by running a 1-dimensional Brownian motion along the integral curves of the vector field X. The general case of a Hörmander-type operator $L = X_0 + \sum_{k=1}^{d} X_k^2$ is treated first for commuting vector fields. For the noncommuting case, the scheme is first explained for smooth paths as motivation, and passing to the stochastic case is justified by Doob's well-known L^p martingale inequality. The chapter ends with discussions on smoothness of the diffusion with respect to the initial condition and on the flows generated by stochastic differential equations.

Chapter 3 deals with various topics related to diffusion processes, still in the setting of a Euclidean space. I found the treatment of the explosion problem in this chapter especially nice. The concept of subordination introduced in this chapter can be a useful tool for constructing diffusions with special generators. The discussion on invariant measures is concise and to the point.

With Chapter 4 the author launches his extrinsic approach to diffusions on manifolds. This approach has the obvious advantage of a big reward after a reasonable initial investment in the first three chapters. Unfortunately, from this approach it may sometimes be difficult to extract meaningful differential-geometric information. Still, it is an economical way of quickly getting oneself into the subject, bearing in mind that until curvature comes into the picture one has not touched differential geometry yet. I found reading the section on Brownian motion on a submanifold a highly rewarding experience, especially the description of the normal component of the ambient Brownian motion in Theorem 4.42. The nonexplosion criterion in terms of the mean curvature tensor of an embedded submanifold in Theorem 4.43 was new to me, although it should be pointed out that in the case of a minimal surface embedded in a Euclidean space, the nonexplosion of Brownian motion can be concluded more quickly from the fact that coordinate functions are harmonic functions, whose gradients are bounded by 1 in absolute value.

Chapter 5 is a natural continuation of Chapter 4 with emphasis on more geometric topics. After a discussion on stochastic parallel transport, the curvature tensor finally makes its appearance. The connection between the rate of escape of Brownian motion and the lower bound of the Ricci curvature is explained by showing that Brownian motion does not explode if the Ricci curvature does not decrease negatively faster than quadratically. Both probabilistic proofs and analytic proofs can be found in the literature, but the proof presented here has the distinct feature of being quantitative because it gives a precise Gaussian-type upper bound for the probability that Brownian motion has an oscillation of a given size within a fixed amount of time.

Chapter 6 deals with Bochner's identity and its consequences. The first part is devoted to various results on the Jacobian of the diffeomorphism semigroup, in particular its connection with the Ricci curvature. If the Ricci curvature is bounded from below by a positive constant K, the analysis on the Jacobian shows that the spectral gap of the Laplacian is not less than K. As explained at the end of Section 6.1, this misses the optimal Lichnerowicz bound $dK/(d-1)$ by a factor of $d/(d-1)$, where d is the dimension of the manifold. The reason is, of course, that the probabilistic method does not take into account the average decay of grad f itself along Brownian paths. This eigenvalue estimate is followed by the well-known probabilistic representation of grad $P_t f(x)$

in terms of a multiplicative functional determined by the Ricci curvature along Brownian paths, Bakry's approach to the logarithmic Sobolev inequality for the heat kernel measure, and an integration-by-parts formula due to Bismut.

With Chapter 7 we start working intrinsically. The chapter itself is a review, in intrinsic terms, of some basic facts about differential geometry and Brownian motion already covered in the previous chapters. Chapter 8 explains the Malliavin-Eells-Elworthy theory of horizontal Brownian motion in the orthonormal frame bundle $O(M)$ (upstairs versus downstairs on M itself). Doing things upstairs is indeed very convenient because the stochastic parallel transport is simply the horizontal Brownian motion itself. Here and in some other places in the book, I took the author's advice and skipped the subsections on measurability considerations. Section 8.3, entitled "Curvature considerations and an explosion criterion", is a quick review of some differential-geometric facts about the orthonormal frame bundle, and the content of the second half of the title is found in the next section, where the important role of the distance function (or the radial part of Brownian motion) is explained. Section 8.4 ends with a result due to Varopoulos: If the Ricci curvature grows more negatively than $-\beta(r)^2$ with $\int_1^\infty dr/\beta(r) < \infty$ (plus some technical conditions), then Brownian motion does explode.

Chapter 9 is devoted to the local behavior of Brownian motion on manifolds. Section 9.4 gives a quantitative version of the folklore to the effect that locally Brownian motion on a Riemannian manifold behaves just like Euclidean Brownian motion. This result is used in the next section to obtain a two-term expansion of the expected exit time from a small geodesic ball due to Pinsky and his collaborators.

The final Chapter 10 brings us to some topics of current research, and in particular to some of the author's recent work on the perturbation of Brownian motion paths on a Riemannian manifold. The idea here is to find perturbations such that the stochastic anti-development of the perturbed motion is still a Euclidean Brownian motion. A heuristic calculation carried out in Section 10.1 shows that the perturbation vector field should satisfy an ODE along a Brownian path involving the Ricci tensor. This vector field in turn determines an ODE on the path space and generates a flow on the same space. Two applications are given. The first one is the proof that, under the direct coupling (the Kendall-Cranston coupling without tangential flipping), the distance of the coupled Brownian motions decays at the rate of e^{-Kt}, where K is the lower bound of the Ricci curvature. The second application is the proof of Bismut's famous formula for the gradient of the solution of the heat equation.

The very last section of the book has the title "An admission of defeat", a rather pessimistic note to end the book. After learning so much from the book, I wished that this was one admission that Stroock had kept to himself. While I agree with the author's general assessment that at the present time probabilistic techniques as applied to differential geometry do not come close in competing with well-established analytic techniques in this field, there are several ways we can justify the continuing research in this area. First of all, while keeping our aim high at solving open problems in differential geometry, we create and solve at the same time new and interesting geometric problems of purely probabilistic interest. The highly interesting recent research on the behavior of Brownian motion on Riemannian manifolds under various geometric conditions by A. Grigoryan and his collaborators is a case in point. Second, when in 1984 two Chinese mathematicians, H. C. Yang and J. Q. Zhong, proved that the spectral gap of a compact Riemannian manifold with nonnegative Ricci curvature is bounded from below by $\pi^2/\mathrm{diam}(M)^2$, not many probabilists knew this result. Ten years later, using the Kendall-Cranston coupling, another pair of Chinese mathematicians, M. F. Chen and F. Y. Wang, succeeded in giving a highly motivated probabilistic proof. Also, the missing factor $d/(d-1)$ in the probabilistic proof of the Lichnerowicz lower bound on pp. 147–148 can be retrieved probabilistically by a similar coupling argument. While the probabilistic methods are in no way simpler than analytic methods in absolute terms, they are certainly so for probabilists and serve as a vehicle to gain a wider audience in the probability community for these beautiful geometric results. We can reasonably hope that new probabilistic techniques will emerge to help us gain further insight into known geometric results and prove new ones. Third, the recent interest and extensive research in the relatively new area of stochastic analysis on path and loop spaces over Riemannian manifolds following Driver's ground-breaking results on the quasi-invariance of the Wiener measure and the integration-by-parts formulas in these spaces are largely the continuation of the research in the so-called "stochastic differential geometry" during the last 25 years.

My odyssey with this highly informative book ended just in time to fend off the last warning from Mathematical Reviews, which threatened that I may have to part with the book. This will not happen, for the book, by now somewhat worn by many months of commuting with me through the unpredictable Chicago weather and traffic, belongs to me.

Reviewed by *Elton Pei Hsu*