PUTNAM TRAINING 11/30/2021
EASY PUTNAM PROBLEMS

Easy Putnam Problems

1. 2019-A1. Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

2. 2018-A1. Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

3. 2016-B1. Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln\left(e^{x_n} - x_n\right)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

4. 2015-B1. Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

5. 2014-A1. Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

6. 2013-A1. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

7. 2012-A1. Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i,d_j, d_k are the side lengths of an acute triangle.

8. 2010-B1. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

9. **2009-A1.** Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

10. **2009-B1.** Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

11. **2008-A1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

12. **2008-A2.** Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

13. **2008-B1.** What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

14. **2004-A1.** Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

15. **2003-A1.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

16. **2002-A2.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

17. **2001-A1.** Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

18. **1997-A5.** Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

Solutions

Easy Putnam Problems

1. 2019-A1. Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

- *Solution.* The answer is all nonnegative integers, except multiples of 3 that are not multiple of 9.
Let $X = A^3 + B^3 + C^3 - 3ABC$.
First, we show that we can make X equal to each of the claimed values.
Write $B = A + b$ and $C = A + c$, so that

$$X = (b^2 - bc + c^2)(3A + b + c).$$

Taking $(b, c) = (0, 1)$ or $(b, c) = (1, 1)$, we obtain respectively $X = 3A + 1$ and $X = 3A + 2$; consequently, as A varies, we achieve every nonnegative integer not divisible by 3. By taking $(b, c) = (1, 2)$, we obtain $X = 9A + 9$; consequently, as A varies, we achieve every positive integer divisible by 9. We may also achieve $X = 0$ by taking $(b, c) = (0, 0)$.
Next, we show that X can take *only* the claimed values.
Note that X is always nonnegative because of the arithmetic mean-geometric mean inequality:

$$\frac{A^3 + B^3 + C^3}{3} \geq \sqrt[3]{A^3B^3C^3} = ABC\,.$$

It thus only remains to show that if X is a multiple of 3, then it is a multiple of 9. We have

$$X = (\underbrace{(b + c)^2 - 3bc}_{b^2-bc+c^2})(3A + b + c) \equiv (b + c)^3 \equiv b + c \pmod{3}\,.$$

Consequently, if X is divisible by 3, then $b + c$ must be divisible by 3, so each factor in $X = ((b + c)^2 - 3bc)(3A + b + c)$ is divisible by 3. This proves the claim.

2. 2018-A1. Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

- *Solution.* By clearing denominators and regrouping, we see that the given equation is equivalent to

$$(3a - 2018)(3b - 2018) = 2018^2.$$

Each of the factors is congruent to $1 \pmod{3}$. There are 6 factors of $2018^2 = 2^2 \cdot 1009^2$ that are congruent to $1 \pmod{3}$: 1, 2^2, 1009, $2^2 \cdot 1009$, 1009^2, $2^2 \cdot 1009^2$. These lead to the 6 possible pairs: $(a, b) = (673, 1358114)$, $(674, 340033)$, $(1009, 2018)$, $(2018, 1009)$, $(340033, 674)$, and $(1358114, 673)$.

3. 2016-B1. Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln\left(e^{x_n} - x_n\right)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

- *Solution.* First we prove that x_n has a limit as $n \to \infty$. The recurrence can be written $e^{x_{n+1}} = e^{x_n} - x_n$. For $x > 0$ we have $e^x > 1 + x$, hence by induction we get $0 < x_{n+1} < x_n \leq 1$, so the sequence is decreasing and bounded. By the Monotone Convergence Theorem that implies that x_n has in fact a limit.
Next we prove that the limit is zero. In fact, let $x = \lim_{n\to\infty} x_n$. Then $x = \ln\left(e^x - x\right)$, $e^x = e^x - x$, hence $x = 0$.
Finally we will prove that the sum is $e - 1$. From the recursive definition of x_n we have $x_n = e^{x_n} - e^{x_{n+1}}$, hence

$$\begin{aligned} x_0 + x_1 + x_2 + \cdots + x_N &= \left(e^{x_0} - e^{x_1}\right) + \left(e^{x_1} - e^{x_2}\right) + \cdots + \left(e^{x_N} - e^{x_{N+1}}\right) \\ &= e^{x_0} - e^{x_{N+1}} \underset{N\to\infty}{\longrightarrow} e^1 - e^0 = e - 1\,. \end{aligned}$$

4. 2015-B1. Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

- *Solution.* The given expression multiplied by $e^{x/2}$ is the third derivative of $g(x) = 8e^{x/2}f(x)$, which has the same zeros as $f(x)$. We know (from Rolle's theorem) that between two distinct real zeros of a function there is a zero of its derivative, hence g''' must have at least $5 - 3 = 2$ distinct real zeros, and the same is true for the given expression.

