PUTNAM TRAINING 11/16/2021
PIGEONHOLE PRINCIPLE
GENERATING FUNCTIONS
TELESCOPING

Pigeonhole Principle

The Pigeonhole Principle: If n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

1. Prove that any $(n+1)$-element subset of $\{1, 2, \dots, 2n\}$ contains two integers that are relatively prime.

2. Prove that if we select $n+1$ numbers from the set $S = \{1, 2, 3, \dots, 2n\}$, among the numbers selected there are two such that one is a multiple of the other one.

3. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1, 4, 7, \dots, 100\}$. Prove that there must be two distinct integers in A whose sum if 104.

4. Let A be the set of all 8-digit numbers in base 3 (so they are written with the digits 0,1,2 only), including those with leading zeroes such as 00120010. Prove that given 4 elements from A, two of them must coincide in at least 2 places.

5. (Putnam, 2006-B2.) Prove that, for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

6. (IMO 1972.) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

7. Prove that among any seven real numbers $y_1, \dots, y_7$, there are two such that

$$0 \leq \frac{y_i - y_j}{1 + y_i y_j} \leq \frac{1}{\sqrt{3}}.$$

8. Prove that among five different integers there are always three with sum divisible by 3.

9. Prove that there exist an integer n such that the first four digits of 2^n are $2, 0, 2, 1$.

10. Prove that every convex polyhedron has at least two faces with the same number of edges.

Generating Functions

Generating Functions: $a_0, a_1, a_2, a_3, \cdots \mapsto \sum_{n=0}^{\infty} a_n x^n$.

1. Prove that for any positive integer n

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1},$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ (binomial coefficient).

2. Prove that for any positive integer n

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

3. Prove that for any positive integers $k \leq m, n$,

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{m+n}{k}.$$

4. Let F_n be the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$, defined recursively $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2.$$

5. How many different sequences are there that satisfy all the following conditions:
(a) The items of the sequences are the digits 0–9.
(b) The length of the sequences is 6 (e.g. 061030)
(c) Repetitions are allowed.
(d) The sum of the items is exactly 10 (e.g. 111322).

Telescoping

Telescoping: $\sum_{k=1}^{n} (a_{k+1} - a_k) = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n+1} - a_n) = a_{n+1} - a_1$.

1. Prove that $\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = 9$.

2. Let N be a positive integer and let S_N be the sum

$$S_N = \frac{1}{2} \sum_{k=1}^{N^2} \frac{1}{\sqrt{n}}.$$

Find $\lfloor S_N \rfloor$, where $\lfloor x \rfloor$ = largest integer less than or equal to x.

3. Find a closed form for $\sum_{n=1}^{N} n \cdot n!$

4. (Putnam 1984) Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

5. (Putnam 1977) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

6. Evaluate the infinite series: $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}$.

Solutions

Pigeonhole Principle

The Pigeonhole Principle: If n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

1. Prove that any $(n+1)$-element subset of $\{1, 2, \ldots, 2n\}$ contains two integers that are relatively prime.

- *Solution.* We divide the set into n-classes $\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}$. By the pigeonhole principle, given $n+1$ elements, at least two of them will be in the same class, $\{2k-1, 2k\}$ $(1 \le k \le n)$. But $2k-1$ and $2k$ are relatively prime because their difference is 1.

2. Prove that if we select $n+1$ numbers from the set $S = \{1, 2, 3, \ldots, 2n\}$, among the numbers selected there are two such that one is a multiple of the other one.

- *Solution.* For each odd number $\alpha = 2k-1$, $k = 1, \ldots, n$, let C_α be the set of elements x in S such that $x = 2^i\alpha$ for some i. The sets $C_1, C_3, \ldots, C_{2n-1}$ are a classification of S into n classes. By the pigeonhole principle, given $n+1$ elements of S, at least two of them will be in the same class. But any two elements of the same class C_α verify that one is a multiple of the other one.

3. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1, 4, 7, \ldots, 100\}$. Prove that there must be two distinct integers in A whose sum if 104.

- *Solution.* The given set can be divided into 18 subsets $\{1\}$, $\{4, 100\}$, $\{7, 97\}$, $\{10, 94\}$,..., $\{49, 55\}$, $\{52\}$. By the pigeonhole principle two of the numbers will be in the same set, and all 2-element subsets shown verify that the sum of their elements is 104.

