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SHORT–TIME ASYMPTOTICS OF THE HEAT KERNEL ON A CONCAVE BOUNDARY*

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Abstract. A probabilistic method is used to study short-time asymptotic behavior of heat kernel in the exterior of an insulated smooth convex body. The expansion of the heat kernel $p(t,a,b)$ when both a and b are on the boundary is obtained by reducing the problem to the computation of a Wiener functional on a Brownian bridge. The leading terms of $\log p(t,a,b)$ are proved to be

$$-\frac{\rho^2}{2t} - \frac{\mu_1 \rho^{1/3}}{t^{1/3}} \int_0^\rho N(s)^{2/3} ds - \left(\frac{d}{2} + \frac{1}{6}\right) \log t + C_0 + o(1)$$

where ρ is the distance between a and b, $N(s)$ is the normal curvature of the geodesic joining a and b, and C_0 is an explicitly identified constant.

Key words. heat kernel, Laplace–Beltrami operator, normal curvature, diffusion process on manifold, Brownian bridge, Feynman–Kac formula, Girsanov formula

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1. Introduction. Let M be the exterior of a smooth, strictly convex body in Euclidean space. Let a, b be two points on the boundary such that there is a unique distance-minimizing curve γ joining them which lies completely in M. Since ∂M is concave viewed from M, it is clear then γ must be the unique geodesic joining a and b in ∂M when ∂M is viewed as a Riemannian manifold with induced metric. Let us denote the length of γ by $\rho = d(a,b)$.

Let $p(t,x,y)$ be the heat kernel of the Laplace operator $\Delta/2$ on the domain M with the Neumann boundary condition on ∂M. In this paper we are interested in the asymptotic behavior of $p(t,a,b)$ as $t \to 0$. Recall the basic result of Varadhan [10]:

$$\lim_{t\to 0} t \log p(t,a,b) = -\frac{1}{2}\rho^2. \tag{1.1}$$

Our problem is to seek an improvement of (1.1) which reflects the geometry of the boundary near the geodesic γ. It has long been recognized in the diffraction theory that the correction to (1.1) takes the following form

$$\log p(t,a,b) = -\frac{\rho^2}{2t} - \frac{C}{t^{1/3}} + o\left(\frac{1}{t^{1/3}}\right) \tag{1.2}$$

where C is a positive constant. In fact, using the idea of path integration, Buslaev [2] was able to give a heuristic argument of (1.2) and identified constant C explicitly. However, to make his argument into a mathematically acceptable proof seems not to be a simple matter. Equation (1.2) has long been known in physics literature as Busleav's conjecture.

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We will study the expansion (1.2) by a probabilistic method initiated by Molchanov [9]. Our result can be briefly described as follows. We parametrize the geodesic γ by arclength. Let $N(s)$ be the normal curvature of γ (as a curve in $\partial M \hookrightarrow M$) at point $\gamma(s)$. $N(s)$ is simply the curvature of γ when viewed as a curve in the Euclidean space. Since ∂M is the exterior of a strictly convex body, $N(s)$ is strictly positive along γ. The asymptotic behavior of $p(t,a,b)$ is described by

$$\log p(t,a,b) = -\frac{\rho^2}{2t} - \frac{\mu_1 \rho^{1/3}}{t^{1/3}} \int_0^\rho N(s)^{2/3} ds - \left(\frac{d}{2} + \frac{1}{6}\right) \log t + C_0 + o(1).$$

Here μ_1 is the first eigenvalue of $\phi''(x)/2 - |x|\phi(x) + \mu\phi(x) = 0$ on R^1 and C_0 is nonzero constant.

Probabilistically, the heat kernel $p(t,x,y)$ is the transition density function of reflecting Brownian motion on M. By a series of asymptotic analyses, we reduce the computation of $p(t,a,b)$ to that of the following Wiener functional on the standard Brownian bridge $\hat{W}$:

$$E\left[\exp\left\{-\lambda \int_0^1 l(s)|\hat{W}_s| ds\right\}\right], \tag{1.3}$$

where l is a smooth, strictly positive function.

Our research is inspired by the work of Ikeda [5], where a special case of the present problem is discussed. In [5], manifold M is assumed to have the form of a warped product (thus the normal curvature $N(s)$ is a constant). This assumption allows us to construct Brownian motion on M by skew product and to simplify the analysis involved. In our present work, we have further explored some ideas from [5]. For a related problem under a different context, see Melrose and Taylor [8].

The plan of this work is as follows. In §2, we make precise our geometric assumptions and state our main theorem. The proof of the main theorem is outlined in §3. In order not to interrupt the main line of argument, verifications of some intermediate results used in §3 are relegated to §§4 and 5. The asymptotic analysis of the Wiener functional (1.3) is carried out in §6.

Note. The author was informed that Professor N. Ikeda has also obtained results related to the present work.

2. Assumptions and the main theorem. Unfortunately the Euclidean coordinate system is not suitable for our work. We therefore need a little elementary differential geometry. Although we may sometimes discuss the problem under general differentio-geometrical setting, the case where M is the exterior of a smooth, strictly convex body is our primary concern. We will see that various geometrical assumptions we make along the way are satisfied in this important case.

So let us assume that (M,g) is a Riemannian manifold with smooth boundary ∂M. We assume that ∂M is strictly concave when viewed from M. Mathematically this means that the second fundamental form (defined below) of ∂M is strictly positive definite. Now let a and b be two points on ∂M such that there is a unique geodesic in ∂M joining them on which they are not conjugate. For example, a and b can be any two nonantipodal points on a sphere. The geodesic is the arc of the great circle passing through a and b of lesser length. We can set up a semigeodesic coordinate system $\tilde{x} = (x^2, \cdots, x^d)$ on ∂M in a neighborhood of γ with a as the origin and x^2 in the direction of the geodesic γ (cf. Molchanov [9, p. 10]). We let $x = (x^1, \tilde{x})$ be the point in M which lies on the geodesic passing through $\tilde{x}$ and perpendicular to ∂M with $x^1 = d(x, \partial M)$.

Instead of M, which has a boundary, we can consider $(M^- \cup M, g)$ the double of M. Here M^- is just a copy of M, and M and M^- are identified along the boundary. The heat kernel on $M^- \cup M$ and the Neumann heat kernel on M are related in a very simple way (see §3 below).

The second fundamental form H of the boundary ∂M can be identified with the matrix

$$H_{ij} = H\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle \nabla_i \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^1} \right\rangle$$

($\nabla_i = \nabla_{\partial/\partial x^i}$ is the covariant derivative). The normal curvature of γ at $\gamma(s)$ is by definition

$$N(s) = H_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))$$

(see [7, p. 44]). For brevity, we sometimes write $H_{ij}(s)$ for $H_{ij}(\gamma(s))$ and $g(s)$ for $g(\gamma(s))$. The following lemma clarifies the geometric meaning of the second fundamental form.

LEMMA 2.1. *Let $g = (g_{ij})$ be the metric matrix in the semigeodesic coordinates.* (a) *We have*

$$g_{1i}(x) = \delta_{1i}, \qquad g_{2i}(0, \tilde{x}) = \delta_{2i}, \quad i = 1, \cdots, d.$$

(b) *Near the boundary ∂M, the metric matrix has the expansion*

$$g_{ij}(x) = g_{ij}(0, \tilde{x}) + 2H_{ij}(\tilde{x})|x^1| + O(|x^1|^2), \qquad 2 \le i, j \le d.$$

Proof. Since the coordinate line $\tilde{x} =$ const. is a geodesic perpendicular to N, we have $g_{1i}(0, \tilde{x}) = \delta_{1i}$ for $x \in \partial M$ and $\nabla_1(\partial/\partial x^1) = 0$. This implies

$$\nabla_1 g_{1i} = \left\langle \frac{\partial}{\partial x^1}, \nabla_1 \frac{\partial}{\partial x^i} \right\rangle = \frac{1}{2} \nabla_i \left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right\rangle = 0.$$

It follows that $g_{1i}(x) = \delta_{1i}$. The same proof applies to $g_{2i}(0, \tilde{x})$. Part (a) is proved.

