PUTNAM TRAINING 10/12/2021
INEQUALITIES

1. The notation $n!^{(k)}$ means take factorial of n k times. For example, $n!^{(3)}$ means $((n!)!)!$ What is bigger, $1999!^{(2000)}$ or $2000!^{(1999)}$?

2. Which is larger, $\log_2 3$ or $\log_3 5$?

3. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

4. Prove that $n! < \left(\frac{n+1}{2}\right)^n$, for $n = 2, 3, 4, \ldots$

5. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers, and let $b_1, b_2, \ldots, b_n$ be any permutation of the first sequence. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n\,.$$

6. Find the minimum value of the function $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, where $x_1, x_2, \ldots, x_n$ are positive real numbers such that $x_1 x_2 \cdots x_n = 1$.

7. Find the minimum of $\sin^3 x / \cos x + \cos^3 x / \sin x$, $0 < x < \pi/2$.

8. Prove that $e^{1/e} + e^{1/\pi} \geq 2e^{1/3}$.

9. (Putnam, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

10. Prove that if the numbers a, b, and c satisfy the inequalities $|a - b| \geq |c|$, $|b - c| \geq |a|$, $|c - a| \geq |b|$, then one of those numbers is the sum of the other two.

11. Find the positive solutions of the system of equations

$$x_1 + \frac{1}{x_2} = 4\,, \quad x_2 + \frac{1}{x_3} = 1\,, \ldots\,, \quad x_{99} + \frac{1}{x_{100}} = 4\,, \quad x_{100} + \frac{1}{x_1} = 1\,.$$