4. Let A be the set of all 8-digit numbers in base 3 (so they are written with the digits 0,1,2 only), including those with leading zeroes such as 00120010. Prove that given 4 elements from A, two of them must coincide in at least 2 places.

- *Solution.* For $k = 1, 2, \ldots, 8$, look at the digit used in place k for each of the 4 given elements. Since there are only 3 available digits, two of the elements will use the same digit in place k, so they coincide at that place. Hence at each place, there are at least two elements that coincide at that place. Pick any pair of such elements for each of the 8 places. Since there are 8 places we will have 8 pairs of elements, but there are only $\binom{4}{2} = 6$ two-element subsets in a 4-element set, so two of the pairs will be the same pair, and the elements of that pair will coincide in two different places.

5. (Putnam, 2006-B2.) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \le \frac{1}{n+1}.$$

- *Solution.* Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. For $i = 0, \ldots, n$, put $s_i = x_1 + \cdots + x_i$ (so that $s_0 = 0$). Sort the numbers $\{s_0\}, \ldots, \{s_n\}$ into ascending order, and call the result $t_0, \ldots, t_n$. Since $0 = t_0 \le \cdots \le t_n < 1$, the differences

$$t_1 - t_0, \ldots, t_n - t_{n-1}, 1 - t_n$$

are nonnegative and add up to 1. Hence (as in the pigeonhole principle) one of these differences is no more than $1/(n+1)$; if it is anything other than $1 - t_n$, it equals $\pm(\{s_i\} - \{s_j\})$ for some $0 \le i < j \le n$. Put $S = \{x_{i+1}, \ldots, x_j\}$ and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$; then

$$\begin{aligned} \left| m + \sum_{s \in S} s \right| &= |m + s_j - s_i| \\ &= |\{s_j\} - \{s_i\}| \\ &\le \frac{1}{n+1}, \end{aligned}$$

as desired. In case $1 - t_n \le 1/(n+1)$, we take $S = \{x_1, \ldots, x_n\}$ and $m = -\lceil s_n \rceil$, and again obtain the desired conclusion.

6. (IMO 1972.) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

- *Solution.* A set of 10 elements has $2^{10} - 1 = 1023$ non-empty subsets. The possible sums of at most ten two-digit numbers cannot be larger than $10 \cdot 99 = 990$. There are more subsets than possible sums, so two different subsets S_1 and S_2 must have the same sum. If $S_1 \cap S_2 = \emptyset$ then we are done. Otherwise remove the common elements and we get two non-intersecting subsets with the same sum.

7. Prove that among any seven real numbers $y_1, \ldots, y_7$, there are two such that

$$0 \le \frac{y_i - y_j}{1 + y_i y_j} \le \frac{1}{\sqrt{3}}.$$

- *Solution.* Writing $y_i = \tan x_i$, with $-\frac{\pi}{2} \le x_i \le \frac{\pi}{2}$ $(i = 1, \ldots, 7)$, we have that

$$\frac{y_i - y_j}{1 + y_i y_j} = \tan(x_i - x_j),$$

so all we need is to do is prove that there are x_i, x_j such that $0 \le x_i - x_j \le \frac{\pi}{6}$. To do so we divide the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ into 6 subintervals each of length $\frac{\pi}{6}$. By the box principle, two of the x_is will be in the same subinterval, and their difference will be not larger than $\frac{\pi}{6}$, as required.

8. Prove that among five different integers there are always three with sum divisible by 3.

- *Solution.* Classify the numbers by their reminder when divided by 3. Either three of them will yield the same reminder, and their sum will be a multiple of 3, or there will be at least a number x_r for each possible reminder $r = 0, 1, 2$, and their sum $x_0 + x_1 + x_2$ will be a multiple of 3 too.