By the definition of the second fundamental form and part (a), we have on ∂M

$$H_{ij}(\tilde{x}) = \left\langle \frac{\partial}{\partial x^j}, \nabla_i \frac{\partial}{\partial x^1} \right\rangle = \left\langle \frac{\partial}{\partial x^i}, \nabla_1 \frac{\partial}{\partial x^j} \right\rangle = \frac{1}{2} \nabla_1 g_{ij}.$$

Part (b) follows immediately.

We will prove our asymptotic formula for the heat kernel under the following two geometrical assumptions.

Assumption (A). The normal curvature $N(s) = H_{22}(s), 0 \le s \le \rho$, is *strictly* positive along the geodesic γ.

Assumption (B). For any neighborhood G of γ in M, there exists $\epsilon > 0$ such that any piecewise smooth curve in M joining a and b with length $\le d(a, b) + \epsilon$ lies completely inside G. Equivalently, $d(a, b) < d(a, \partial G) + d(b, \partial G)$ for any neighborhood G of γ.

It is easy to verify that in the case where M is the exterior of a strictly convex body in the Euclidean space, the above assumptions (A) and (B) are satisfied.

Let (μ_1, ϕ_1) be the first normalized eigenpairs of the eigenvalue problem

$$\frac{1}{2}\phi''(x) - |x|\phi(x) + \mu\phi(x) = 0, \qquad x \in R^1.$$

We are in a position to state our main result.

THEOREM. *Let M be a Riemannian manifold with boundary and $p(t,x,y)$ the heat kernel of the Laplace–Beltrami operator $\Delta/2$ on M under the Neumann boundary condition (insulated boundary). Suppose that a and b are two points on the boundary such that there is a unique geodesic in the boundary ∂M joining them along which they are not conjugate. Then under further assumptions* (A) *and* (B), *we have as $t \to 0$,*

$$p(t,a,b) \approx \\ \gamma H(a,b)\rho^{2/3}[N(a)N(b)]^{1/6}t^{-(d/2+1/6)} \exp\left\{-\frac{\rho^2}{2t} - \frac{\mu_1\rho^{1/3}}{t^{1/3}}\int_0^\rho N(s)^{2/3}ds\right\},$$

where

$$\gamma = 2(2\pi)^{-(d-1)/2}|\phi_1(0)|^2$$

and

$$H(a,b) = \frac{[g(a)g(b)]^{-1/4}}{\left[\det \int_0^\rho g(s)^{-1}ds\right]^{1/2}}\rho^{2/3}.$$

Remark. $H(a,b)$ has an intrinsic geometric meaning, cf. Molchanov [9, p. 14–15].

Before proving this theorem, we need to transform Assumption (B) into a form more suitable for computation. Let G be any neighborhood of γ; by Assumption (B) we have $d(a,b) < d(a,\partial G) + d(b,\partial G)$. One important consequence of this assumption is that the computation of the asymptotic behavior of $p(t,a,b)$ can be localized inside G. This means that the metric outside G has no effect on the asymptotics of $p(t,a,b)$. In fact if p_{g_1} and p_{g_2} are two heat kernels for the metrics g_1 and g_2 which coincide on G, then we have

$$\lim_{t\to 0}\frac{p_{g_1}(t,a,b)}{p_{g_2}(t,a,b)} = 1$$

(See Azencott [1, p. 157]). Note that in [1], the above relation is proved under the assumption $d(a,b) < \max\{d(a,\partial G), d(b,\partial G)\}$. The result holds, however, under the more relaxed condition $d(a,b) < d(a,\partial G)+d(b,\partial G)$. See Hsu [4] for details. Therefore, for the purpose of computing the asymptotics of $p(t,a,b)$, we may arbitrarily alter the metric outside G to facilitate the computation. Thus we can assume that $M = R^n_+ = \{x = (x^1, x^2, \cdots, x^n) : x^1 \geq 0\}$, $M^- \cup M = R^n$, and that the metric is Euclidean outside a small neighborhood G of γ. Let $g^{-1} = (g^{ij})$ be the inverse of the metric matrix. From Lemma 2.1 a simple calculation shows

$$g^{22}(x) = 1 - 2H_{22}(\tilde{x})|x^1| + O(|x^1|^2).$$

We can then impose the following global assumptions on g^{22}:

Assumption (B1). For all $x \in R^n$, we have $g^{22}(x) \leq 1$.

Assumption (B2). There exists a constant $\gamma > 0$ such that for all $x \in R^n$,

$$1 - 2H_{22}(\tilde{x})|x^1| - \gamma|x^1|^2 \leq g^{22}(x) \leq 1 - 2H_{22}(\tilde{x})|x^1| + \gamma|x^1|^2.$$

The reason that we can make Assumptions (B1) and (B2) is simple: These two assumptions hold on a small neighborhood G of the geodesic γ. We can then choose the metric so that they also hold outside G. Let us emphasize once more that (B1) and (B2) are derived from (A) and (B) and the above mentioned localization principle. We may prove our main theorem under these assumptions without losing the generality of our result.

Remark. A casual reader might think Assumption (B) is redundant because it should always hold. (B) may fail if M is not complete in its Riemannian metric. Since the Euclidean space is complete, (B) indeed holds in this case. On the other hand, Assumption (A) is essential.

Finally let us look at a simple example where (B1) and (B2) are satisfied by the obvious choice of coordinates.

Example. Let $M \subset R^2$ be the exterior component of the ellipse: $x = a\cos\theta, y = \sin\theta$. Introduce coordinates (θ, t) on M:

$$x = \left(a + \frac{bt}{\lambda(\theta)}\right)\cos\theta, \qquad y = \left(b + \frac{at}{\lambda(\theta)}\right)\sin\theta$$

where $\lambda(\theta)^2 = a^2\sin^2\theta + b^2\cos^2\theta$. A simple calculation shows

$$dx^2 + dy^2 = dt^2 + \lambda(\theta)^2\left(1 + \frac{ab}{\lambda(\theta)^3}\,t\right)^2 d\theta^2.$$

Let

$$x^1 = t, \qquad x^2 = \int_0^\theta \lambda(u)du.$$

We have

$$g^{22}(x) = \left(1 + \frac{ab}{\lambda(\theta)^3}\,x^1\right)^{-2}$$

and

$$H_{22}(x^2) = \frac{ab}{\lambda(\theta)}.$$

Clearly, Assumptions (A), (B), (B1), and (B2) are satisfied.

