PUTNAM TRAINING 10/12/2021
INEQUALITIES - SOLUTIONS

1. The notation $n!^{(k)}$ means take factorial of n k times. For example, $n!^{(3)}$ means $((n!)!)!$ What is bigger, $1999!^{(2000)}$ or $2000!^{(1999)}$?

- *Solution.*We have that $n!$ is increasing for $n \geq 1$, i.e., $1 \leq n < m \implies n! < m!$ So $1999! > 2000 \implies (1999!)! > 2000! \implies ((1999!)!)! > (2000!)! \implies \dots \implies 1999!^{(2000)} > 2000!^{(1999)}$.

2. Which is larger, $\log_2 3$ or $\log_3 5$?

- *Solution.* Let $x = \log_2 3$ and $y = \log_3 5$, so $2^x = 3$, $3^y = 5$. Then, $27 = 3^3 = (2^x)^3 = 8^x$, and $25 = 5^2 = (3^y)^2 = 9^y$, hence $8^x > 9^y$, but $8 < 9$, hence $x > y$, i.e., $\log_2 3 > \log_3 5$.

3. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

- *Solution.* Using the Arithmetic Mean-Geometric Mean Inequality on each factor of the LHS we get

$$\left(\frac{a^2b + b^2c + c^2a}{3}\right)\left(\frac{ab^2 + bc^2 + ca^2}{3}\right) \geq \left(\sqrt[3]{a^3b^3c^3}\right)\left(\sqrt[3]{a^3b^3c^3}\right) = a^2b^2c^2 \,.$$

Multiplying by 9 we get the desired inequality.

4. Prove that $n! < \left(\frac{n+1}{2}\right)^n$, for $n = 2, 3, 4, \dots$

- *Solution.* This result is the Arithmetic Mean-Geometric Mean applied to the set of numbers $1, 2, \dots, n$:

$$\sqrt[n]{1 \cdot 2 \cdot \dots \cdot n} < \frac{1 + 2 + \dots + n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2} \,.$$

Raising both sides to the nth power we get the desired result.

5. Let $a_1, a_2, \dots, a_n$ be a sequence of positive numbers, and let $b_1, b_2, \dots, b_n$ be any permutation of the first sequence. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \geq n \,.$$

- *Solution.* Using the Arithmetic Mean-Geometric Mean inequality we get:

$$\frac{1}{n}\left\{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n}\right\} \geq \sqrt[n]{\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdots \frac{a_n}{b_n}} = 1 \,.$$

From here the desired result follows.

6. Find the minimum value of the function $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$, where $x_1, x_2, \dots, x_n$ are positive real numbers such that $x_1 x_2 \cdots x_n = 1$.

- *Solution.* By the Arithmetic Mean-Geometric Mean Inequality

$$1 = \sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

Hence $f(x_1, x_2, \dots, x_n) \geq n$. On the other hand $f(1, 1, \dots, 1) = n$, so the minimum value is n.

7. Find the minimum of $\sin^3 x/\cos x + \cos^3 x/\sin x$, $0 < x < \pi/2$.

- *Solution.* The answer is 1. In fact, the sequences $(\sin^3 x, \cos^3 x)$ and $(1/\sin x, 1/\cos x)$ are oppositely sorted, hence by the rearrangement inequality:

$$\begin{aligned}\sin^3 x/\cos x + \cos^3 x/\sin x &\geq \sin^3 x/\sin x + \cos^3 x/\cos x \\ &= \sin^2 x + \cos^2 x = 1\,.\end{aligned}$$

Equality is attained at $x = \pi/4$.

8. Prove that $e^{1/e} + e^{1/\pi} \geq 2e^{1/3}$.

- *Solution.* Consider the function $f(x) = e^{1/x}$ for $x > 0$. We have $f'(x) = -\frac{1}{x^2} e^{1/x} < 0$, $f''(x) = e^{1/x}(\frac{2}{x^3} + \frac{1}{x^4}) > 0$, hence f is decreasing and convex.

By convexity, we have

$$\frac{1}{2}(f(e) + f(\pi)) \geq f\left(\tfrac{e+\pi}{2}\right)\,.$$

On the other hand we have $(e + \pi)/2 < 3$, and since f is decreasing, $f(\frac{e+\pi}{2}) > f(3)$, and from here the result follows.

9. (Putnam, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

- *Solution.* The given inequality is equivalent to

$$\frac{(m+n)!}{m!\,n!}\,m^m n^n = \binom{m+n}{n} m^m n^n < (m+n)^{m+n},$$

which is obviously true because the binomial expansion of $(m+n)^{m+n}$ includes the term on the left plus other terms.

10. Prove that if the numbers a, b, and c satisfy the inequalities $|a - b| \geq |c|$, $|b - c| \geq |a|$, $|c - a| \geq |b|$, then one of those numbers is the sum of the other two.

- *Solution.* Squaring the inequalities and moving their left hand sides to the right we get

$$0 \geq c^2 - (a-b)^2 = (c+a-b)(c-a+b)$$
$$0 \geq a^2 - (b-c)^2 = (a+b-c)(a-b+c)$$
$$0 \geq b^2 - (c-a)^2 = (b+c-a)(b-c+a)\,.$$

Multiplying them together we get:

$$0 \geq (a+b-c)^2(a-b+c)^2(-a+b+c)^2\,,$$

hence, one of the factors must be zero.

11. Find the positive solutions of the system of equations

$$x_1 + \frac{1}{x_2} = 4\,, \quad x_2 + \frac{1}{x_3} = 1\,, \ldots\,, \quad x_{99} + \frac{1}{x_{100}} = 4\,, \quad x_{100} + \frac{1}{x_1} = 1\,.$$

- *Solution.* By the Geometric Mean-Arithmetic Mean inequality

$$x_1 + \frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}\,, \ldots\,, x_{100} + \frac{1}{x_1} \geq 2\sqrt{\frac{x_{100}}{x_1}}\,.$$

Multiplying we get

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{100} + \frac{1}{x_1}\right) \geq 2^{100}\,.$$

From the system of equations we get

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{100} + \frac{1}{x_1}\right) = 2^{100}\,,$$

so all those inequalities are equalities, i.e.,

$$x_1 + \frac{1}{x_2} = 2\sqrt{\frac{x_1}{x_2}} \quad \Longrightarrow \quad \left(\sqrt{x_1} - \frac{1}{\sqrt{x_2}}\right)^2 = 0 \quad \Longrightarrow \quad x_1 = \frac{1}{x_2}\,,$$

and analogously: $x_2 = 1/x_3$, $\ldots$, $x_{100} = 1/x_1$. Hence $x_1 = 1/x_2$, $x_2 = 1/x_3$, $\ldots$, $x_{100} = 1/x_1$, and from here we get $x_1 = 2$, $x_2 = 1/2$,$\ldots$, $x_{99} = 2$, $x_{100} = 1/2$.