9. Prove that there exist an integer n such that the first four digits of 2^n are $2, 0, 2, 1$.

- *Solution.* We must prove that there are positive integers n, k such that

$$2021 \cdot 10^k \leq 2^n < 2022 \cdot 10^k\,.$$

That double inequality is equivalent to

$$\log_{10}(2021) + k \leq n \log_{10}(2) < \log_{10}(2022) + k\,.$$

where $\log_{10}$ represents the decimal logarithm. Writing $\alpha = \log_{10}(2021) - 3$, $\beta = \log_{10}(2022) - 3$, we have $0 < \alpha < \beta < 1$, and the problem amounts to showing that for some integer n, the fractional part of $n \log_{10}(2)$ is in the interval $[\alpha, \beta)$. This is true because $\log_{10}(2)$ is irrational, and the integer multiples of an irrational number are dense modulo 1 (their fractional parts are dense in the interval $[0, 1)$).

10. Prove that every convex polyhedron has at least two faces with the same number of edges.

- *Solution.* Let F be a face with the largest number m of edges. Then for the $m+1$ faces consisting of F and its m neighbors the possible number of edges are $3, 4, \dots, m$. These are only $m - 2$ possibilities, hence the number of edges must occur more than once.

Generating Functions

Generating Functions: $a_0, a_1, a_2, a_3, \dots \mapsto \sum_{n=0}^{\infty} a_n x^n$.

1. Prove that for any positive integer n

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}\,,$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ (binomial coefficient).

- *Solution.* We have

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (1+x)^n\,.$$

Differentiating respect to x:

$$\binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n} = n(1+x)^{n-1}\,.$$

Plugging in $x = 1$ we get the desired identity

2. Prove that for any positive integer n

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

- *Solution.* The desired expression states the equality between the coefficient of x^n in each of the following expansions:

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k ,$$

and

$$\{(1+x)^n\}^2 = \left\{ \sum_{k=0}^{n} \binom{n}{k} x^k \right\}^2 = \sum_{k=0}^{n} \sum_{i+j=k} \binom{n}{i}\binom{n}{j} x^k .$$

Taking into account that $\binom{n}{j} = \binom{n}{n-j}$, for $k = n$ we get

$$\sum_{i+j=n} \binom{n}{i}\binom{n}{j} = \sum_{i+j=n} \binom{n}{i}\binom{n}{n-j} = \sum_{i=1}^{n} \binom{n}{i}^2 ,$$

and that must be equal to the coefficient of x^n in $(1+x)^{2n}$, which is $\binom{2n}{n}$.

3. Prove that for any positive integers $k \le m, n$,

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{m+n}{k}.$$

- *Solution.* This is just a generalization of the previous problem. We have

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k ,$$

and

$$\begin{aligned}(1+x)^m(1+x)^n &= \left\{ \sum_{i=0}^{m} \binom{m}{i} x^i \right\} \left\{ \sum_{j=0}^{n} \binom{n}{j} x^j \right\} \\ &= \sum_{k=0}^{m+n} \sum_{\substack{i+j=k \\ 0\le i,j\le k}} \binom{n}{j}\binom{m}{i} x^k .\end{aligned}$$

The coefficient of x^k must be the same on both sides, so:

$$\binom{m+n}{k} = \sum_{\substack{i+j=k \\ 0\le i,j\le k}} \binom{n}{j}\binom{m}{i} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} ,$$

where we replace $i = k - j$ in the last step.

4. Let F_n be the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$, defined recursively $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2\,.$$

- *Solution.* The generating function for the Fibonacci sequence is

$$0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots = \frac{x}{1 - x - x^2}\,.$$

The desired sum is the left hand side with $x = 1/2$, hence its value is

$$0 + \frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{2} - \frac{1}{2^2}} = \boxed{2}\,.$$

5. How many different sequences are there that satisfy all the following conditions:
(a) The items of the sequences are the digits 0–9.
(b) The length of the sequences is 6 (e.g. 061030)
(c) Repetitions are allowed.
(d) The sum of the items is exactly 10 (e.g. 111322).