3. Proof of the theorem. Let $g^{-1} = (g^{ij})$ be the inverse of the metric matrix g. The Laplace–Beltrami operator on $(M^- \cup M = R^d, g)$ is given by

$$\Delta = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\,g^{ij}\frac{\partial}{\partial x^j}\right) = g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + 2b^i\frac{\partial}{\partial x^i}$$

where

$$b^i = \frac{1}{2}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}\,g^{ij}).$$

Let $X = \{X_t = (X_t^1, \cdots, X_t^d) : t \geq 0\}$ be the Riemannian Brownian motion on $(M^- \cup M, g)$, i.e., the diffusion process generated by $\Delta/2$ (cf. Ikeda–Watanabe [6, Chaps. IV, V]). Denote by $\tau : M^- \cup M \to M$ the natural projection. Then the process $\tau(X)$ is the reflecting Brownian motion on (M, g). Let p^X and $p^{\tau(X)} = p$ be the respective transition density. Then obviously

$$p\,(t, x, y) = p^X(t, x, y) + p^X(t, x, y^*)$$

where $\{y, y^*\} = \tau^{-1}(y)$. In particular, since $b^* = b$

$$p(t,a,b) = 2\,p^X(t,a,b). \tag{3.1}$$

Diffusion process X can be obtained as the solution of the stochastic differential equation on R^d:

$$dX_s = \sigma(X_s)\,dB_s + b(X_s)\,ds.$$

Here σ is a smooth square root of g and B is a standard Brownian motion in R^d.

The behavior of the heat kernel $p(t,a,b)$ depends on the law of the Brownian bridge from a and conditioned to reach b at time t. As $t \to 0$, the Brownian bridge tends to travel along the geodesic γ with uniform speed ρ/t. Let

$$Y_s = X_s - \frac{s\rho}{t}e_2.$$

(e_2 is the unit vector $(0,1,0,\cdots,0)$ in x^2-direction.) We therefore expect Y to be a process with small magnitude. The equation for Y is

$$dY_s = \sigma\left(Y_s + \frac{s\rho}{t}e_2\right)\,dB_s + \left[b\left(Y_s + \frac{s\rho}{t}e_2\right) - \frac{\rho}{t}e_2\right]\,ds.$$

We now alter the drift of this equation by the Girsanov transform. Consider a new equation

$$dZ_s = \sigma\left(Z_s + \frac{s\rho}{t}e_2\right)\,dB_s + c\left(Z_s + \frac{s\rho}{t}e_2; t\right)\,ds. \tag{3.2}$$

Let P^Y and P^Z denote the laws of the processes Y and Z on the sample path space $C([0,t] \to R^n)$. By the Girsanov formula (Ikeda–Watanabe [6, p. 180]), we have

$$\frac{dP^Y}{dP^Z} = \exp\left[\int_0^t \left\langle h\left(Z_s + \frac{s\rho}{t}e_2; t\right), dB_s\right\rangle - \frac{1}{2}\int_0^t \left\|h\left(Z_s + \frac{s\rho}{t}e_2; t\right)\right\|^2 ds\right] \stackrel{\text{def}}{=} N_t.$$

where

$$h(x;t) = \sigma(x)^{-1}\left[b(x) - c(x;t) - \frac{\rho}{t}e_2\right]. \tag{3.3}$$

Let D be a neighborhood of a (the origin of the coordinates). We have

$$\begin{aligned} P[X_t \in D + \rho e_2] &= P[Y_t \in D] = E[N_t; Z_t \in D] \\ &= \int_D E\left[N_t \middle| Z_t = y\right] P\left[Z_t \in dy\right]. \end{aligned}$$

It follows from this and (3.2) that

$$p(t,a,b) = \frac{2}{\sqrt{\det\, g(\rho)}} E[N_t | Z_t = 0]\, p^Z(0,0;t,0). \tag{3.4}$$

where $p^Z(s,z;v,x)$ is the transition density of the process Z (with respect to the Lebesgue measure) defined by (3.2). Formula (3.4) is the key to the subsequent discussion.

We now choose the drift c in (3.2):

$$c(x;t) = b(x) - b^1(x)\, e_1 - \frac{\rho}{t}\,[I - g(x)^{-1}]\, e_2. \tag{3.5}$$

Or, what is the same thing (see (3.3))

$$h(x;t) = \sigma(x)\left[b^1(x)\, e_1 - \frac{\rho}{t} e_2\right]. \tag{3.6}$$

The advantage of this choice will be clear later. Note that

$$c^1\left(z + \frac{s\rho}{t} e_2\right) \equiv 0.$$

This means that the first component Z^1 of (3.2) is simply a one-dimensional Brownian motion.

The last two factors on the right-hand side of (3.4) will now be analyzed separately. First of all, we have the following lemma.

LEMMA 3.1. *As $t \to 0$, we have*

$$p^Z(0,0;t,0) = \left(\frac{1}{2\pi t}\right)^{d/2} H_1\left[1 + O(\sqrt{t})\right]$$

where

$$H_1 = \rho^{d/2}\left[\det \int_0^\rho g(s)^{-1} ds\right]^{-1/2}.$$

To study $E[N_t | Z_t = 0]$, set

$$Z_s^* = Z_s + \frac{s\rho}{t} e_2$$

for brevity. Using (3.6), we verify easily

$$\begin{aligned}
&\int_0^t \langle h(Z_s^*;t), dB_s\rangle \\
&\quad = \int_0^t \langle \sigma(Z_s^*)^{-1} h(Z_s^*;t), dZ_s - c(Z_s^*;t)\, ds\rangle \\
&\quad = -\frac{\rho^2}{t^2}\int_0^t [1 - g^{22}(Z_s^*)]\, ds - \frac{\rho}{t} Z_t^2 + \int_0^t b^1(Z_s^*)\, dZ_s^1 + \frac{\rho}{t}\int_0^t b^2(Z_s^*)\, ds.
\end{aligned}$$

We also have

$$\|h(Z_s^*;t)\|^2 = \frac{\rho^2}{t^2} - \frac{\rho^2}{t^2}[1 - g^{22}(Z_s^*)] + |b^1(Z_s^*)|^2.$$

It follows that

$$\log N_t = -\frac{\rho^2}{2t} - \Theta_t + \log H_2 + F_t \tag{3.7}$$

with

$$\Theta_t = \frac{\rho^2}{2\, t^2}\int_0^t [1 - g^{22}(Z_s^*)]\, ds \tag{3.8}$$

$$H_2 = \exp\left\{\int_0^\rho b^2(se_2)\,ds\right\} = \left[\frac{\det g(\rho)}{\det g(0)}\right]^{1/4}$$

and

(3.9)

$$F_t = -\frac{\rho}{t}Z_t^2 + \int_0^t b^1(Z_s^*)\,dZ_s^1 + \frac{\rho}{t}\int_0^t \left[b^2(Z_s^*) - b^2\left(\frac{s\rho}{t}e_2\right)\right]\,ds - \frac{1}{2}\int_0^t |b^1(Z_s^*)|^2 ds.$$

It is clear now that the proof of the main theorem in §2 will be completed if we show the following three lemmas.

LEMMA 3.2. *Let $\hat{W}$ be the standard Brownian bridge. We have*

$$\lim_{t\to 0}\frac{E\left[\exp\{-\Theta_t\}\middle| Z_t = 0\right]}{E\left[\exp\left\{-\frac{\rho^2}{\sqrt{t}}\int_0^1 N(s\rho)|\hat{W}_s|ds\right\}\right]} = 1.$$

LEMMA 3.3. *We have*

$$\lim_{t\to 0}\frac{E\left[\exp\{-\Theta_t + F_t\}\middle| Z_t = 0\right]}{E\left[\exp\{-\Theta_t\}\middle| Z_t = 0\right]} = 1.$$

Let

$$S(\lambda; l) \stackrel{\text{def}}{=} E\left[\exp\left\{-\lambda\int_0^1 l(s)|\hat{W}_s|ds\right\}\right].$$

LEMMA 3.4. *Let $l : [0,1] \to R_+$ be twice continuously differentiable and strictly positive on the closed interval $[0,1]$, then for any $k \geq 0$*

$$S(\lambda; l) = \sqrt{2\pi}|\phi_1(0)|^2[l(0)\,l(1)]^{1/6}\lambda^{1/3}\exp\left\{-\mu_1\lambda^{2/3}\int_0^1 l(s)^{2/3}ds\right\}[1 + O(\lambda^{-k})].$$

The next three sections are devoted to the proof of Lemmas 3.1 to 3.4.

