PUTNAM TRAINING 11/23/2021
MISCELANOUS PROBLEMS

1. (Putnam 1986) What is the units (i.e., rightmost) digit of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$?

2. (IMO 1975) Prove that there are infinitely many points on the unit circle $x^2+y^2=1$ such that the distance between any two of them is a rational number.

3. (Putnam 1988) Prove that if we paint every point of the plane in one of three colors, there will be two points one inch apart with the same color. Is this result necessarily true if we replace "three" by "nine"?

4. Imagine an infinite chessboard that contains a positive integer in each square. If the value of each square is equal to the average of its four neighbors to the north, south, west and east, prove that the values in all the squares are equal.

5. (Putnam 1984) Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!\,.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

6. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

7. On a table there is a row of fifty coins, of various denominations (the denominations could be of any values). Alice picks a coin from one of the ends and puts it in her pocket, then Bob chooses a coin from one of the ends and puts it in his pocket, and the alternation continues until Bob pockets the last coin. Prove that Alice can play so that she guarantees at least as much money as Bob.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \circ f$ has a fixed point, i.e., there is some real number x_0 such that $f(f(x_0)) = x_0$. Prove that f also has a fixed point.

9. Prove that $\tan 1^\circ$ is irrational.

10. Consider a set X and a binary operation $*$, i.e., for each $x, y \in X$, $x * y \in X$. Assume
(1) $(x * y) * y = x$ for all $x, y \in X$, and
(2) $x * (x * y) = y$ for all $x, y \in X$.
Prove that $x * y = y * x$ for all $x, y \in X$.

11. (Ilan Vardi, Mekh-mat entrance examinations problems, No. 21) A circle is inscribed in a face of a cube of side 2. Another circle is circumscribed about a neighboring face of the cube. Find the least distance between points of the circles.

Solutions

1. (Putnam 1986) What is the units (i.e., rightmost) digit of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$?

- *Solution.*
Let $I = \frac{10^{20000} - 3^{200}}{10^{100}+3} = \frac{(10^{100})^{200} - 3^{200}}{10^{100}+3}$
$= (10^{100})^{199} - (10^{100})^{198} \cdot 3 + \cdots + 10^{100} \cdot 3^{198} - 3^{199}$
so I is an integer. On the other hand since $\frac{3^{200}}{10^{100}+3} < 1$ we have that $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor = I$. Finally the rightmost digit of I can be found as the 1-digit number congruent to $-3^{199} \pmod{10}$. The sequence $3^n \bmod 10$ has period 4 and $199 = 3 + 4 \cdot 49$, hence $-3^{199} \bmod 10 = -3^3 \bmod 10 = -27 \bmod 10 = 3$. Hence the units digit of I is 3.

2. (IMO 1975) Prove that there are infinitely many points on the unit circle $x^2 + y^2 = 1$ such that the distance between any two of them is a rational number.

- *Solution.* Let α be any (say the smallest) acute angle of a right triangle with sides 3, 4 and 5 (or any other Pythagorean triple). Next, place an infinite sequence of points on the unit circle at coordinates $(\cos(2n\alpha), \sin(2n\alpha))$, $n = 0, 1, 2, \dots$ (The sequence contains in fact infinitely many points because α cannot be a rational multiple of π.) The distance from $(\cos(2n\alpha), \sin(2n\alpha))$ to $(\cos(2m\alpha), \sin(2m\alpha))$ is $2\sin(|n-m|\alpha)$, so all we need to prove is that $\sin(k\alpha)$ is rational for any k. This can be done by induction using that $\sin\alpha$ and $\cos\alpha$ are rational, and if $\sin u$, $\cos u$, $\sin v$ and $\cos(v)$ are all rational so are $\sin(u+v) = \sin u \cos v + \cos u \sin v$ and $\cos(u+v) = \cos u \cos v - \sin u \sin v$.

3. (Putnam 1988) Prove that if we paint every point of the plane in one of three colors, there will be two points one inch apart with the same color. Is this result necessarily true if we replace "three" by "nine"?