- *Solution.*
The answer equals the coefficient of x^{10} in the expansion of

$$(1 + x + x^2 + \cdots + x^9)^6\,.$$

Since $1+x+x^2+\cdots = 1/(1-x)$ the answer can be obtained also from the coefficient of x^{10} in the Maclaurin series of $1/(1-x)^6 = (1-x)^{-6}$. Since that includes six sequences of the form $0, 0, \cdots, 10, \cdots, 0$ we need to subtract 6, so the final answer is

$$\binom{-6}{10} - 6 = \frac{(-6)(-7)(-8)(-9)(-10)(-11)(-12)(-13)(-14)(-15)}{10!} - 6$$
$$= 3003 - 6 = \boxed{2997}\,.$$

Telescoping

Telescoping: $\displaystyle\sum_{k=1}^{n}(a_{k+1} - a_k) = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n+1} - a_n) = a_{n+1} - a_1.$

1. Prove that $\displaystyle\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = 9\,.$

- *Solution.* After rationalizing we get a telescopic sum:

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{100}-\sqrt{99})$$
$$= 10 - 1 = 9\,.$$

2. Let N be a positive integer and let S_N be the sum

$$S_N = \frac{1}{2}\sum_{k=1}^{N^2} \frac{1}{\sqrt{n}}\,.$$

Find $\lfloor S_N \rfloor$, where $\lfloor x \rfloor$ = largest integer less than or equal to x.

- *Solution.* For every $n \geq 1$ we have

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n} + \sqrt{n+1}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n-1} + \sqrt{n}} = \sqrt{n} - \sqrt{n-1}\,,$$

hence

$$\sum_{k=1}^{N^2} (\sqrt{n+1} - \sqrt{n}) < S_N < \sum_{k=1}^{N^2} (\sqrt{n} - \sqrt{n-1})\,.$$

The sums on the left and right sides telescope:

$$\sum_{k=1}^{N^2} (\sqrt{n+1} - \sqrt{n}) = \sqrt{N^2+1} - 1 > N - 1\,,$$

$$\sum_{k=1}^{N^2} (\sqrt{n} - \sqrt{n-1}) = \sqrt{N^2} - \sqrt{0} = N\,,$$

hence $\lfloor S_N \rfloor = N - 1$.

3. Find a closed form for $\displaystyle\sum_{n=1}^{N} n \cdot n!$

- *Solution.* We have

$$\sum_{n=1}^{N} n \cdot n! = \sum_{n=1}^{N} \{(n+1) - 1\} \cdot n! = \sum_{n=1}^{N} \{(n+1)! - n!\} =$$
$$(2! - 1!) + (3! - 2!) + \cdots + ((N+1)! - N!) = (N+1)! - 1\,.$$

4. (Putnam 1984) Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

- *Solution.* We have

$$\frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} = \frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}}\,.$$

So this is a telescopic sum:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} &= \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ \frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}} \right\} \\ &= \lim_{n\to\infty} \left\{ 3 - \frac{3^{n+1}}{3^{n+1} - 2^{n+1}} \right\} \\ &= 3 - 1 = 2\,. \end{aligned}$$

5. (Putnam 1977) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}\,.$$

- *Solution.* This is a telescopic product:

$$\frac{n^3-1}{n^3+1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} = \frac{(n-1)\{n\,(n+1)+1\}}{(n+1)\{(n-1)\,n+1\}}\,,$$

hence

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \lim_{N\to\infty} \prod_{n=2}^{N} \frac{(n-1)\{n\,(n+1)+1\}}{(n+1)\{(n-1)\,n+1\}}$$

$$= \lim_{N\to\infty} \frac{2\{N(N+1)+1\}}{3N(N+1)} = \frac{2}{3}\,.$$

6. Evaluate the infinite series: $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}$.

- *Solution.*
We have

$$\begin{aligned}\frac{n}{n^4+n^2+1} &= \frac{n}{(n^2+1)^2-n^2} \\ &= \frac{1/2}{n^2-n+1} - \frac{1/2}{n^2+n+1} \\ &= \frac{1/2}{(n-1)\,n+1} - \frac{1/2}{n\,(n+1)+1}\,.\end{aligned}$$

So

$$\begin{aligned}\sum_{n=0}^{N} \frac{n}{n^4+n^2+1} &= \frac{1/2}{(-1)\cdot 0+1} - \frac{1/2}{0\cdot 1+1} + \frac{1/2}{0\cdot 1+1} - \frac{1/2}{1\cdot 2+1} + \cdots \\ &\qquad \cdots + \frac{1/2}{(N-1)\,N+1} - \frac{1/2}{N\,(N+1)+1} \\ &= \frac{1}{2} - \frac{1/2}{N\,(N+1)+1} \quad \underset{N\to\infty}{\longrightarrow} \quad \frac{1}{2}\,.\end{aligned}$$

Hence, the sum is 1/2.