4. Proof of Lemma 3.1. Throughout the rest of this paper, letters $c_1, c_2, \cdots$, whose values may change from one appearance to another, represent constants depending only on the geometry of the manifold.

By (3.2), the function $p^Z(s, z; v, y)$ is the fundamental solution of the parabolic operator

$$L = \frac{\partial}{\partial s} + \frac{1}{2}g^{ij}\left(z + \frac{s\rho}{t}e_2\right)\frac{\partial^2}{\partial z^i\partial z^j} + c^i\left(z + \frac{s\rho}{t}e_2; t\right)\frac{\partial}{\partial z^i}. \tag{4.1}$$

Let us investigate the coefficients c more carefully. First of all, as we have pointed out before, $c^1 \equiv 0$. By Lemma 2.1

$$g(x)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{g}(\tilde{x})^{-1} - 2\tilde{g}(\tilde{x})^{-1}H(\tilde{x})\tilde{g}(\tilde{x})^{-1}|x^1| + O(|x^1|^2)\end{pmatrix}.$$

($\tilde{g}$ is the last $(n-1)\times(n-1)$ principal minor of g). Hence near the geodesic γ

$$\tilde{c}(z + se_2) = \tilde{b}(z + se_2) - \frac{2\rho}{t}D(s)\,e_2|z^1| + \frac{1}{t}O(|z^1|\|z\|) \tag{4.2}$$

where

$$D(s) = \tilde{g}(se_2)^{-1} H(se_2)\,\tilde{g}(se_2)^{-1}.$$

We prove Lemma 3.1 by the method of parametrix (cf. Friedman [3]). We need to pay special attention to the dependence of the coefficients on t.

Let L^x be the operator obtained from L by freezing the coefficients of L at $z = x$. Set for a positive definite matrix A

$$\Gamma(A, y) = \frac{1}{(2\pi)^{d/2}\sqrt{\det A}} \exp\left\{-\frac{1}{2}\langle y, A^{-1}y\rangle\right\}.$$

Let

$$u_0(s, z; v, x) = \Gamma(\lambda_x(s, v), x - z)$$

with

$$\lambda_x(s, v) = \int_s^v g\left(x + \frac{l\rho}{t} e_2\right)^{-1} dl.$$

We have

$$u_0(0, 0; t, 0) = \left(\frac{1}{2\pi t}\right)^{d/2} H_1 \tag{4.3}$$

with the same H_1 as in the statement of Lemma 3.1.

Now $p^Z = u$ can be obtained by iteration from the equation

$$u(s, z; v, x) = u_0(s, z; v, x) + \int_s^v dl \int_{R^d} u(s, z; l, y)(L - L^x)\, u_0(l, y; v, x)dy.$$

We thus obtain an absolutely convergent series $p^Z = \sum_{m=0}^{\infty} u_m$. Using the easy estimate

$$\|g(z) - g(x)\| + t\|c(z; t) - c(x; t)\| \le c_3\|z - x\| \tag{4.4}$$

which follows from (3.5), we verify by induction the following estimate:

$$|u_m(s, z; v, x)| \le \frac{c_1 c_2^m}{\Gamma(m/2 + 1)} \left[1 + \frac{\sqrt{v - s}}{t}\right] (v - s)^{(m-d)/2} \exp\left\{-\frac{\|z - x\|^2}{c_1(v - s)}\right\}.$$

It follows immediately that

$$p^Z(s, z; x, v) \le \frac{c_4}{(v - s)^{d/2}} \left[1 + \frac{v - s}{t}\right] \exp\left\{-\frac{\|z - x\|^2}{c_4(v - s)}\right\}. \tag{4.5}$$

Now that we have

$$\sum_{m=2}^{\infty} |u_m(0, 0; t, 0)| \le c_5 t^{-(d-1)/2},$$

it is easy to see from (4.3) that the assertion of Lemma 3.1 is implied by the inequality

$$|u_1(0, 0; t, 0)| \le c_6 t^{-(d-1)/2}, \tag{4.6}$$

which we are about to show. By the iteration formula

$$u_1(0,0;t,0) = \int_0^t dl \int_{R^d} \Gamma(\lambda_y(0,l),y)[L - L^0]\Gamma(\lambda_0(l,t),y)\,dy. \tag{4.7}$$

From (4.2), we have

$$L - L^0 = \alpha^{ij}\frac{\partial^2}{\partial y^i \partial y^j} + \beta^i \frac{\partial}{\partial y^i} + \frac{\rho}{t}\gamma^i \frac{\partial}{\partial y^i} - \frac{2\rho}{t}\sum_{i=2}^{n} D_{2i}\left(\frac{s\rho}{t}\right)|y^1|\frac{\partial}{\partial y^i}$$

with $\alpha = \alpha(y,l,t)$, etc., satisfying

$$\|\alpha\| + \|\beta\| + \|\gamma\|^{1/2} \le c_7\|y\|. \tag{4.8}$$

This fact together with (4.7) gives

$$\begin{aligned} u_1(0,0;t,0) = &-\frac{2\rho}{t}\int_0^t D_{2i}\left(\frac{l\rho}{t}\right) dl \int_{R^n} [\Gamma(\lambda_y(0,l),y) - \Gamma(\lambda_0(0,l),y)]|y^1| \\ &\times \frac{\partial}{\partial y_i}\Gamma(\lambda_0(l,t),y)dy + O(t^{-(d-1)/2}). \end{aligned} \tag{4.9}$$

(Inserting $\Gamma(\lambda_0(0,l),y)$ creates a term equal to zero after integration.) Finally using the inequality

$$|\Gamma(\lambda_y(0,l),y) - \Gamma(\lambda_0(0,l),y)| \le c_8 l^{-d/2}\|y\|e^{-\|y\|/c_8 l}$$

we obtain (4.6) from (4.9) by simple estimation. The proof of Lemma 3.1 is therefore complete.

5. Proof of Lemma 3.2 and Lemma 3.3. We adopt the following notational convention. If $G(Z)$ is a functional of the process Z, the same functional of Z conditioned by $Z_t = 0$ is denoted by $\hat{G}$, i.e., $\hat{G} = G(\hat{Z})$. Also if $x \in R^n$, then $\tilde{x} = (x^2, \cdots, x^d)$.

Set

$$Z_s^t = \frac{Z_{st}}{\sqrt{t}}, \qquad M^t = \max_{0\le s\le 1}\|Z_s^t\|, \qquad \tilde{M}^t = \max_{0\le s\le 1}\|\tilde{Z}_s^t\|.$$

As mentioned immediately before Lemma 3.1, the process

$$\{W_s \stackrel{\text{def}}{=} Z_s^{t,1}; 0 \le s \le 1\}$$

is a one-dimensional Brownian motion.

Let P_W be the law of $\tilde{Z}^t$ conditioned by the process W. This means that under the probability P_W, the process $\tilde{Z}^t$ is the solution of the stochastic differential equation

$$d\tilde{Z}_s^t = \tilde{\sigma}(\sqrt{t}W_s e_1 + \sqrt{t}\tilde{Z}_s^t + s\rho e_2)\,d\tilde{B}_s + \sqrt{t}\tilde{c}(\sqrt{t}W_s e_1 + \sqrt{t}\tilde{Z}_s^t + s\rho e_2)\,ds.$$

In this equation $W = \{W_s; 0 \le s \le 1\}$ is assumed to be deterministic. Let p^{Z^t} be the transition density of Z^t and let $p_W^{\tilde{Z}^t}$ be that of the process $\tilde{Z}^t$ under the probability P_W.