- *Solution.* We can prove the first part by way of contradiction. Assume that we have colored the points of the plane with three colors such that any two points at distance 1 have different colors. Consider any two points A and B at distance $\sqrt{3}$ (see figure 1). The circles of radius 1 and center A and B meet at two points P and Q, forming equilateral triangles APQ and BPQ. Since the vertices of each triangle must have different colors that forces A and B to have the same color. So any two points at distance $\sqrt{3}$ have the same color. Next consider a triangle DCE with $CD = CE = \sqrt{3}$ and $DE = 1$. The points D and E must have the same color as C, but since they are at distance 1 they should have different colors, so we get a contradiction.
For the second part, if we replace "three" by "nine" then we can color the plane with nine different colors so that any two points at distance 1 have different colors: we can arrange them periodically in a grid of squares of size $2/3 \times 2/3$ as shown in figure 2. If two points P and Q have the same color then either they belong to the same square and $PQ < (2/3)\sqrt{2} < 1$, or they belong to different squares and $PQ \geq 4/3 > 1$.

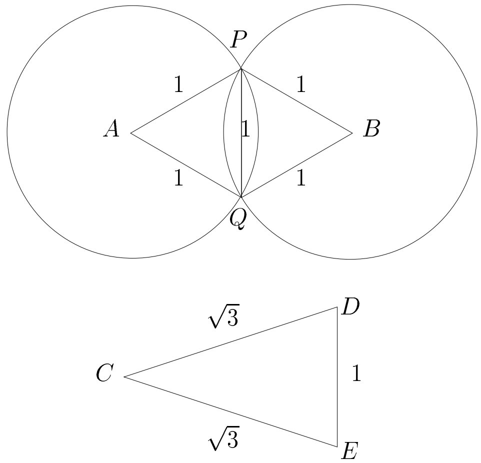

FIGURE 1

A	B	C	A	B	C
D	E	F	D	E	F
G	H	I	G	H	I
A	B	C	A	B	C
D	E	F	D	E	F

FIGURE 2

4. Imagine an infinite chessboard that contains a positive integer in each square. If the value of each square is equal to the average of its four neighbors to the north, south, west and east, prove that the values in all the squares are equal.

- *Solution.* Since the values are positive integers, one of them, say n, will be the smallest one. Look at any square with that value n. Since the values of its four neighbors must be at least n and their average is n, all four will have value n. By the same reasoning the neighbors of these must be n too, and so on, so all the squares must have the same value n.

5. (Putnam 1984) Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!\,.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

- *Solution.* We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)(f(n+1) - f(n))\,,$$

hence

$$\begin{aligned} f(n+2) &= (n+2)(f(n+1) - f(n)) + f(n+1) \\ &= (n+3)f(n+1) - (n+2)f(n)\,, \end{aligned}$$

and we can take $P(x) = x + 3$, $Q(x) = -x - 2$.

6. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

- *Solution.* We can group the terms of the sequence in the following way:

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1}_{b_0} + \underbrace{(a_2 + a_3)}_{b_1} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{b_2} + \cdots + \underbrace{(a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})}_{b_k} + \cdots$$

The condition implies that $b_k \leq b_{k+1}$ for every $k \geq 0$, hence the sequence diverges.

7. On a table there is a row of fifty coins, of various denominations (the denominations could be of any values). Alice picks a coin from one of the ends and puts it in her pocket, then Bob chooses a coin from one of the ends and puts it in his pocket, and the alternation continues until Bob pockets the last coin. Prove that Alice can play so that she guarantees at least as much money as Bob.

- *Solution.* Alice adds the values of the coins in odd positions 1st, 3rd, 5th, etc., getting a sum S_{odd}. Then she does the same with the coins placed in even positions 2nd, 4th, 6th, etc., and gets a sum S_{even}. Assume that $S_{odd} \geq S_{even}$. Then she will pick all the coins in odd positions, forcing Bob to pick only coins in the even positions. To do so she stars by picking the coin in position 1, so Bob can pick only the coins in position 2 or 50. If he picks the coin in position 2, Alice will the pick coin in position 3, if he picks the coin in position 50 she picks the coin in position 49, and so on, with Alice always picking the coin at the same side as the coin picked by Bob.
If $S_{odd} \leq S_{even}$, then Alice will use a similar strategy ensuring that she will end up picking all the coins in the even positions, and forcing Bob to pick the coins in the odd positions—this time she will pick first the 50th coin, and then at each step she will pick a coin at the same side as the coin picked by Bob.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \circ f$ has a fixed point, i.e., there is some real number x_0 such that $f(f(x_0)) = x_0$. Prove that f also has a fixed point.

