# PUTNAM TRAINING 10/19/2021
# MATHEMATICAL INDUCTION AND RECURRENCES

**1.** The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ is defined as a sequence whose first two terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.

**2.** A chessboard is a $8 \times 8$ grid (64 squares arranged in 8 rows and 8 columns), but here we will call "chessboard" any $m \times m$ square grid. We call *defective* a chessboard if one of its squares is missing. Prove that any $2^n \times 2^n$ ($n \geq 1$) defective chessboard can be tiled (completely covered without overlapping) with L-shaped triominos occupying exactly 3 squares, like this  .

**3.** Let $a_n$ be the following expression with $n$ nested radicals:

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}\,.$$

Prove that $a_n = 2\cos\frac{\pi}{2^{n+1}}$.

**4.** Prove that, for every positive integer $n$, $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

**5.** We need to put $n$ cents of stamps on an envelop, but we have only (an unlimited supply of) 5¢ and 12¢ stamps. Prove that we can perform the task if $n \geq 44$.

**6.** We define recursively the *Ulam numbers* by setting $u_1 = 1$, $u_2 = 2$, and for each subsequent integer $n$, we set $n$ equal to the next Ulam number if it can be written uniquely as the sum of two distinct Ulam numbers; e.g.: $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, etc. Prove that there are infinitely many Ulam numbers.

**7.** Find the number of subsets of $\{1, 2, \dots, n\}$ that contain no consecutive elements of $\{1, 2, \dots, n\}$.

**8.** Determine the maximum number of regions in the plane that are determined by $n$ "vee"s. A "vee" is two rays which meet at a point. The angle between them is any positive number.

**9.** Define a *domino* to be a $1 \times 2$ rectangle. In how many ways can an $2 \times n$ rectangle be tiled by dominoes?

**10.** Let $t_1, t_2, t_3$ be integers, and let $\lambda_1, \lambda_2, \lambda_3$ be real or complex numbers. Define the sequence $a_n = \lambda_1 t_1^n + \lambda_2 t_2^n + \lambda_3 t_3^n$ for $n = 0, 1, 2$. Prove that if $a_0$, $a_1$, and $a_2$ are integers then $a_n$ is an integer for every $n \geq 0$.

## Solutions

**1.** The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is defined as a sequence whose first two terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.

- *Solution.*
We prove it by strong induction. First we notice that the result is true for $n = 0$ ($F_0 = 0 < 1 = 2^0$), and $n = 1$ ($F_1 = 1 < 2 = 2^1$). Next, for the inductive step, assume that $n \geq 1$ and assume that the claim is true, i.e. $F_k < 2^k$, for every $k$ such that $0 \leq k \leq n$. Then we must prove that the result is also true for $n + 1$. In fact:

$$F_{n+1} = F_n + F_{n-1} \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{<} 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}\,,$$

and we are done.

**2.** A chessboard is a $8 \times 8$ grid (64 squares arranged in 8 rows and 8 columns), but here we will call "chessboard" any $m \times m$ square grid. We call *defective* a chessboard if one of its squares is missing. Prove that any $2^n \times 2^n$ ($n \geq 1$) defective chessboard can be tiled (completely covered without overlapping) with L-shaped triominos occupying exactly 3 squares, like this  .

- *Solution.* We prove it by induction on $n$. For $n = 1$ the defective chessboard consists of just a single L and the tiling is trivial. Next, for the inductive step, assume that a $2^n \times 2^n$ defective chessboard can be tiled with L's. Now, given a $2^{n+1} \times 2^{n+1}$ defective chessboard, we can divide it into four $2^n \times 2^n$ chessboards as shown in the figure below. One of them will have a square missing and will be defective, so it can be tiled with L's. Then we place an L covering exactly one corner of each of the other $2^n \times 2^n$ chessboards (see figure). The remaining part of each of those chessboards is like a defective chessboard and can be tiled in the desired way too. So the whole $2^{n+1} \times 2^{n+1}$ defective chessboard can be tiled with L's.

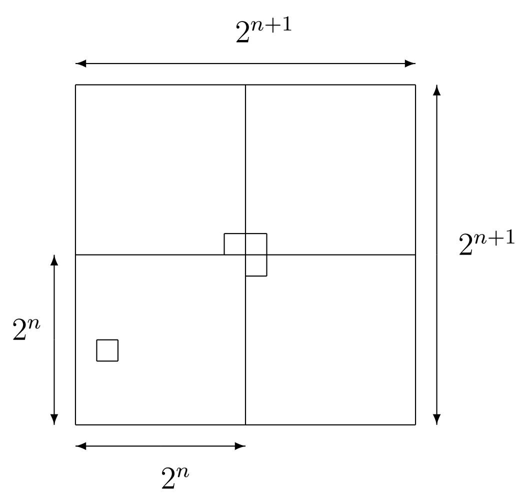


Figure 1. A $2^{n+1} \times 2^{n+1}$ defective chessboard.