Inequality (4.6) and Lemma 3.1 can be paraphrased as follows:

$$p^{Z^t}(s,z;v,x) \le \frac{c_2}{(v-s)^{d/2}} e^{-\|z-x\|^2/c_2(v-s)} \le \frac{c_3}{\|z-x\|^d}$$

$$p^{Z^t}(0,0;1,0) = \left(\frac{1}{2\pi}\right)^{d/2} [1+O(\sqrt{t})] \ge c_4.$$

The proof of Lemma 3.1 can be applied to obtain the following estimates for function $p_W^{\tilde{Z}^t}$:

$$p_W^{\tilde{Z}^t}(s,\tilde{z};v,\tilde{x}) \le \frac{c_2}{(v-s)^{(d-1)/2}} e^{-\|\tilde{z}-\tilde{x}\|^2/c_2(v-s)} \le \frac{c_3}{\|\tilde{z}-\tilde{x}\|^{(d-1)}} \tag{5.1a}$$

and

$$p_W^{\tilde{Z}^t}(0,0;t,0) \ge c_4 \left[1 - c_5 \int_0^1 |W_s| ds\right]. \tag{5.1b}$$

To see this, we only need to observe that (4.2) and estimates (4.4) and (4.8), which are crucial to the proof there, should be replaced by

$$\begin{aligned}\tilde{c}(\sqrt{t}W_{s/t}e_1 + \tilde{z} + se_2) = &\tilde{b}(\sqrt{t}W_{s/t}e_1 + \tilde{z} + se_2) \\ &- \frac{2\rho}{t}\left[\tilde{I} - \tilde{g}(\sqrt{t}W_{s/t}e_1 + \tilde{z} + se_2)^{-1}\right] e_2 + \frac{1}{\sqrt{t}} O(|W_{s/t}|\,\|\tilde{z}\|)\end{aligned}$$

$$\left\|\tilde{g}(\sqrt{t}W_{s/t}e_1 + \tilde{z}) - \tilde{g}(\sqrt{t}W_{s/t}e_1 + \tilde{x})\right\| \le c_6\|\tilde{z} - \tilde{x}\|$$

$$\left\|\tilde{c}(\sqrt{t}W_{s/t}e_1 + \tilde{z};t) - \tilde{c}(\sqrt{t}W_{s/t}e_1 + \tilde{x};t)\right\| \le \frac{c_6}{t}\|\tilde{z} - \tilde{x}\|$$

and

$$\|\alpha\| + \|\beta\| \le c_7\|\tilde{y}\|, \qquad \|\gamma\| \le c_7\sqrt{t}|W_{s/t}|\,\|\tilde{y}\|.$$

with constants c_6, c_7 independent of W and t.

LEMMA 5.1. *There exist constants c_0, c_1 independent of t such that for sufficiently large $a \gg 1$,*

$$P_W\left[\hat{\tilde{M}}^t > a\right] \le e^{-c_1 a^2} \qquad if \int_0^1 |W_s| ds \le c_0. \tag{a}$$

and

$$P\left[\hat{M}^t > a\right] \le e^{-c_1 a^2}. \tag{b}$$

Proof. Let

$$\tau_a = \inf\left\{s : \|\tilde{Z}_s^t\| \ge a\right\}.$$

By the Markov property, we have, for any neighborhood D of the origin in $R^{(d-1)}$,

$$P_W\left[\tilde{M}^t > a, \tilde{Z}_1^t \in D\right] = E_W\left[\int_D p_W^{\tilde{Z}^t}(a,\tau_a;1,y)\,dy; \tau_a < 1\right]. \tag{5.2}$$

Divide (5.2) by $P\left[\tilde{Z}_1^t \in D\right]$ and use (5.1). Letting $|D| \to 0$, we see that for $a \gg 1$,

$$P_W\left[\hat{\tilde{M}}^t > a\right] \le \frac{1}{2}\left[1 - c_5\int_0^1 |W_s|ds\right]^{-1} P_W\left[\tilde{M}^t > a\right]. \tag{5.3}$$

To estimate the last probability, we note that the equation of $\tilde{Z}^t$ is

$$d\tilde{Z}_s^t = dQ_s + \sqrt{t}\tilde{c}(\sqrt{t}W_s e_1 + \sqrt{t}\tilde{Z}_s^t + s\rho e_2)\,ds \tag{5.4}$$

where

$$dQ_s = \tilde{\sigma}(\sqrt{t}W_s e_1 + \sqrt{t}\tilde{Z}_s^t + s\rho e_2)\,d\tilde{B}_s.$$

By Lemma 2.1 and (3.5) the drift in (5.4) is bounded by

$$\sqrt{t}\left\|c(\sqrt{t}W_s e_1 + \sqrt{t}\tilde{Z}_s^t + s\rho e_2; t)\right\| \le c_8\sqrt{t} + c_8|W_s|.$$

Also note that $Q_s^i, i = 2, \cdots$ are martingale with bounded characteristic: $[Q^i]_1 \le d\|\sigma\|_\infty^2$. It follows that for some $a \ge 2c_8$ and all $t \le 1$

$$\begin{aligned} P_W\left[\tilde{M}^t > a\right] \le & P_W\left[\max_{0\le s\le 1}\|Q_s\| > \frac{a}{2}\left(1 - \int_0^1 |W_s|ds\right)\right] \\ \le & dP_W\left[\max_{0\le s\le d\|\sigma\|_\infty^2}\|\beta_s\| > \frac{a}{2}\left(1 - \int_0^1 |W_s|ds\right)\right] \\ \le & d\exp\left\{-c_9 a^2\left(1 - \int_0^1 |W_s|ds\right)^2\right\} \end{aligned}$$

(β is an independent one-dimensional Brownian motion). Part (a) follows immediately from (5.3) by choosing, for example, $c_0 < \min(c_5^{-1}, 1)$ and $c_1 > c_9(1 - c_0)^2$. The proof of part (b) is similar and easier.

LEMMA 5.2. *For any positive ϵ, K and $0 < \delta < 1/6$, there exists a positive constant $t_0 = t_0(\epsilon, K, \delta)$ such that for all $t \le t_0$,*

$$P\left[\int_0^1 |\hat{W}_s|ds \le Kt^{1/6}, \quad \max_{0\le s\le 1}|\hat{W}_s| \ge \epsilon\, t^{-1/6}\right] \le \exp\left\{-t^{-(1/3+\delta)}\right\}.$$

Proof. This lemma is proved in Lemma 5.4 of Ikeda [5, p. 188–189].

We now turn to the following proof.