5. 2014-A1. Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

- *Solution.* The coefficient of x^n in the Taylor series of $(1 - x + x^2)e^x$ for $n = 0, 1, 2$ is $1, 0, \frac{1}{2}$, respectively. For $n \geq 3$, the coefficient of x^n is

$$\begin{aligned} \frac{1}{n!} - \frac{1}{(n-1)!} + \frac{1}{(n-2)!} &= \frac{1 - n + n(n-1)}{n!} \\ &= \frac{n-1}{n(n-2)!}. \end{aligned}$$

If $n - 1$ is prime, then since $n - 1$ is relatively prime to n and to $(n-2)!$, the lowest-terms numerator is $n - 1$, which is prime. If $n - 1 = ab$ is composite, then if $a \neq b$, both a and b appear separately in $(n-2)!$, and so the lowest-terms numerator is 1. If $n - 1 = a^2$, then either $a = 2$, in which case the coefficient is $\frac{4}{30} = \frac{2}{15}$; or $a > 2$, in which case $n-1=a^2>2a$, whence both a and $2a$ appear in $(n-2)!$, and so $n-1=a^2$ divides $(n-2)!$ and the lowest-terms numerator is 1.

6. 2013-A1. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

- *Solution.* If the numbers on the faces having a common vertex v have different numbers $a_0<a_1<a_2<a_3<a_4$, then $a_k\geq k$ for $k=0,1,2,3,4$, and $S_v=a_0+a_1+a_2+a_3+a_4\geq 0+1+2+3+4=10$. Adding over the 12 vertices we get $\sum_v S_v\geq 12\cdot 10=120$. In that sum each number occurs three times, one per each vertex of the face, so the sum of the numbers written on the faces of the icosahedron will be greater than or equal to $120/3=40$, contradicting the hypothesis that it is 39.

7. 2012-A1. Let $d_1,d_2,\ldots,d_{12}$ be real numbers in the interval $(1,12)$. Show that there exist distinct indices i, j, k such that d_i,d_j, d_k are the side lengths of an acute triangle.

- *Solution.* Assume without loss of generality that $1<d_1\leq d_2\leq\cdots\leq d_{12}<12$. Note that three numbers $0<a\leq b\leq c$ are the side lengths of an acute triangle precisely if $a^2+b^2>c^2$, so if not such three indices exist we would have $d_i^2+d_{i+1}^2\leq d_{i+2}^2$ for $i=1,\ldots,10$. Consequently $1<d_1,d_2$, $d_3^2\geq d_1^2+d_2^2>2$, $d_4^2\geq d_2^2+d_3^2>3$, and analogously $d_5^2>5$, $d_6^2>8$, $d_7^2>13$, $d_8^2>21$, $d_9^2>34$, $d_{10}^2>55$, $d_{11}^2>89$, $d_{12}^2>144$, but this last inequality implies $d_{12}>12$, which is a contradiction.

8. 2010-B1. Is there an infinite sequence of real numbers $a_1,a_2,a_3,\ldots$ such that

$$a_1^m+a_2^m+a_3^m+\cdots=m$$

for every positive integer m?

- *Solution.*
First solution: No such sequence exists. If it did, then the Cauchy-Schwartz inequality would imply

$$\begin{aligned}8&=(a_1^2+a_2^2+\cdots)(a_1^4+a_2^4+\cdots)\\&\geq(a_1^3+a_2^3+\cdots)^2=9,\end{aligned}$$

contradiction.
Second solution: Suppose that such a sequence exists. If $a_k^2\in[0,1]$ for all k, then $a_k^4\leq a_k^2$ for all k, and so

$$4=a_1^4+a_2^4+\cdots\leq a_1^2+a_2^2+\cdots=2,$$

contradiction. There thus exists a positive integer k for which $a_k^2>1$. However, in this case, for m large, $a_k^{2m}>2m$ and so $a_1^{2m}+a_2^{2m}+\cdots\neq 2m$.

9. 2009-A1. Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane?

- *Solution.* Yes, it does follow. Let P be any point in the plane. Let $ABCD$ be any square with center P. Let E, F, G, H be the midpoints of the segments AB, BC, CD, DA, respectively. The function f must satisfy the equations

$$\begin{aligned} 0 &= f(A) + f(B) + f(C) + f(D) \\ 0 &= f(E) + f(F) + f(G) + f(H) \\ 0 &= f(A) + f(E) + f(P) + f(H) \\ 0 &= f(B) + f(F) + f(P) + f(E) \\ 0 &= f(C) + f(G) + f(P) + f(F) \\ 0 &= f(D) + f(H) + f(P) + f(G). \end{aligned}$$

If we add the last four equations, then subtract the first equation and twice the second equation, we obtain $0 = 4f(P)$, whence $f(P) = 0$.