- *Solution.* By contradiction. If the equality $f(x) = x$ never holds then $f(x) > x$ for every x, or $f(x) < x$ for every x. Then $f(f(x)) > f(x) > x$ for every x, or $f(f(x)) < f(x) < x$ for every x, contradicting the hypothesis that $f \circ f$ has a fixed point.

9. Prove that $\tan 1^\circ$ is irrational.

- *Solution.* By contradiction. Assume $\tan 1^\circ$ is rational. Then $\tan 2^\circ = \tan(1^\circ + 1^\circ) = \frac{\tan 1^\circ + \tan 1^\circ}{1 - \tan 1^\circ \tan 1^\circ}$ would be rational too. Same for $\tan 3^\circ = \tan(2^\circ + 1^\circ) = \frac{\tan 2^\circ + \tan 1^\circ}{1 - \tan 2^\circ \tan 1^\circ}$, $\dots$, $\tan(n+1)^\circ = \tan(n^\circ + 1^\circ) = \frac{\tan n^\circ + \tan 1^\circ}{1 - \tan n^\circ \tan 1^\circ}$, $\dots$, $\tan 30^\circ = \tan(29^\circ + 1^\circ) = \frac{\tan 29^\circ + \tan 1^\circ}{1 - \tan 29^\circ \tan 1^\circ}$. But $\tan 30^\circ = \frac{1}{\sqrt{3}}$ is irrational.

10. Consider a set X and a binary operation $*$, i.e., for each $x, y \in X$, $x * y \in X$. Assume
(1) $(x * y) * y = x$ for all $x, y \in X$, and
(2) $x * (x * y) = y$ for all $x, y \in X$.
Prove that $x * y = y * x$ for all $x, y \in X$.

- *Solution.* For any $x, y \in X$, letting $z = y * x$ and using (1) we have

$$x * ((y * \underbrace{(y * x)}_{z}) * \underbrace{(y * x)}_{z}) = x * (\underbrace{(y * z) * z}_{y}) \overset{(1)}{=} x * y \,,$$

and using (2) twice:

$$x * ((\underbrace{y * (y * x)}_{x}) * (y * x)) \overset{(2)}{=} x * (x * \underbrace{(y * x)}_{z}) = x * (x * z) \overset{(2)}{=} z = y * x \,.$$

Since $x * y$ and $y * z$ are equal to the same expression, they are equal to each other: $x * y = y * x$, Q.E.D.

11. (Ilan Vardi, Mekh-mat entrance examinations problems, No. 21) A circle is inscribed in a face of a cube of side 2. Another circle is circumscribed about a neighboring face of the cube. Find the least distance between points of the circles.

- *Solution.* The answer is $\sqrt{3} - \sqrt{2}$.
Consider two spheres with center at the center of the cube with each containing one of the circles mentioned in the problem. Clearly, the distance between the circles cannot be less than the distances between the spheres. On the other hand, it is easy to see that there is a ray from the center that intersects both circles. It follows that this distance is minimal. Hence the answer is the difference between the radii of the spheres.
Using a system of Cartesian coordinates, assume the center of the cube is at the origin $(0, 0, 0,)$, and the vertices are at $(\pm 1, \pm 1, \pm 1)$. The spheres mentioned above will be centered at the origin too. The inner sphere will touch the sides of the cube at their center, one such point will be the one with coordinates $(0, 1, 1)$, so the inner sphere has radius $\sqrt{2}$. The outer sphere will contain the vertices of the cube, in particular $(1, 1, 1)$, so its radius will be $\sqrt{3}$. Hence the difference between their radii is $\sqrt{3} - \sqrt{2}$.