Proof of Lemma 3.2. Set

$$A_{t,K} = \left\{\omega : \int_0^1 |\hat{W}_s|ds \le Kt^{1/6}\right\}$$

$$B_{t,\epsilon} = \left\{\omega : \max_{0\le s\le 1}|\hat{W}_s| \ge \epsilon\, t^{-1/6}\right\}.$$

Also set

$$G_t = \hat{\tilde{M}}^t \int_0^1 |\hat{W}_s|ds.$$

We have from (3.8)

$$\hat{\Theta}_t = \frac{\rho^2}{2t}\int_0^1 [1 - g^{22}(\sqrt{t}\hat{Z}_s^t + s\rho e_2)]\, ds.$$

Assumption (B2) implies

$$2H_{22}(\tilde{z} + se_2)|z^1| - \gamma|z^1|^2 \le 1 - g^{22}(z + se_2) \le 2H_{22}(\tilde{z} + se_2)|z^1| + \gamma|z^1|^2.$$

Since

$$|H_{22}(\tilde{z} + se_2) - N(s)| \le c_1\|\tilde{z}\|$$

and $N(s)$ is strictly positive by Assumption (A), we have

$$\hat{\Theta}_t \le \frac{\rho^2}{\sqrt{t}}\left[1 + c_4\epsilon\gamma t^{1/3}\right]\int_0^1 N(s\rho)|\hat{W}_s|ds + c_3G_t \text{ on } B_{t,\epsilon}^c. \tag{5.5}$$

Symmetrically, we have

$$\hat{\Theta}_t \ge \frac{\rho^2}{\sqrt{t}}\left[1 - c_4\epsilon\gamma\|_\infty t^{1/3}\right]\int_0^1 N(s\rho)|\hat{W}_s|ds - c_3G_t \quad \text{on } B_{t,\epsilon}^c. \tag{5.6}$$

Now Lemma 5.1(a) implies that if $t \le (c_0/K)^{1/6}$

$$E_{\hat{W}}\left[\exp\{c_3G_t\}\right] \le \exp\left\{c_5\int_0^1 |\hat{W}_s|ds + c_5\left(\int_0^1 |\hat{W}_s|ds\right)^2\right\} \quad \text{on } A_{t,K}. \tag{5.7}$$

(Integration by parts!) By the Schwartz inequality, (5.7) gives

$$E_{\hat{W}}\left[\exp\{-c_3G_t\}\right] \ge \exp\left\{-c_5\int_0^1 |\hat{W}_s|ds - c_5\left(\int_0^1 |\hat{W}_s|ds\right)^2\right\} \quad \text{on } A_{t,K}. \tag{5.8}$$

We also have

$$\hat{\Theta}_t \ge 2\,c_6Kt^{-1/3} - \frac{\rho^2\gamma}{2}\int_0^1 |\hat{W}_s|^2ds \quad \text{on } A_{t,K}^c.$$

Hence for $\eta = [2\,c_6/\rho^2\|f\|_\infty]^{1/2}$, we have

$$\hat{\Theta}_t \ge c_6Kt^{-1/3} \quad \text{on } A_{t,K}^c \cap B_{t,\eta\sqrt{K}}^c. \tag{5.9}$$

Observe that

$$E\left[G(\hat{Z})\right] = E\left[E_{\hat{W}}\left[G(\hat{W}e_1 + \hat{\tilde{Z}})\right]\right].$$

Thus, on the one hand, using (5.5), (5.8), and Lemma 5.2, we have

$$\begin{aligned}
&E\left[\exp\left\{-\hat{\Theta}_t\right\}\right]\\
&\ge E\left[\exp\left\{-\hat{\Theta}_t\right\}; A_{t,K}\cap B_{t,\epsilon}^c\right]\\
&\ge E\left[\exp\left\{-\frac{\rho^2}{\sqrt{t}}[1 + c_4\epsilon\,\gamma t^{1/3} + c_7t^{1/2} + c_7Kt^{2/3}]\int_0^1 N(s\rho)|\hat{W}_s|ds\right\}; A_{t,K}\cap B_{t,\epsilon}^c\right]\\
&\ge E\left[\exp\{\cdots\}\right] - E\left[\exp\{\cdots\}; A_{t,K}^c\right] - P\left[A_{t,K}\cap B_{t,\epsilon}\right]\\
&\ge \left[\exp\left\{-\frac{\rho^2}{\sqrt{t}}[1 + c_4\epsilon\,\gamma t^{1/3} + c_7t^{1/2} + c_7Kt^{2/3}]\int_0^1 N(s\rho)|\hat{W}_s|ds\right\}\right]\\
&\quad - \exp\left\{-c_8Kt^{-1/3}\right\} - \exp\left\{-c_8t^{-(1/3+\delta)}\right\}.
\end{aligned}$$

On the other hand, by Assumption (B1), we have $\hat{\Theta}_t \geq 0$; hence using (5.6), (5.7), (5.9), and Lemma 5.2, we have

$$\begin{aligned} E\left[\exp\left\{-\hat{\Theta}_t\right\}\right] \leq & E\left[\exp\left\{-\hat{\Theta}_t\right\}; A_{t,K} \cap B^c_{t,\epsilon}\right] + E\left[\exp\left\{-\hat{\Theta}_t\right\}; A^c_{t,K} \cap B^c_{t,\eta\sqrt{K}}\right] \\ & + P\left[B_{t,\eta\sqrt{K}}\right] + P[A_{t,K} \cap B_{t,\epsilon}] \\ \leq & E\left[\exp\left\{-\frac{\rho^2}{\sqrt{t}}[1 - c_4\epsilon\gamma t^{1/3} - c_7 t^{1/2} - c_7 K t^{2/3}] \int_0^1 N(s\rho)|W_s| ds\right\}\right] \\ & + \exp\left\{c_6 K t^{-1/3}\right\} + \exp\left\{-c_8 K t^{-1/3}\right\} + \exp\left\{-t^{-(1/3+\delta)}\right\}. \end{aligned}$$

By Lemma 3.4, which we will prove independently in §6, there exist constants c_9 and c_{10} such that as $t \to 0$, for any $k \geq 0$,

$$E\left[\exp\left\{-\frac{\rho^2}{\sqrt{t}}\int_0^1 N(s\rho)|\hat{W}_s| ds\right\}\right] \sim c_9\, t^{-1/6} \exp\left\{-c_{10} t^{-1/3}\right\}. \tag{5.10}$$

Choose $K > c_{10}/\min(c_6, c_8)$. Using (5.10) and the above bounds for $E\left[\exp\{-\hat{\Theta}_t\}\right]$, we obtain

$$\exp\left\{-c_{11}\epsilon\gamma\right\} \leq \lim_{t\to 0}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right\} \frac{E\left[\exp\{-\hat{\Theta}_t\}\right]}{E\left[\exp\left\{-\frac{\rho^2}{\sqrt{t}}\int_0^1 N(s\rho)|\hat{W}_s| ds\right\}\right]} \leq \exp\left\{c_{11}\epsilon\gamma\right\}.$$

($c_{11} = 2\,c_4 c_{10}/3$.) Letting $\epsilon \to 0$, we obtain Lemma 3.2.

Proof of Lemma 3.3. Let us first prove: There exist constants c_1 and c_2 such that for any $|q| \geq 1$

$$E\left[\exp\{q\hat{F}_t\}\right] \leq c_1 e^{c_2 q^2 t}. \tag{5.11}$$

Set

$$\begin{aligned} C_u = & \sqrt{t}\int_0^u b^1(\sqrt{t}\hat{Z}^t_s + s\rho e_2)\, d\hat{W}_s + \int_0^u \left[b^2(\sqrt{t}\hat{Z}^t_s + s\rho e_2) - b^2(s\rho e_2)\right] ds \\ & + \int_0^u \left|b^1(\sqrt{t}\hat{Z}^t_s + s\rho e_2)\right|^2 ds. \end{aligned}$$

Then we have $\hat{F}_t = C_1$. Obviously

$$\left\{E\left[\exp\{q\hat{F}_t\}\right]\right\}^2 \leq E\left[\exp\{2qC_{1/2}\}\right] E\left[\exp\{2q(C_1 - C_{1/2})\}\right].$$