**3.** Let $a_n$ be the following expression with $n$ nested radicals:

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}\,.$$

Prove that $a_n = 2\cos\frac{\pi}{2^{n+1}}$.

- *Solution.* Note that $a_n$ can be defined recursively like this: $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. We proceed by induction.
*1. Base Case:* For $n = 1$ we have in fact $a_1 = \sqrt{2}$, and $2\cos\frac{\pi}{4} = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$.
*2. Induction Step:* Assuming the result is true for some $n \geq 1$, we have

$$\begin{aligned} a_{n+1} = \sqrt{2 + a_n} &= \sqrt{2 + 2\cos\frac{\pi}{2^{n+1}}} \\ &= \sqrt{2 + 2\cos 2\frac{\pi}{2^{n+2}}} \\ &= \sqrt{2 + 2(2\cos^2\frac{\pi}{2^{n+2}} - 1)} \qquad (\cos 2x = 2\cos^2 x - 1) \\ &= \sqrt{4\cos^2\frac{\pi}{2^{n+2}}} \\ &= 2\cos\frac{\pi}{2^{n+2}}\,, \end{aligned}$$

hence the stament is also true for $n + 1$ and this completes the proof.

**4.** Prove that, for every positive integer $n$, $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

- *Solution.* Let $S_n = \sum_{k=1}^{n} k^3$ and $T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. We want to prove $S_n = T_n^2$. By induction:
*1. Base Case:* For $n = 1$ we have $S_1 = 1 = T_1^2$.
*2. Induction Step:* Assume $S_n = T_n^2$ for some $n \geq 1$. We need to prove that this implies $S_{n+1} = T_{n+1}^2$. In fact:

$$\begin{aligned} T_{n+1}^2 &= (T_n + (n+1))^2 = T_n^2 + 2T_n(n+1) + (n+1)^2 = \\ &\qquad [\text{apply inducion hypotesis } T_n^2 = S_n \text{ and replace } T_n = \tfrac{n(n+1)}{2}] \\ &= S_n + 2\frac{n(n+1)}{2}(n+1) + (n+1)^2 = S_n + (n+1)^3 = S_{n+1}\,. \end{aligned}$$

This completes the induction step, and the statement is true for every $n \geq 1$.

**5.** We need to put $n$ cents of stamps on an envelop, but we have only (an unlimited supply of) 5¢ and 12¢ stamps. Prove that we can perform the task if $n \geq 44$.

- *Solution.*
We proceed by induction. For the basis step, i.e. $n = 44$, we can use four 5¢ stamps and two 12¢ stamps, so that $5 \cdot 4 + 12 \cdot 2 = 44$. Next, for the induction step, assume that for a given $n \geq 44$ the task is possible by using $x$ 5¢ stamps and $y$ 12¢ stamps,

i.e, $n = 5x + 12y$. We must now prove that we can find some combination of $x'$ 5¢ stamps and $y'$ 12¢ stamps so that $n + 1 = 5x' + 12y'$. First note that either $x \geq 7$ or $y \geq 2$ — otherwise we would have $x \leq 6$ and $y \leq 1$, hence $n \leq 5 \cdot 6 + 12 \cdot 1 = 42 < 44$, contradicting the hypothesis that $n \geq 44$. So we consider the two cases:

1. If $x \geq 7$, then we can accomplish the goal by setting $x' = x - 7$ and $y' = y + 6$:

$$5x' + 12y' = 5(x - 7) + 12(y + 6) = 5x + 12y + 1 = n + 1\,.$$

2. On the other hand, if $y \geq 2$ then, we can do it by setting $x' = x+5$ and $y' = y-2$:

$$5x' + 12y' = 5(x + 5) + 12(y - 2) = 5x + 12y + 1 = n + 1\,.$$

**6.** We define recursively the *Ulam numbers* by setting $u_1 = 1$, $u_2 = 2$, and for each subsequent integer $n$, we set $n$ equal to the next Ulam number if it can be written uniquely as the sum of two distinct Ulam numbers; e.g.: $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, etc. Prove that there are infinitely many Ulam numbers.