10. **2009-B1.** Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

- *Solution.* Every positive rational number can be uniquely written in lowest terms as a/b for a, b positive integers. We prove the statement in the problem by induction on the largest prime dividing either a or b (where this is considered to be 1 if $a = b = 1$). For the base case, we can write $1/1 = 2!/2!$. For a general a/b, let p be the largest prime dividing either a or b. Assume p divides a (the other case, with p dividing b, is analogous). Then $a/b = p^k a'/b$ for some $k > 0$ and positive integers a', b whose largest prime factors are strictly less than p. Writing $p = p!/(p-1)!$ we have $a/b = (p!)^k \frac{a'}{(p-1)!^k b}$, and all prime factors of a' and $(p-1)!^k b$ are strictly less than p. By the induction hypothesis, $\frac{a'}{(p-1)!^k b}$ can be written as a quotient of products of prime factorials, and so $a/b = (p!)^k \frac{a'}{(p-1)!^k b}$ can as well. This completes the induction.

11. **2008-A1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

- *Solution.* The function $g(x) = f(x, 0)$ works. Substituting $(x, y, z) = (0, 0, 0)$ into the given functional equation yields $f(0, 0) = 0$, whence substituting $(x, y, z) = (x, 0, 0)$ yields $f(x, 0) + f(0, x) = 0$. Finally, substituting $(x, y, z) = (x, y, 0)$ yields $f(x, y) = -f(y, 0) - f(0, x) = g(x) - g(y)$.

12. **2008-A2.** Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

- *Solution.*

First solution: Pair each entry of the first row with the entry directly below it in the second row. If Alan ever writes a number in one of the first two rows, Barbara writes the same number in the other entry in the pair. If Alan writes a number anywhere other than the first two rows, Barbara does likewise. At the end, the resulting matrix will have two identical rows, so its determinant will be zero.
Second solution: Whenever Alan writes a number x in an entry in some row, Barbara writes $-x$ in some other entry in the same row. At the end, the resulting matrix will have all rows summing to zero, so it cannot have full rank.

13. 2008-B1. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

- *Solution.* There are at most two such points. For example, the points $(0, 0)$ and $(1, 0)$ lie on a circle with center $(1/2, x)$ for any real number x, not necessarily rational. On the other hand, with three point A, B, C, we could find the center of the circle as the intersection of the perpendicular bisectors of the segments AB and BC. If A, B, and C are rational, the middle points of AB and BC will be rational, the bisectors will be rational lines (representable by equations with rational coefficients), and their intersection will be rational.

14. 2004-A1. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

- *Solution.* Yes. Suppose otherwise. Then there would be an N such that $S(N) <$ 80% and $S(N+1) >$ 80%; that is, O'Keal's free throw percentage is under 80% at some point, and after one subsequent free throw (necessarily made), her percentage is over 80%. If she makes m of her first N free throws, then $m/N < 4/5$ and $(m+1)/(N+1) > 4/5$. This means that $5m < 4N < 5m + 1$, which is impossible since then $4N$ is an integer between the consecutive integers $5m$ and $5m + 1$.

15. 2003-A1. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

- *Solution.* There are n such sums. More precisely, there is exactly one such sum with k terms for each of $k = 1, \dots, n$ (and clearly no others). To see this, note that if $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$, then

$$\begin{aligned} ka_1 &= a_1 + a_1 + \cdots + a_1 \\ &\le n \le a_1 + (a_1 + 1) + \cdots + (a_1 + 1) \\ &= ka_1 + k - 1. \end{aligned}$$

However, there is a unique integer a_1 satisfying these inequalities, namely $a_1 = \lfloor n/k \rfloor$. Moreover, once a_1 is fixed, there are k different possibilities for the sum $a_1+a_2+\cdots+a_k$:

if i is the last integer such that $a_i = a_1$, then the sum equals $ka_1 + (i-1)$. The possible values of i are $1, \ldots, k$, and exactly one of these sums comes out equal to n, proving our claim.

16. 2002-A2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

- *Solution.* Draw a great circle through two of the points. There are two closed hemispheres with this great circle as boundary, and each of the other three points lies in one of them. By the pigeonhole principle, two of those three points lie in the same hemisphere, and that hemisphere thus contains four of the five given points.

17. 2001-A1. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

- *Solution.* The hypothesis implies $((b * a) * b) * (b * a) = b$ for all $a, b \in S$ (by replacing a by $b * a$), and hence $a * (b * a) = b$ for all $a, b \in S$ (using $(b * a) * b = a$).

18. 1997-A5. Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

- *Solution.* We may discard any solutions for which $a_1 \neq a_2$, since those come in pairs; so assume $a_1 = a_2$. Similarly, we may assume that $a_3 = a_4$, $a_5 = a_6$, $a_7 = a_8$, $a_9 = a_{10}$. Thus we get the equation

$$2/a_1 + 2/a_3 + 2/a_5 + 2/a_7 + 2/a_9 = 1.$$

Again, we may assume $a_1 = a_3$ and $a_5 = a_7$, so we get $4/a_1 + 4/a_5 + 2/a_9 = 1$; and $a_1 = a_5$, so $8/a_1 + 2/a_9 = 1$. This implies that $(a_1 - 8)(a_9 - 2) = 16$, which by counting has 5 solutions. Thus N_{10} is odd.