Thus it is enough to prove the estimate for each of the factors on the right-hand side of the above inequality. The proofs for the two factors are the same. Take, for example, the first factor. Since $\hat{W}$ is a standard Brownian bridge, we can write

$$d\hat{W}_s = dW_s - \frac{\hat{W}_s}{1 - s} ds$$

for a Brownian motion W. Set

$$A_u = 2q\sqrt{t}\int_0^u b^1(\sqrt{t}\hat{Z}_s^t + s\rho e_2)\,dW_s - 2q^2t\int_0^u \left|b^1(\sqrt{t}\hat{Z}_s^t + s\rho e_2)\right|^2 ds$$

and

$$D_u = 2qC_u - A_u.$$

We have

$$\left\{E\left[\exp\{2qC_{1/2}\}\right]\right\}^2 \le E\left[\exp\{2A_{1/2}\}\right]E\left[\exp\{2D_{1/2}\}\right]. \tag{5.12}$$

The first factor on the right-hand side is equal to 1 by the choice of A_u. As for the second factor, we have the bound

$$2|D_{1/2}| \le c_3q^2t + c_3q\sqrt{t}M^t.$$

It follows immediately from Lemma 5.1(b) that the second factor in (5.12) is bounded by $c_4e^{2c_2q^2t}$. This implies

$$E\left[\exp\{qC_{1/2}\}\right] \le c_1e^{c_2q^2t}.$$

Inequality (5.11) is proved.

We now complete the proof of Lemma 3.3. By Lemma 3.2, we have

$$E\left[\exp\left\{-c\hat{\Theta}_t\right\}\right] \ge \exp\left\{-c_5^{-1}t^{-1/3}\right\} \tag{5.13}$$

for fixed c. We use the cases $c = 1, 2$. Let $p > 1$ and $1/p + 1/q = 1$. By (5.11), (5.13) and the Schwartz inequality we obtain

$$\begin{aligned}
&E\left[\exp\left\{-\hat{\Theta}_t + \hat{F}_t\right\}\right]\\
&\le \left\{E\left[\exp\left\{-\hat{\Theta}_t\right\}\right]\right\}^{1/p}\left\{E\left[\exp\left\{-2\hat{\Theta}_t\right\}\right]\right\}^{1/2q}\left\{E\left[\exp\left\{2q\hat{F}_t\right\}\right]\right\}^{1/2q}\\
&\le E\left[\exp\left\{-\hat{\Theta}_t\right\}\right]c_1^{1/2q}\exp\left\{\frac{c_5}{2}(q^3t)^{-1/3} + 2c_2qt\right\}
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\exp\left\{-\hat{\Theta}_t + \hat{F}_t\right\}\right]\\
&\ge \left\{E\left[\exp\left\{-\hat{\Theta}_t\right\}\right]\right\}^{p}\left\{E\left[\exp\left\{-2\hat{\Theta}_t\right\}\right]\right\}^{-p/2q}\left\{E\left[\exp\left\{-\frac{2q}{p}\hat{F}_t\right\}\right]\right\}^{-p/2q}\\
&\ge E\left[\exp\left\{-\hat{\Theta}_t\right\}\right]c_1^{-p/2q}\exp\left\{-\frac{c_5}{2}p(q^3t)^{-1/3} - 2c_2p^{-1}qt\right\}.
\end{aligned}$$

Taking $q = t^{-1/2}$ and letting $t \to 0$, we obtain immediately Lemma 3.3.

6. Proof of Lemma 3.4. Assume first that l is a constant. Using the scaling property of Brownian motion, we have

$$S(\lambda;t) = E\left[\exp\left\{-\int_0^{(\lambda l)^{2/3}}|W_s|ds\right\}\middle| W_{(\lambda l)^{2/3}} = 0\right]$$

(W is one-dimensional Brownian motion). By the Feynmann–Kac formula,

$$S(\lambda; l) = \sqrt{2\pi}(\lambda l)^{1/3} q((\lambda l)^{2/3}, 0, 0)$$

where $q(s, x, y)$ is the fundamental solution of

$$L = \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + |x|.$$

We have the eigenexpansion

$$q(s, x, y) = \sum_{m=0}^{\infty} \exp\{-\mu_m s\}\phi_m(x)\phi_m(y).$$

It follows easily that for any $k \geq 0$,

$$S(\lambda; l) = \sqrt{2\pi}|\phi_1(0)|^2(\lambda l)^{1/3} \exp\left\{-\mu_1(\lambda l)^{2/3}\right\}\left[1 + O(\lambda^{-k})\right]. \tag{6.1}$$

Thus Lemma 3.4 holds in this special case. For the general case, let

$$\begin{aligned}
L(s) &= \int_0^s l(u)^{2/3} du \\
\mu(s) &= \frac{1}{3}\frac{d}{ds}\left[\log l(L^{-1}(s))\right] \\
\psi(s) &= L(1)\, \mu(L(1)\, s) \\
T_\lambda &= \lambda^{2/3} L(1) \\
J_\psi &= \frac{1}{2}\int_0^1 [\psi'(s) + \psi(s)^2]|\hat{W}_s|^2 ds \\
\Omega(\lambda, \psi) &= J_\psi + \lambda \int_0^1 |\hat{W}_s| ds
\end{aligned}$$

(L^{-1} is the inverse function of L). It is not difficult to see that Lemma 3.4 is implied by the following two relations:

$$S(\lambda; l) = \left[\frac{l(0)\, l(1)}{L(1)^3}\right]^{1/6} E\left[\exp\left\{-\Omega(T_\lambda^{3/2}, \psi)\right\}\right] \tag{6.2}$$

$$\lim_{\lambda\to\infty} \frac{E\left[\exp\left\{-\Omega(\lambda, \psi)\right\}\right]}{E\left[\exp\left\{-\Omega(\lambda, 0)\right\}\right]} = 1. \tag{6.3}$$

To show (6.2), let δ be the δ-function at $x = 0$, and set

$$u_\lambda(s, x) = E_x\left[\exp\left\{-\lambda\int_s^1 l(s)|W_s| ds\right\}\delta(W_1)\right].$$

We have $S(\lambda; l) = \sqrt{2\pi} u_\lambda(0, 0)$. Function $u_\lambda(s, x)$ satisfies

$$\frac{\partial u}{\partial s} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \lambda l(s)|x| u = 0, \qquad u(1, \cdot) = \delta.$$

Introduce a new function $w_\lambda(s,x)$ by

$$u_\lambda(s,x) = [\lambda l(1)]^{1/3} w_\lambda\left(\lambda^{2/3}L(s), [\lambda l(s)]^{1/3}x\right).$$

Then

$$S(\lambda; l) = \sqrt{2\pi}[\lambda l(1)]^{1/3} w_\lambda(0,0). \tag{6.4}$$

We verify directly that $w_\lambda(s,x)$ satisfies the equation

$$\frac{\partial w}{\partial s} + \frac{1}{2}\frac{\partial^2 w}{\partial x^2} + \lambda^{-2/3}\mu(\lambda^{-2/3}s)\, x\,\frac{\partial w}{\partial x} - |x|w = 0, \qquad w(T_\lambda, \cdot) = \delta$$

By the Girsanov formula and the Feynman–Kac formula, we can write

$$w_\lambda(0,0) = E\left[\exp\{A(\lambda)\}\delta(W_{T_\lambda})\right]$$

where

$$A(\lambda) = \lambda^{-2/3}\int_0^{T_\lambda} \mu\left(\frac{s}{\lambda^{2/3}}\right) W_s dW s - \frac{1}{2}\lambda^{-4/3}\int_0^{T_\lambda} \mu\left(\frac{s}{\lambda^{2/3}}\right)^2 |W_s|^2 ds - \int_0^{T_\lambda} |W_s| ds.$$

Using the scaling property of Brownian motion, we can write

$$w_\lambda(0,0) = \frac{1}{\sqrt{2\pi T_\lambda}} E\left[\exp\{B(\lambda)\}\right] \tag{6.5}$$

with

$$B(\lambda) = \int_0^1 \psi(s)\hat{W}_s d\hat{W}_s - \frac{1}{2}\int_0^1 \psi(s)^2 |\hat{W}_s|^2 ds - T_\lambda^{3/2}\int_0^1 |\hat{W}_s| ds.$$

By Itô's formula,

$$\begin{aligned} B(\lambda) &= -\frac{1}{2}\int_0^1 [\psi'(s) + \psi(s)^2]|\hat{W}_s|^2 ds - T_\lambda^{3/2}\int_0^1 |\hat{W}_s| ds - \frac{1}{2}\int_0^1 \psi(s)\, ds \\ &= -\ \Omega(T_\lambda, \psi) - \frac{1}{6}\log\frac{l(1)}{l(0)}. \end{aligned}$$

The desired formula (6.2) follows from (6.4) and (6.5).