\- *Solution.*
Let $U_m = \{u_1, u_2, \dots, u_m\}$ $(m \geq 2)$ be the first $m$ Ulam numbers (written in increasing order). Let $S_m$ be the set of integers greater than $u_m$ that can be written uniquely as the sum of two different Ulam numbers from $U_m$. The next Ulam number $u_{m+1}$ is precisely the minimum element of $S_m$, unless $S_m$ is empty, but it is not because $u_{m-1} + u_m \in S_m$.

**7.** Find the number of subsets of $\{1, 2, \dots, n\}$ that contain no consecutive elements of $\{1, 2, \dots, n\}$.

\- *Solution.* Let $f(n)$ be that number. Then we easily find $f(0) = 1$ (the empty subset), $f(1) = 2$ (including the empty subset), $f(2) = 3$, $f(3) = 5$, $f(4) = 8$, ... suggesting that $f(n) = F_{n+2}$ (shifted Fibonacci sequence). We prove this by showing that $f(n)$ verifies the same recurrence as the Fibonacci sequence. The subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing $n$, and the ones containing $n$. The number of the ones not containing $n$ is just $f(n-1)$. On the other hand the ones containing $n$ cannot contain $n - 1$, so their number equals $f(n - 2)$. Hence $f(n) = f(n - 1) + f(n - 2)$, QED.

**8.** Determine the maximum number of regions in the plane that are determined by $n$ "vee"s. A "vee" is two rays which meet at a point. The angle between them is any positive number.

\- *Solution.* Let $x_n$ be the number of regions in the plane determined by $n$ "vee"s. Then $x_1 = 2$, and $x_{n+1} = x_n + 4n + 1$. We justify the recursion by noticing that the $(n + 1)$th "vee" intersects each of the other "vee"s at 4 points, so it is divided into $4n + 1$ pieces, and each piece divides one of the existing regions of the plane into two, increasing the total number of regions by $4n + 1$. So the answer is

$$x_n = 2 + (4 + 1) + (4 \cdot 2 + 1) + \cdots + (4 \cdot (n - 1) + 1) = 2n^2 - n + 1\,.$$

**9.** Define a *domino* to be a $1 \times 2$ rectangle. In how many ways can an $2 \times n$ rectangle be tiled by dominoes?

- *Solution.* Let $x_n$ be the number of tilings of an $n \times 2$ rectangle by dominoes. We easily find $x_1 = 1$, $x_2 = 2$. For $n \geq 3$ we can place the rightmost domino vertically and tile the rest of the rectangle in $x_{n-1}$ ways, or we can place two horizontal dominoes to the right and tile the rest in $x_{n-2}$ ways, so $x_n = x_{n-1} + x_{n-2}$. So the answer is the shifted Fibonacci sequence, $x_n = F_{n+1}$.

**10.** Let $t_1, t_2, t_3$ be integers, and let $\lambda_1, \lambda_2, \lambda_3$ be real or complex numbers. Define the sequence $a_n = \lambda_1 t_1^n + \lambda_2 t_2^n + \lambda_3 t_3^n$ for $n = 0, 1, 2$. Prove that if $a_0$, $a_1$, and $a_2$ are integers then $a_n$ is an integer for every $n \geq 0$.

- *Solution.*
Consider the polynomial

$$p(x) = (x - t_1)(x - t_2)(x - t_3) = x^3 - c_1 x^2 - c_2 x - c_3\,,$$

where $c_1 = t_1 + t_2 + t_3$, $c_2 = -t_1 t_2 - t_1 t_3 - t_2 t_3$, $c_3 = t_1 t_2 t_3$. Note that since $t_1, t_2, t_3$ are integers, then $c_1, c_2, c_3$ are also integers. Also, the $t_i$'s are roots of $p$, i.e., $p(t_i) = 0$, hence $t_i^3 = c_1 t_i^2 + c_2 t_i + c_3$ $(i = 1, 2, 3)$. Multiplying by $t_i^n$ we get $t_i^{n+3} = c_1 t_i^{n+2} + c_2 t_i^{n+1} + c_3 t_i^n$. By definition $a_n = \lambda_1 t_1^n + \lambda_2 t_2^n + \lambda_3 t_3^n$, so by linearity:

$$a_{n+3} = c_1 a_{n+2} + c_2 a_{n+1} + c_3 a_n \qquad (n = 0, 1, 2, \dots)\,.$$

Finally, given that $a_0, a_1, a_2$ are integers by hypothesis, and that $c_1, c_2, c_3$ are also integers, the result follows by induction.