It remains to prove (6.3). We claim

$$\exists p > 1: \qquad C(p) \stackrel{\text{def}}{=} E\left[\exp\{-pJ_\psi\}\right] < \infty. \tag{6.6}$$

Let $\{X_1, X_2, \cdots\}$ be a sequence of independently and identically distributed random variables with standard normal distribution $N(0,1)$. Then the Brownian bridge $\hat{W}_s$ can be expanded as

$$\hat{W}_s = \frac{\sqrt{2}}{\pi}\sum_{k=1}^{\infty} X_k \frac{\sin k\pi s}{k}.$$

We have

$$J_\psi = \frac{1}{2}\sum_{k,l=1}^{\infty} a_{kl} X_k X_l$$

with

$$a_{kl} = \frac{2}{\pi^2} \int_0^1 [\psi'(s) + \psi(s)^2] \frac{\sin k\pi s}{k} \frac{\sin l\pi s}{l} ds.$$

Let H be the Hilbert space

$$H = \left\{ f \in AC[0,1] : \quad f(0) = f(1) = 0, \quad \|f\|_H^2 \overset{\text{def}}{=} \int_0^1 |f'(s)|^2 ds < \infty \right\}.$$

Let $\{e_1, e_2, \cdots\}$ be an orthonormal basis for H. Define $A : H \to H$ by $Ae_k = \sum_{l=1}^\infty a_{kl} e_l$. Let $\alpha_1, \alpha_2, \cdots$ ($\alpha_i \to 0$) be the eigenvalues of A with normalized eigenvectors $f_1, f_2, \cdots$. Define (c_{kl}) by $e_k = \sum_{l=1}^\infty c_{kl} f_l$. The random variables $Y_i = \sum_{l=1}^\infty c_{li} X_l$, $i = 1, 2, \cdots$ are again i.i.d. with standard normal $N(0,1)$. Furthermore $J_\psi = \frac{1}{2} \sum_{i=1}^\infty \alpha_i |Y_i|^2$. It follows that as long as $1 + p\alpha_i \geq 0$ for all i, we have

$$C(p) = E\left[\exp\left\{-\frac{p}{2}\sum_{i=1}^\infty \alpha_i |Y_i|^2\right\}\right] = \prod_{i=1}^\infty (1 + p\alpha_i)^{-1/2}. \tag{6.7}$$

The infinite product (6.7) converges to a finite value if and only if the series $\sum_{i=1}^\infty \alpha_i$ converges and $1 + p\alpha_i > 0$ for all i. On the other hand, from the definition of $C(p)$, we know $C(p)$ is finite for small p. Thus the the series $\sum_{i=1}^\infty \alpha_i$ indeed converges. It is now clear that $C(p)$ is finite for those p such that $1 + p\alpha_i > 0$ for all i. Thus $C(p) < \infty$ for some $p > 1$ if and only if all eigenvalues $\alpha_i > -1$ (note that $\alpha_i \to 0$), or what is the same thing,

$$\forall f \in H : \qquad \langle Af, f \rangle_H > -\|f\|_H^2. \tag{6.8}$$

A direct computation shows that

$$\langle Af, f \rangle = \int_0^1 [\psi'(s) + \psi(s)^2] \, |f(s)|^2 ds.$$

Relation (6.8) follows then from the elementary fact: for all $f \in H$

$$\int_0^1 [\psi'(s) + \psi(s)^2] |f(s)|^2 ds + \int_0^1 |f'(s)|^2 ds = \int_0^1 [\psi(s) f(s) - f'(s)]^2 ds > 0.$$

Equation (6.6) is proved.

We can now finish the proof of (6.3). Let

$$C_{\lambda,\epsilon} = \left\{ \omega : |J_\psi| \leq \epsilon \lambda^{1/3} \int_0^1 |\hat{W}_s| ds \right\}$$

and

$$D_{\lambda,K} = \left\{ \omega : \int_0^1 |\hat{W}_s| ds > K \lambda^{-1/3} \right\}.$$

Then on $C_{\lambda,\epsilon}$

$$\lambda[1 - \epsilon\lambda^{-2/3}] \int_0^1 |\hat{W}_s| ds \leq \Omega(\lambda, \psi) \geq \lambda[1 + \epsilon\lambda^{-2/3}] \int_0^1 |\hat{W}_s| ds.$$

On the set $D_{\lambda,K}$

$$\Omega(\lambda,\psi) \geq J_\psi + K\lambda^{2/3}.$$

It follows that on the one hand

$$(6.9)\qquad \begin{aligned} E\left[\exp\left\{-\Omega(\lambda,\psi)\right\}\right] \leq & E\left[\exp\left\{-\lambda[1-\epsilon\lambda^{-2/3}]\int_0^1 |\hat{W}_s|ds\right\}\right] \\ & + C(1)\exp\{-K\lambda^{2/3}\} + C(p)^{1/p}P\left[C^c_{\lambda,\epsilon}\cap D^c_{\lambda,K}\right]^{1/q}. \end{aligned}$$

Note that we have proved $C(1)$ and $C(p)$ are finite. On the other hand, we have

$$(6.10)\qquad \begin{aligned} E\left[\exp\left\{-\Omega(\lambda,\psi)\right\}\right] \geq & E\left[\exp\left\{-\lambda[1+\epsilon\lambda^{-2/3}]\int_0^1 |\hat{W}_s|ds\right\}\right] \\ & - \exp\{-K\lambda^{2/3}\} - P\left[C^c_{\lambda,\epsilon}\cap D^c_{\lambda,K}\right]. \end{aligned}$$

Note that

$$C^c_{\lambda,\epsilon}\cap D^c_{\lambda,K} \subset \left\{\omega : \int_0^1 |\hat{W}_s|ds \leq K\lambda^{-1/3}, \max_{0\leq s\leq 1}|\hat{W}_s| \geq \epsilon c(\psi)\,\lambda^{1/3}\right\}.$$

with $c(\psi) = \|\psi' + \psi^2\|^{-1}_\infty$. Take $K > \mu_1 L(1)$. By Lemma 5.2 and (6.1), (6.9), and (6.10),

$$e^{-2\mu_1\epsilon/3} \leq \lim_{\lambda\to\infty}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right\}\frac{E\left[\exp\{-\Omega(\lambda,\psi)\}\right]}{E\left[\exp\{-\Omega(\lambda,0)\}\right]} \leq e^{2\mu_1\epsilon/3}.$$

Letting $\epsilon \to 0$ we obtain (6.3). The proof is complete.

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